# WHY IS $\pi < 2\phi$?

Abstract. We give a proof of the inequality in the title in terms of *Fibonacci numbers* and *Euler numbers* via a combinatorial argument and asymptotics for these numbers. The result is motivated by Sidorenko's theorem on the number of linear extensions of a partially ordered set and its complement. We conclude with some open problems.

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## 1. Introduction

We start with the inequality

$$(*) \qquad \pi < 2\phi, \quad \text{where} \quad \phi = \frac{1+\sqrt{5}}{2}$$

is the *golden ratio*. The question in the title may seem rather innocent. Of course, $\pi \approx 3.141593 < 2\phi \approx 3.236068$. How deep can this be? Inequality $(*)$ has a conceptual proof in terms of two classical combinatorial sequences. Let us set this up first.

Our first sequence $\{F_n\}$ is the *Fibonacci numbers*, defined by $F_0 = F_1 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. This is perhaps one best known integer sequence which begins

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ \ldots$$

See [K] and [OEIS, A000045] for a trove of information about this wonderful sequence.

Our second sequence $\{E_n\}$ is the sequence of *Euler numbers*. This is a sequence which begins

$$1,\ 1,\ 1,\ 2,\ 5,\ 16,\ 61,\ 272,\ 1385,\ 7936,\ 50521,\ \ldots$$

Our favorite definition of the sequence is via the *Seidel–Entringer triangle* of Seidel [Se]:

$$\begin{array}{c}
\mathbf{1} \\
0 \ \to \ \mathbf{1} \\
\mathbf{1} \ \leftarrow \ 1 \ \leftarrow \ 0 \\
0 \ \to \ 1 \ \to \ 2 \ \to \ \mathbf{2} \\
\mathbf{5} \ \leftarrow \ 5 \ \leftarrow \ \mathbf{4} \ \leftarrow \ 2 \ \leftarrow \ 0 \\
0 \ \to \ 5 \ \to \ \mathbf{10} \ \to \ \mathbf{14} \ \to \ 16 \ \to \ \mathbf{16}
\end{array}$$

Here one alternates direction, following the *ox-plowing* and *boustrophedon* order, start the row with zero, and each new number equal to the previous number plus the number above. For example, $\mathbf{14 = 10 + 4}$ as in the last row of the triangle above. The numbers in this triangle are called *Entringer numbers*. The nonzero first and last number in each row are the Euler numbers. We refer to [S2] for an extensive survey and to [OEIS, A000111] for numerous result and further references.

**Theorem 1.** *For all $n \geq 1$, we have:*

$$E_n \cdot F_n \geq n!$$

For example, $F_3 \cdot E_3 = 2 \cdot 3 = 3!$, $F_4 \cdot E_4 = 5 \cdot 5 = 25 > 4! = 24$, $F_5 \cdot E_5 = 8 \cdot 16 = 128 > 5! = 120$, etc. To understand the connection, recall the classical generating functions for each sequence:

$$\mathcal{F}(t) = \sum_{n=0}^{\infty} F_n t^n = \frac{1}{1-t-t^2} \qquad \text{and}$$

$$\mathcal{E}(t) = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!} = \tan(t) + \sec(t) = \frac{1+\sin(t)}{\cos(t)}.$$

See [S2, Thm. 1.1], [GJ, §3.2.22], [Me] for different proofs of the statement about $\mathcal{E}(t)$. These formulas imply the following (also classical) asymptotics of the numbers

$$F_n \sim \frac{1}{\sqrt{5}}\,\phi^{n+1} \qquad \text{and} \qquad \frac{E_n}{n!} \sim \frac{4}{\pi}\left(\frac{2}{\pi}\right)^n.$$

Here we use $a_n \sim b_n$ as a notation for $a_n/b_n \to 1$ as $n \to \infty$.

In fact, we only need the base of the exponent and not the leading constants. Here $\phi$ is the smallest root of $1 - t - t^2 = 0$. Similarly, $\pi/2$ is the smallest (in absolute value) solution of $\cos(t) = 0$. While the formula for Fibonacci numbers is written in most combinatorics textbooks, the asymptotic formula for Euler numbers is not as well known. We refer to a marvelous monograph [FlS, p. 269] where this is one of the main examples and to the survey [S2, Eq. 1.10].

Now, the theorem and the asymptotics above give

$$1 \le \frac{F_n \cdot E_n}{n!} \sim \frac{4\phi}{\sqrt{5}\pi}\left(\frac{2\phi}{\pi}\right)^n.$$

This implies inequality ($*$). See below why the inequality has to be strict.

The rest of the paper is structured as follows. First, we give a combinatorial proof of the theorem in the next section. We then discuss the origin of the theorem, state exercises that provide details for our proofs, and give some curious open problems (Section 3).

## 2. Combinatorial proof of Theorem 1

We start with classical combinatorial interpretations of Euler and Fibonacci numbers. These will be used to obtain a combinatorial proof of Theorem 1.

First, consider words in the symbols $\{\diamond, \subset, \supset\}$, where each open bracket "$\subset$" is followed by a closed bracket "$\supset$". Denote by $\mathcal{B}_n$ the set of such sequences of length $n$. For example,

$$\mathcal{B}_4 = \big\{\diamond\diamond\diamond\diamond,\ \diamond\diamond\subset\supset,\ \diamond\subset\supset\diamond,\ \subset\supset\diamond\diamond,\ \subset\supset\subset\supset\big\}$$

**Proposition 2.** *We have $|\mathcal{B}_n| = F_n$, for all $n \ge 1$.*

Let $S_n$ denote the set of all permutations of $\{1, 2, \ldots, n\}$, so $|S_n| = n!$. A permutation $\sigma \in S_n$ is called *alternating* if $\sigma(1) < \sigma(2) > \sigma(3) < \sigma(4) > \ldots$ Let $\mathcal{A}_n$ be the set of alternating permutations in $S_n$.

**Proposition 3.** *We have $|\mathcal{A}_n| = E_n$, for all $n \ge 1$.*

These results are well known. The proof of Proposition 2 is an easy exercise in induction. Proposition 3 follows as a corollary of Exercise 5 below using the Seidel–Entringer triangle.

We can now reformulate Theorem 1 as:

$$\big|\mathcal{A}_n\big| \cdot \big|\mathcal{B}_n\big| \ge \big|S_n\big|.$$

Consider now the map $\Phi : \mathcal{A}_n \times \mathcal{B}_n \to S_n$ defined as follows: $\Phi(\sigma, w) = \omega$, where $\omega$ is a permutation obtained from $\sigma \in \mathcal{A}_n$ by swapping numbers in the positions of a pair of consecutive brackets "$\subset\supset$" in $w \in \mathcal{B}_n$. For example,

$$\Phi\big((3,6,2,5,4,7,1,8),\ \diamond\diamond\subset\supset\diamond\subset\supset\diamond\big) \,=\, (3,6,5,2,4,1,7,8).$$

The theorem now follows from the following lemma.

**Lemma 4.** *The map $\Phi : \mathcal{A}_n \times \mathcal{B}_n \to S_n$ is a surjection.*

*Proof.* We need to show that for every $\omega \in S_n$ there exist $\sigma \in \mathcal{A}_n$ and $w \in \mathcal{B}_n$ such that $\omega = \Phi(\sigma, w)$. Denote by $J = \{\omega(2), \omega(4), \ldots\}$ the set of entries in even positions, and let $b = \omega(i)$ be the smallest entry in $J$. Locally, permutation $\omega$ looks as follows:

$$\omega \,=\, (\ldots, x, a, b, c, y \ldots).$$

Now, if $b > a, c$, do nothing. Since $x, y > b$, locally we have the desired inequalities $x > a < b > c < y$. Then repeat the procedure by induction for sub-permutations $\sigma_1 = (\ldots, x, a)$ and $\sigma_2 = (c, y, \ldots)$.

If $b < \max\{a, c\}$, swap $b$ with the largest of these elements. Say this is $a$. Again, locally we have the desired inequalities $x > b < a > c$. Make the word $w$ have a pair of brackets $\subset\supset$ indicating that $a$ and $b$ are swapped. Then repeat the procedure by induction for sub-permutations $\sigma_1 = (\ldots, x)$ and $\sigma_2 = (c, y, \ldots)$. In the case when $\max\{a, c\} = c$, proceed symmetrically with permutations $\sigma_1 = (\ldots, x, a)$ and $\sigma_2 = (b, y, \ldots)$. Let $\sigma$ denote the resulting permutation at the end of the process.

Observe that elements that move ($b$ and possibly $a/c$) move at most once, so the bracket sequence $w$ is well defined. Note also that at every move elements at even positions could only increase and at odd – decrease, and that the parity of positions translates to $\sigma_1$ and $\sigma_2$.

By induction, we obtain alternating inequalities for both $\sigma_1$ and $\sigma_2$; the last element of $\sigma_1$ increases if the last position of $\sigma_1$ is even, and decreases if odd; similarly the first element of $\sigma_2$ is followed by an increase if its position in $w$ is odd, and is followed by a decrease if even. The last element of $\sigma_1$ and the first element of $\sigma_2$ are also smaller or larger than the elements adjacent to them in the middle ($b$ and possibly $a$ and $c$) with inequalities matching the parity of the position (increase if odd and decrease if even). Thus $\sigma$ is alternating, as desired. Finally, note that $\Phi(\sigma, w) = \omega$, by construction. This completes the proof. □

**Exercise 5.** *Denote by $E_{n,k} = |\mathcal{A}_{n+1,k}|$ for $n = 0, 1, \ldots$ and $k = 0, \ldots, n$, where $\mathcal{A}_{n,k} = \{\sigma \in \mathcal{A}_n, \sigma(1) = k\}$ is the set of alternating permutations $\sigma \in S_m$ such that $\sigma(1) = k$. Note that $E_{n,n} = E_n$. Place these numbers in the Seidel–Entringer triangle in the ox-plowing order and prove that they satisfy equations as in the triangle. Deduce Proposition 3.*

**Exercise 6.** *The goal of this exercise is to use the Seidel–Entringer triangle to show that the generating function $\mathcal{E}(t) = \sum_{n=0}^{\infty} E_n x^n/n!$ equals* $\tan(t) + \sec(t)$, *see e.g.* [A] *and* [KGP, Ex. 6.75].

(a) *Consider a triangular array of integers*

$$\begin{array}{ccccc} & & a_{00} & & \\ & a_{10} & & a_{11} & \\ a_{20} & & a_{11} & & a_{22} \\ & & \cdots & & \end{array}$$

*satisfying* $a_{ij} = a_{ij-1} + a_{i-1,j-1}$, i.e. *each entry, except those on the left diagonal, is a sum of the entry to its left and the entry above it to its left. Show that*

$$a_{nn} = \sum_{k=0}^{n} \binom{n}{k} a_{k0} \quad \text{and} \quad \sum_{n=0}^{\infty} a_{nn} \frac{t^n}{n!} = e^t \sum_{n=0}^{\infty} a_{n0} \frac{t^n}{n!}.$$

(b) *Change the signs of the Seidel–Entringer triangle so that we have two positive rows, two negative rows, two positive rows, etc.:*

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 0 & & 1 & & & & \\ & & & -1 & & -1 & & 0 & & & \\ & & 0 & & -1 & & -2 & & -2 & & \\ & 5 & & 5 & & 4 & & 2 & & 0 & \\ 0 & & 5 & & 10 & & 14 & & 16 & & 16 \end{array}$$

*Let* $u(t)$ *and* $v(t)$ *be the exponential generating functions for the left and right diagonals of this signed Seidel–Entringer triangle. Deduce that* $v(t) = e^t u(t)$.

(c) *For* $u(t)$ *and* $v(t)$ *as defined above show that* $-v(t) + 2 = e^{-t} u(t)$.

(d) *Show that* $u(t) = \cosh(t)$, $v(t) = 1 + \tanh(t)$ *and that* $\mathcal{E}(t) = \tan(t) + \sec(t)$.

**Exercise 7.** *Find a pair of permutations* $\sigma, \sigma' \in S_4$ *such that* $\Phi(\sigma) = \Phi(\sigma')$. *Use the proof above to show that* $E_n \cdot F_n > n!(1+\varepsilon)^n$ *for some explicit* $\varepsilon > 0$.

**Exercise 8.** *Denote by* $g(\sigma)$ *the number of times* $\sigma \in S_n$ *appears as the image of* $\Phi$. *Give an explicit combinatorial interpretation of* $g(\sigma)$. *Find* $\sigma \in S_n$ *for which* $g(\sigma)$ *is maximal.*

## 3. Linear extensions of partially ordered sets

We denote by $\mathcal{P}$ a partially ordered set, or *poset* for short, on a set $X$ of $n = |X|$ elements, its order relation is denoted by $\preceq$. Let $e(\mathcal{P})$ be the number of *linear extensions* of $\mathcal{P}$, defined as bijections $f : X \to \{1, \ldots, n\}$ such that $f(u) < f(v)$ for all $u, v \in X$ with $u \preceq v$. For example, if the poset $\mathcal{P}$ forms a single $n$-chain (every two elements are comparable), we have $e(\mathcal{P}) = 1$. On the other hand, if the poset $\mathcal{P}$ forms a single $n$-antichain (no two elements are comparable), we have $e(\mathcal{P}) = n!$. We refer to [T1], [T2, Ch. 8] and [S1, Ch. 3] for standard definitions and notation.

The following geometric construction is our main source of examples. Let $S \subset \mathbb{R}^2$ be a finite set of points. Define an ordering $(x_1, y_1) \preccurlyeq (x_2, y_2)$ when $x_1 \le x_2$ and $y_1 \le y_2$. The resulting poset $\mathcal{P}_S$ is called *two-dimensional*. For example a poset $\mathcal{H}_{p,q}$ with $p+q+1$ elements forming a *hook* (two incomparable chains with $p$ and $q$ elements with an extra minimal element) has $\binom{p+q}{p}$ linear extensions. Similarly, poset $\mathcal{U}_n$ forming a zigzag pattern with $n$ points as in in Figure 1, has Euler number $e(\mathcal{U}_n) = E_n$ of linear extensions.

Another notable example is the poset $\mathcal{C}_k$ with $2 \times k$ elements forming a grid. It has *Catalan number* of linear extensions:

$$e(\mathcal{C}_k) \;=\; \frac{1}{k+1}\binom{2k}{k}$$

(see e.g. [S1, S3] and [OEIS, A000108]).

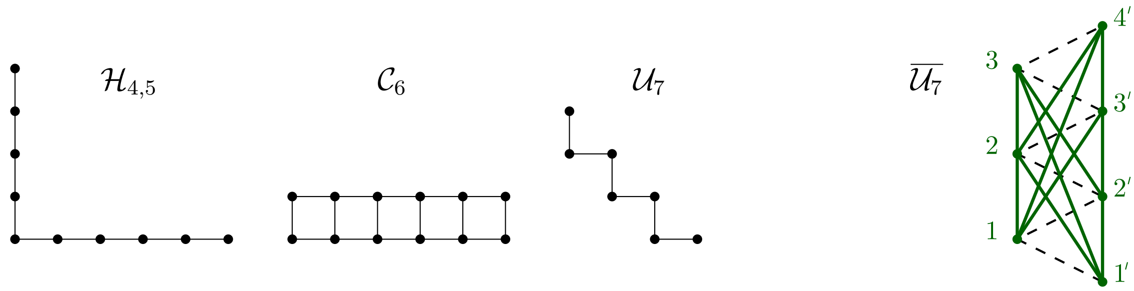


FIGURE 1. Two-dimensional posets $\mathcal{H}_{4,5}$, $\mathcal{C}_6$, $\mathcal{U}_7$ and a complement of $\mathcal{U}_7$.

For a poset $\mathcal{P}$ on set $S$, denote by $\mathrm{C}(\mathcal{P})$ the comparability graph of $\mathcal{P}$, that is the graph with vertices $\mathcal{P}$ and edges $\{x, y\}$ if $x$ and $y$ are comparable in the poset. A poset $\overline{\mathcal{P}}$ on $S$ is called a *complement* if its comparability graph $\mathrm{C}(\overline{\mathcal{P}})$ is the complement of $\mathrm{C}(\mathcal{P})$. Note that a poset can have more than one complement.

**Proposition 9.** *Every two-dimensional poset $\mathcal{P}$ has a complement poset $\overline{\mathcal{P}}$.*

We leave the proof of the proposition to the reader with a hint given in Figure 2.

**Example 10.** *A complement poset $\overline{\mathcal{U}}_n$ is described as follows: elements $X = X_1 \cup X_2$ where $X_1 = \{1, \ldots, \lfloor n/2 \rfloor\}$ and $X_2 = \{1', \ldots, \lceil n/2 \rceil'\}$, and relations $i \preceq j$ and $i' \preceq j'$ if $i < j$, $i \preceq j'$ if $j - i > 1$, and $i' \preceq j$ if $j - i > 0$. See Figure 1 for an example.*

*Next, we use induction to prove that $e(\overline{\mathcal{U}}_n) = F_n$. First note that $e(\overline{\mathcal{U}}_0) = e(\overline{\mathcal{U}}_1) = 1$. For $n \geq 1$, the minimal elements of $\overline{\mathcal{U}}_{n+1}$ are $1$ and $1'$, the minimal elements of $\overline{\mathcal{U}}_{n+1} - \{1'\}$ are $1$ and $2'$, and the minimal element of $\overline{\mathcal{U}}_{n+1} - \{1\}$ is $1'$. Thus, the linear extensions of $\overline{\mathcal{U}_{n+1}}$ either start with $1'$ or with both $11'$. Thus*

$$e(\overline{\mathcal{U}}_{n+1}) \;=\; e(\overline{\mathcal{U}}_{n+1} - \{1'\}) + e(\overline{\mathcal{U}}_{n+1} - \{1, 1'\}).$$

*Since $\overline{\mathcal{U}}_{n+1} - \{1'\}$ and $\overline{\mathcal{U}}_{n+1} - \{1, 1'\}$ are isomorphic to $\overline{\mathcal{U}}_n$ and $\overline{\mathcal{U}}_{n-1}$ respectively, then we obtain that $e(\overline{\mathcal{U}}_{n+1})$ satisfies the Fibonacci recurrence.*

**Exercise 11.** *Describe the complement poset $\overline{\mathcal{H}}_{p,q}$. Show that $e(\overline{\mathcal{H}}_{p,q}) = (p+q+1)p!q!$.*

**Exercise 12.** *Describe the complement poset $\overline{\mathcal{C}}_k$. Prove that $Q_k := e(\overline{\mathcal{C}}_k)$ is the number of permutations $(a_1, \ldots, a_k, b_1, \ldots, b_k) \in S_{2k}$ such that $a_i < b_j$ for all $1 \leq i < j \leq k$.*

**Remark 13.** The problem of computing $e(\mathcal{P})$ is known to be #P-complete [BW], and is difficult even in some seemingly simple cases (see e.g. [ERZ, MPP]).

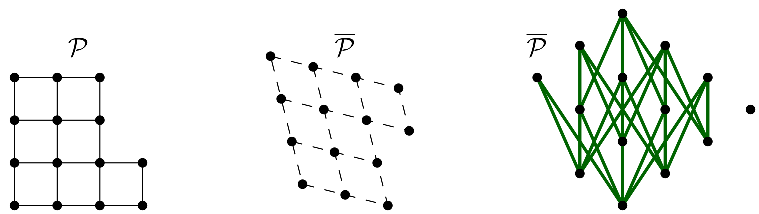


FIGURE 2. The Hasse diagram of a two-dimensional poset $\mathcal{P}$, its complement $\overline{\mathcal{P}}$ as a set of points in $\mathbb{R}^2$ , and the Hasse diagram of $\overline{\mathcal{P}}$.

We are now getting to the heart of the motivation behind Theorem 1.

**Theorem 14** (Sidorenko [Si])**.** *Let $\mathcal{P}$ be a two-dimensional poset with $n$ elements, and let $\overline{\mathcal{P}}$ be a complement of $\mathcal{P}$. We have*

$$e(\mathcal{P})\, e(\overline{\mathcal{P}}) \, \geq \, n!$$

Clearly, when $\mathcal{P}$ is an $n$-chain, we have $\overline{\mathcal{P}}$ is an $n$-antichain, and the inequality is tight. Similarly, by Exercise 11, we have $e(\mathcal{H}_{p,q})e(\overline{\mathcal{H}}_{p,q}) = n!$ since $n = |\mathcal{H}_{p,q}| = p+q+1$ in this case, so the inequality is tight again.

Observe that Exercise 10 and Sidorenko's theorem immediately imply Theorem 1. Note that the proof of Sidorenko's theorem is non-bijective and uses Stanley's interpretation of $e(\mathcal{P})$ as volumes of certain polytopes. The following exercise gives an idea of this connection.

**Exercise 15.** *Consider a polytope $\mathsf{P}_n \subset \mathbb{R}^n$ defined by the following inequalities:*

$$x_i \geq 0, \;\; \text{for all} \;\; 1 \leq i \leq n,$$
$$x_i + x_{i+1} \leq 1, \;\; \text{for all} \;\; 1 \leq i \leq n-1.$$

*Describe* $\mathsf{P}_3$. *Prove that* $\mathsf{P}_n$ *has* $F_{n+1}$ *has vertices. Prove that* $\operatorname{vol}(\mathsf{P}_n) = E_n/n!$.

Our proof of Theorem 1 suggests that there might be a direct combinatorial proof for all two-dimensional posets. If this is too much to hope for, perhaps the following problem can be resolved.

**Open Problem 16.** Give a combinatorial proof that $Q_k C_k \geq (2k)!$, where $Q_k = e(\overline{\mathcal{C}}_k)$. A direct computation shows that the sequence $\{Q_k\}$ starts with $2, 12, 150, 3192, 106290$, etc. Find the generating function

$$\mathcal{Q}(t) \, = \, 1 \, + \, \sum_{k=1}^{\infty} Q_k \, \frac{t^k}{k!}$$

and exact asymptotics for $Q_k$. Note that by Sidorenko's theorem and Exercise 12, we have $Q_k \geq (2k)!/4^k$.

**Remark 17.** We should mention that Sidorenko's theorem can be reduced to a special case of the *Mahler conjecture*, see [BBS]. This leads to a counterpart to Sidorenko's theorem, giving the following upper bound:

$$e(\mathcal{P})\, e(\overline{\mathcal{P}}) \;\le\; n!\, \left(\frac{\pi}{2}\right)^n \big(1+o(1)\big).$$

The proof uses *Santaló's inequality* for polar polytopes, which is sharp for convex bodies. The authors of [BBS] suggest that this bound can be further improved, although not by much.

**Open Problem 18.** Denote by $\mathcal{R}_k$ the poset corresponding to $[k\times k]$ square of points in the grid. It is known that

$$\log e(\mathcal{R}_k) \;=\; \frac{1}{2}\, n\log n \;+\; \left(\frac{1}{2} - 2\log 2\right) n \;+\; O\big(\sqrt{n}\,\log n\big).$$

where $n = k^2$ (see e.g. [MPP] and [OEIS, A039622]). Find the asymptotics of $e(\overline{\mathcal{R}}_k)$. Note that since $e(\mathcal{R}_k) \le \sqrt{n!}$, we have $e(\overline{\mathcal{R}}_k) \ge \sqrt{n!}$. Note also that by the the remark above we have:

$$\log e(\overline{\mathcal{R}}_k) \;=\; \frac{1}{2}\, n\log n \;+\; \Theta(n).$$

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