Some Guesses in the Theory of Partitions

By F. J. Dyson

Professor Littlewood, when he makes use of an algebraic identity, always saves himself the trouble of proving it; he maintains that an identity, if true, can be verified in a few lines by anybody obtuse enough to feel the need of verification. My object in the following pages is to confute this assertion.

In order to save space, I must refer my readers to the first three pages of chapter XIX of Hardy and Wright's *Introduction to the Theory of Numbers* for a detailed account of the idea of a partition, and for a description of the way in which the properties of partitions are represented in the form of algebraic identities. I will always refer to this chapter by the symbol (A). The plan of my argument is as follows. After a few preliminaries I state certain properties of partitions which I am unable to prove: these guesses are then transformed into algebraic identities which are also unproved, although there is conclusive numerical evidence in their support; finally, I indulge in some even vaguer guesses concerning the existence of identities which I am not only unable to prove but also unable to state. I think this should be enough to disillusion anyone who takes Professor Littlewood's innocent view of the difficulties of algebra. Needless to say, I strongly recommend my readers to supply the missing proofs, or, even better, the missing identities.

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The total number of partitions of an integer n into a sum of positive integral parts is denoted by $p(n)$. The "generating function" of $p(n)$ is the infinite series

$$(1)\quad P = \sum_{n=0}^{\infty} p(n)x^n,$$

which is a function of the variable x regular in $|x| < 1$. The form of P is given by two identities of Euler

$$(2)\quad P^{-1} = (1-x)(1-x^2)(1-x^3)(1-x^4)\ldots,$$

$$(3)\quad P^{-1} = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(3n+1)} = 1 - x - x^2 + x^5 + x^7 - \ldots,$$

which are proved in (A).

There are three beautiful arithmetical properties of $p(n)$, which were discovered, and later proved, by Ramanujan, namely:—

$$(4)\quad p(5n+4) \equiv 0 \pmod 5,$$

$$(5)\quad p(7n+5) \equiv 0 \pmod 7,$$

$$(6)\quad p(11n+6) \equiv 0 \pmod{11}.$$

They appear as theorems 359–361 in (A), and can be proved

analytically without much difficulty, using identities like (3); in fact, there are at least four different proofs of (4) and (5).

It would be satisfying to have a direct proof of (4). By this I mean, that although we can prove (in four ways) that the partitions of $5n+4$ can be divided into five equally numerous subclasses, it is unsatisfactory to receive from the proofs no concrete idea of how the division is to be made. We require a proof which will not appeal to generating functions, but will demonstrate by cross-examination of the partitions themselves the existence of five exclusive, exhaustive and equally numerous subclasses. In what follows I shall not give such a proof, but I shall take the first step towards it, as will appear.

The result of subtracting the number of parts in a partition from the largest part we call the "*rank*" of the partition. It is easy to see that the ranks of partitions of n will take the values

$$n-1,\ n-3,\ n-4,\ \ldots,\ 2,\ 1,\ 0,\ -1,\ -2,\ \ldots,\ 4-n,\ 3-n,\ 1-n,$$

and no others. The number of partitions of n with rank m we denote by $N(m, n)$. The number of partitions of n whose rank is congruent to m modulo q we denote by $N(m, q, n)$. Thus

$$(7)\quad N(m, q, n) = \sum_{r=-\infty}^{\infty} N(m + rq, n).$$

The conjecture which I am making is

$$(8)\quad N(0, 5, 5n+4) = N(1, 5, 5n+4) = N(2, 5, 5n+4) = N(3, 5, 5n+4) = N(4, 5, 5n+4);$$

or, in words, the partitions of $5n+4$ are divided into five equally numerous classes according to the five possible values of the least positive residue of their ranks modulo 5. In the same way we have

$$(9)\quad N(0, 7, 7n+5) = N(1, 7, 7n+5) = \ldots = N(6, 7, 7n+5).$$

The truth of (4) and (5) would follow at once, if (8) and (9) could be proved. But the corresponding conjecture with modulus 11 is definitely false.

There is in the theory of partitions a "principle of conjugacy," explained in (A), p. 272. This principle includes a duality relation between the number of parts and the largest part in a partition, and thus partitions of rank m are in a relation of duality with partitions of rank $-m$. It can thus easily be proved that

$$(10)\quad N(m, n) = N(-m, n),$$

$$(11)\quad N(m, q, n) = N(q - m, q, n).$$

Hence (8) reduces to only two independent identities, and (9) to three.

Fortunately, this reduction of our capital is more than offset by other considerations. In fact, (8) and (9) are only the leading and most interesting members in a whole series of similar identities, as listed below:—

(12) $N(1, 5, 5n + 1) = N(2, 5, 5n + 1)$,

(13) $N(0, 5, 5n + 2) = N(2, 5, 5n + 2)$,

(8) $N(0, 5, 5n + 4) = N(1, 5, 5n + 4) = N(2, 5, 5n + 4)$,

(14) $N(2, 7, 7n) = N(3, 7, 7n)$,

(15) $N(1, 7, 7n + 1) = N(2, 7, 7n + 1) = N(3, 7, 7n + 1)$,

(16) $N(0, 7, 7n + 2) = N(3, 7, 7n + 2)$,

(17) $N(0, 7, 7n+3) = N(2, 7, 7n+3), N(1, 7, 7n+3) = N(3, 7, 7n+3)$,

(18) $N(0, 7, 7n + 4) = N(1, 7, 7n + 4) = N(3, 7, 7n + 4)$,

(9) $N(0, 7, 7n+5) = N(1, 7, 7n+5) = N(2, 7, 7n+5) = N(3, 7, 7n+5)$,

(19) $N(0, 7, 7n+6) + N(1, 7, 7n+6) = N(2, 7, 7n+6) + N(3, 7, 7n+6)$.

Of these relations, only (8) and (9) give any arithmetical properties of $p(n)$. The rest of the series is interesting only because it may throw some light on (8) and (9); as yet, however, I have been unable to find any plan behind the apparently haphazard distribution of these identities.

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I now proceed to put the equations into algebraic form by means of generating functions. The algebraic form is useful for numerical computations, and also seems to offer the best prospect of arriving at proofs I shall omit the calculations, but on the basis of formulae to be found in (A) the generating function $G(m) = \sum_{n=0}^{\infty} N(m, n)x^n$ takes the form

(20) $$G(m) = P\sum_{r=1}^{\infty} (-1)^{r-1} (x^{\frac{1}{2}r(3r-1)} - x^{\frac{1}{2}r(3r+1)})x^{mr},$$

where P is given by (1). This form is valid when $m \geq 0$ and, with certain reservations, when $m < 0$ also; but when $m < 0$ it is simpler to use the relation

(21) $G(m) = G(-m)$,

deducible from (10). (20) and (21) can thus be combined in the formula

(22) $$G(m) = P\sum_{r=1}^{\infty} (-1)^{r-1} (x^{\frac{1}{2}r(3r-1)} - x^{\frac{1}{2}r(3r+1)})x^{r|m|},$$

valid for all values of m. The series on the right of (22) is simple in form, and is of the type called "false theta-functions" by Professor Rogers, if that is any consolation.

The generating function of $N(m, q, n)$ is

(23) $$G(m, q) = \sum_{n=0}^{\infty} N(m, q, n)x^n = \sum_{s=-\infty}^{\infty} G(m + sq),$$

by (7). We suppose that q is a positive integer, and that $0 < m < q$. Then we substitute from (22) into (23), and the summation with respect to s can be performed in finite terms, giving the final result

(24) $$G(m, q) = P\sum_{r=1}^{\infty} (-1)^{r-1} \frac{(x^{\frac{1}{2}r(3r-1)} - x^{\frac{1}{2}r(3r+1)})(x^{mr} + x^{(q-m)r})}{(1 - x^{qr})}.$$

The coefficients in P have been tabulated as far as x^{600}, and the coefficients in the series on the right of (24) are all very small; (24) therefore affords much the quickest way of calculating the values of $N(m, q, n)$ numerically. The equations (12) — (19) can be expressed in analytical form by means of (24); as an example we take the equation $N(1, 7, n) = N(3, 7, n)$, which leads to the following statement.

(25) *In the power-series*

$$P\sum_{r=1}^{\infty} (-1)^{r-1} \frac{(x^{\frac{1}{2}r(3r-1)} - x^{\frac{1}{2}r(3r+1)})(x^r - x^{3r} - x^{4r} + x^{6r})}{(1 - x^{7r})}$$

the coefficients of $x^{7n+1}, x^{7n+3}, x^{7n+4}, x^{7n+5}$ *vanish identically.*

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It is interesting for several reasons to examine the numerical evidence in some detail. First comes a table of the values of the two differences

$$a = N(0, 5, n) - N(2, 5, n), \; b = N(1, 5, n) - N(2, 5, n)$$

for values of n up to 50.

n	a	b	n	a	b	n	a	b	n	a	b	n	a	b
1	1	0	2	0	1	3	0	−1	4	0	0	5	0	1
6	1	0	7	0	0	8	−1	−1	9	0	0	10	0	1
11	1	0	12	0	1	13	0	−2	14	0	0	15	−1	1
16	1	0	17	0	1	18	−1	−2	19	0	0	20	0	1
21	2	0	22	0	1	23	−1	−2	24	0	0	25	−1	2
26	1	0	27	0	0	28	−1	−3	29	0	0	30	0	2
31	2	0	32	0	2	33	−1	−3	34	0	0	35	−1	2
36	2	0	37	0	1	38	−2	−4	39	0	0	40	−1	2
41	3	0	42	0	2	43	−1	−4	44	0	0	45	−2	3
46	3	0	47	0	2	48	−2	−5	49	0	0	50	0	3

What is remarkable about this table, apart from the columns of zeros, is the regularity of behaviour of a and b within each arithmetic progression of common difference 5, and also the smallness of the values. If the partitions of 48 were distributed "at random" into five classes, we should expect statistically that the numbers of partitions in each pair of classes would differ by anything from 100 to 250. Clearly, then, the values of a and b, namely −2 and −5, require some explanation. It seems certain that there remain to be discovered alternative forms for the generating functions of a and b, which will make it intuitive when these coefficients vanish, when they are positive, when negative, and why in general they are so small. And exactly the same remarks apply to the coefficients relating to the modulus 7.

In the case of modulus 7, we obtain from equations (12)–(19) some striking congruence properties of $p(n)$. We write

$$c = \mathrm{N}(0, 7, n) - \mathrm{N}(3, 7, n),\ d = \mathrm{N}(1, 7, n) - \mathrm{N}(3, 7, n),$$
$$e = \mathrm{N}(2, 7, n) - \mathrm{N}(3, 7, n).$$

Then, by (11), $p(n) \equiv c + 2d + 2e \pmod 7$.

Now using (12)–(19), we find

$$(26)\begin{cases} \text{when } n \equiv 1,\ p(n) \equiv c \pmod 7, \\ \text{when } n \equiv 2,\ p(n) \equiv 2d + 2e \pmod 7, \\ \text{when } n \equiv 3,\ p(n) \equiv 3c \pmod 7, \\ \text{when } n \equiv 4,\ p(n) \equiv -5e \pmod 7. \end{cases}$$

Below is a table of the actual least positive residues of $p(n)$ (mod 7) for various values of n.

n	1	8	15	22	29	36	43	50	57	64
lpr	1	1	1	1	1	1	2	1	2	2
n	2	9	16	23	30	37	44	51	58	65
lpr	2	2	0	2	4	0	2	4	2	2
n	3	10	17	24	31	38	45	52	59	66
lpr	3	0	3	0	3	3	3	0	3	3
n	4	11	18	25	32	39	46	53	60	67
lpr	5	0	0	5	5	0	5	0	5	5

It will be seen that these residues exhibit a strong regularity, which is sufficiently explained by the congruence relations (26) together with the fact that the values of c, d and e are initially very small.

For comparison I append a similar table of the least positive residues of $p(n)$ (mod 11) for various values of n.

n	1	12	23	34	45	56	67	78	89	100
lpr	1	0	1	1	1	0	1	0	1	1
n	2	13	24	35	46	57	68	79	90	101
lpr	2	2	2	0	2	2	2	2	2	2
n	3	14	25	36	47	58	69	80	91	102
lpr	3	3	0	3	3	0	3	3	3	3
n	4	15	26	37	48	59	70	81	92	103
lpr	5	0	5	0	5	0	5	0	5	5
n	5	16	27	38	49	60	71	82	93	104
lpr	7	0	7	0	0	7	7	0	7	0

The regularity of this table is of precisely the same character as the regularity of the previous one. One is thus led irresistibly to the

conclusion that there must be some analogue modulo 11 to the relations (26).

I hold in fact:

That there exists an arithmetical coefficient similar to, but more recondite than, the rank of a partition; I shall call this hypothetical coefficient the "crank" of the partition, and denote by $\mathrm{M}(m, q, n)$ the number of partitions of n whose crank is congruent to m modulo q;

that $\mathrm{M}(m, q, n) = \mathrm{M}(q - m, q, n)$;

that

$$\mathrm{M}(0, 11, 11n + 6) = \mathrm{M}(1, 11, 11n + 6) = \mathrm{M}(2, 11, 11n + 6)$$
$$= \mathrm{M}(3, 11, 11n + 6) = \mathrm{M}(4, 11, 11n + 6);$$

that numerous other relations exist analogous to (12)–(19), and in particular

$$\mathrm{M}(1, 11, 11n+1) = \mathrm{M}(2, 11, 11n+1) = \mathrm{M}(3, 11, 11n+1)$$
$$= \mathrm{M}(4, 11, 11n + 1);$$

that $\mathrm{M}(m, 11, n)$ has a generating function not completely different in form from (24);

that the values of the differences such as $\mathrm{M}(0, 11, n) - \mathrm{M}(4, 11, n)$ are always extremely small compared with $p(n)$.

Whether these guesses are warranted by the evidence, I leave to the reader to decide. Whatever the final verdict of posterity may be, I believe the "crank" is unique among arithmetical functions in having been named before it was discovered. May it be preserved from the ignominious fate of the planet Vulcan!

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Short Vision

By A. C. Falconer

Thought is the only way which leads to life
All else is hollow spheres
Reflecting back
In heavy imitation
And blurred degeneration
A senseless image of our world of thought.

Man *thinks* he is the thought which gives him life!
He binds a sheaf and claims it as himself!
He *is* a ring through which pass swinging ropes
Which merely move a little as he slips.

The Ropes are Thought
The Space is Time
Could he but see, then he might climb.