Positive dependence for colored percolation

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For uniform random 4-colorings of graph edges with colors $\{a, b, c, d\}$, every two colors form a $\frac{1}{2}$-percolation, and every two overlapping pairs of colors form independent $\frac{1}{2}$-percolations. We show positive mutual dependence for pairs of colors ab, ac and ad, and negative mutual dependence for pairs of colors ab, ac and bc. The proof is based on a generalization of the Harris–Kleitman inequalities. We apply the results to crossing probabilities for the colored bond and site percolation, and to colored critical percolation that we also define.

Introduction. The study of *percolation* goes back to the 1957 paper by Broadbent and Hammersley [1] and has been incredibly popular in the last few decades across the sciences. It remains one of the most applied statistical models, reaching far corners of statistical physics and probability, and fields as disparate a materials science, network theory and seismology, see e.g. [2–4].

Despite remarkable recent advances, many problems remain open and continued to be actively pursued, see e.g. [5–8]. Note that specific models of percolation wary greatly depending on the scientific context and applications. Here we consider the *colored* bond (site) percolation, where each graph edge (vertex) takes random color, see e.g. [3, 9, 10].

As one studies random events, one is naturally concerned about their correlations. This led to correlation inequalities, the first of which, *Harris–Kleitman inequality* [11, 12] was discovered independently in probability and graph theory. It shows that every two increasing (or two decreasing) random events on the same probability space are positively correlated. On the other hand, when one event is increasing and another is decreasing, such events are negatively correlated. Outside of its fundamental applications to statistical physics and probability, this result has numerous applications in graph theory [13, 14], order theory [15, 16] and algebraic combinatorics [17].

There are many generalizations and variations on the Harris–Kleitman inequality, see e.g. [18–20], including intensely studied but largely mysterious generalizations to multiple functions [21–23]. In this paper we consider k events $\mathcal{U}_i$ such that every $(k-1)$ of them are mutually independent. To quantify correlations we study the ratio

$$\mu := \frac{\mathbb{P}(\mathcal{U}_1 \cap \cdots \cap \mathcal{U}_k)}{\mathbb{P}(\mathcal{U}_1) \cdots \mathbb{P}(\mathcal{U}_k)}$$

which, we call *mutual dependence*. We prove a general result extending the Harris–Kleitman inequality from $k = 2$ to all k. We concentrate on the case $k = 3$, which is the first nontrivial example and is of independent interest.

Our main application is to 4-colored percolation on infinite graphs and graphs with symmetry. We show that $\mu \geq 1$ or that $\mu \leq 1$ depending on a situation, and in some cases conjecture that our bounds are asymptotically tight. Additionally, we introduce a new colored critical probability for infinite graphs which turns out to be closely related to the usual critical probability.

Positive correlation in percolation. We first illustrate the Harris–Kleitman inequality. Let $G = (V, E)$ be a simple graph, which can be finite or infinite. Consider a *p-percolation* defined by independently at random deleting edges of G with probability $(1 - p)$. We write $\mathbb{P}_p(x \leftrightarrow y)$ for the probability that vertices $x, y \in V$ are connected.

In its basic application, the Harris–Kleitman inequality proves a positive correlation of connectivity of two pairs of vertices:

$$\mathbb{P}_p(x \leftrightarrow y, u \leftrightarrow v) \geq \mathbb{P}_p(x \leftrightarrow y)\, \mathbb{P}_p(u \leftrightarrow v), \quad (1)$$

for all $x, y, u, v \in V$. Equivalently, this says that the probability that two vertices are connected increases if *some other* two vertices are connected, even if these two vertices are far apart in the graph: $\mathbb{P}_p(x \leftrightarrow y \,|\, u \leftrightarrow v) \geq \mathbb{P}_p(x \leftrightarrow y)$. This implies that the *critical probability* $p_c := \sup\{p : \mathbb{P}_p(x \leftrightarrow \infty) = 0\}$ is independent on the vertex x in every connected graph, see e.g. [2, 24]. The idea is that for two vertices x, y, the ratio $\frac{\mathbb{P}_p(x \leftrightarrow \infty)}{\mathbb{P}_p(y \leftrightarrow \infty)}$ can not go below $\mathbb{P}_p(x \leftrightarrow y)$. For the case when $G = \mathbb{Z}^2$ is a square lattice, Harris

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used the inequality to prove that $p_c \geq \frac{1}{2}$ [11]. Famously, Kesten [25] established the equality $p_c = \frac{1}{2}$ twenty years later.

Denote by 2^E the collection of all subsets of E. A subcollection $\mathcal{A} \subseteq 2^E$ is called *closed upward*, if $A + e \in \mathcal{A}$ for every $A \in \mathcal{A}$ and $e \in E \smallsetminus A$. Similarly, $\mathcal{A}$ is *closed downward*, if $A - e \in \mathcal{A}$ for every $A \in \mathcal{A}$ and $e \in A$. We think of $\mathcal{A}$ as *graph property*, and write $\mathbb{P}_p(\mathcal{A})$ for the probability that the property holds for a p-percolation. In this notation, the *Harris–Kleitman inequality* states:

$$\mathbb{P}_p(\mathcal{A} \cap \mathcal{B}) \geq \mathbb{P}_p(\mathcal{A})\, \mathbb{P}_p(\mathcal{B}), \qquad (2)$$

for every two closed upward subcollections $\mathcal{A}, \mathcal{B}$. For $\mathcal{A} = \{H : x \leftrightarrow y\}$ and $\mathcal{B} = \{H : u \leftrightarrow v\}$ we obtain (1). Note that (2) holds also for every two closed downward $\mathcal{A}, \mathcal{B}$. Indeed, their complements $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$ will be closed upwards and

$$\mathbb{P}_p(\mathcal{A} \cap \mathcal{B}) = 1 - \mathbb{P}_p(\bar{\mathcal{A}}) - \mathbb{P}_p(\bar{\mathcal{B}}) + \mathbb{P}_p(\bar{\mathcal{A}} \cap \bar{\mathcal{B}})$$
$$\geq 1 - \mathbb{P}_p(\bar{\mathcal{A}}) - \mathbb{P}_p(\bar{\mathcal{B}}) + \mathbb{P}_p(\bar{\mathcal{A}})\mathbb{P}_p(\bar{\mathcal{B}}) = \mathbb{P}_p(\mathcal{A})\mathbb{P}_p(\mathcal{B}).$$

When $\mathcal{A}$ is closed upward and $\mathcal{B}$ is closed downward, the negative correlation follows by the same argument.

Now, let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be pairwise independent events. We say that they have *positive mutual dependence* if $\mathbb{P}(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) \geq \mathbb{P}(\mathcal{U})\, \mathbb{P}(\mathcal{V})\, \mathbb{P}(\mathcal{W})$. Similarly, we say that they have *negative mutual dependence* if $\mathbb{P}(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) \leq \mathbb{P}(\mathcal{U})\, \mathbb{P}(\mathcal{V})\, \mathbb{P}(\mathcal{W})$.

Examples of mutual dependence. To get some idea of mutual dependence, consider the simplest example of three events, which are pairwise independent but not mutually independent. Roll a tetrahedral die with sides labeled $\{a, b, c, d\}$. Then three events $\mathcal{U} = \{a, b\}$, $\mathcal{V} = \{a, c\}$ and $\mathcal{W} = \{b, c\}$ are pairwise independent, but their intersection has probability 0, which is less than $\frac{1}{8}$ if they were mutually independent. In other words, events $\mathcal{U}, \mathcal{V}, \mathcal{W}$ have negative mutual dependence. On the other hand, if we replace $\mathcal{W}$ with the complement $\mathcal{W}' = \{a, d\}$, then $\mathcal{U}, \mathcal{V}, \mathcal{W}'$ are still pairwise independent, but their intersection has probability $\frac{1}{4}$. In other words, events $\mathcal{U}, \mathcal{V}, \mathcal{W}'$ have positive mutual dependence.

Generalizing previous example, let $n = 2m+1$ be an odd number of tetrahedral die rolls. Denote by $\mathcal{U}, \mathcal{V}, \mathcal{W}$ the events that labels $\{a, b\}$, $\{a, c\}$, $\{b, c\}$ are a majority of the samples, respectively. As before, $\mathcal{U}, \mathcal{V}, \mathcal{W}$ are pairwise independent, and $\mathbb{P}(\mathcal{U}) = \mathbb{P}(\mathcal{V}) = \mathbb{P}(\mathcal{W}) \to \frac{1}{2}$ and $m \to \infty$. It is easy to see by a direct calculation that $\mathbb{P}(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) \to \frac{1}{8}$. It is less obvious that $\mathcal{U}, \mathcal{V}, \mathcal{W}$ have negative mutual dependence for all m: $\mathbb{P}(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) < \frac{1}{8}$. Similarly, for the event $\mathcal{W}' = \overline{\mathcal{W}}$ that labels $\{a, d\}$ are a majority of the samples, we have positive mutual dependence for all m: $\mathbb{P}(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}') = \frac{1}{4} - \mathbb{P}(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) > \frac{1}{8}$.

Moving to random graphs, consider a finite graph $G = (V, E)$ and a uniform random coloring of E with $\{a, b, c, d\}$. Denote by E_{ab}, E_{ac}, E_{bc}, random subsets of E with the corresponding colors. Observe that these are also are pairwise independent. Now let $\mathcal{U}, \mathcal{V}, \mathcal{W}$, be the events that graphs $G_{ab} = (V, E_{ab})$, $G_{ac} = (V, E_{ac})$, $G_{bc} = (V, E_{bc})$, are connected, respectively. Denote $p = \mathbb{P}(\mathcal{U}) = \mathbb{P}(\mathcal{V}) = \mathbb{P}(\mathcal{W})$. Then $\mathcal{U}, \mathcal{V}, \mathcal{W}$ are also pairwise independent, and we have $\mathbb{P}(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) \leq p^2$. The theorem below shows a negative correlation: $\mathbb{P}(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) \leq p^3$. Similarly, let $\mathcal{W}'$ be the event that the graph $G_{ad} = \overline{G_{bc}}$ is connected. The theorem below shows $\mathcal{U}, \mathcal{V}, \mathcal{W}'$ have a positive correlation: $\mathbb{P}(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}') \geq p^3$.

As a graph property, connectivity is closed upward. In the opposite direction, the theorem below reverses the inequality for closed downward properties. Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$, be the events that graphs G_{ab}, G_{ac}, G_{bc}, are triangle-free, respectively. Denote $p = \mathbb{P}(\mathcal{U}) = \mathbb{P}(\mathcal{V}) = \mathbb{P}(\mathcal{W})$. The theorem below shows a positive mutual dependence: $\mathbb{P}(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}) \geq p^3$. Similarly, let $\mathcal{W}'$ be the event that G_{ad} is triangle-free. The theorem below shows a negative correlation: $\mathbb{P}(\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}') \leq p^3$.

The theorem below also applies to nonisomorphic events. Fix six vertices $x, x', y, y', z, z' \in V$. We use $x \leftrightarrow_{ab} x'$ to mean connectivity in graph (V, E_{ab}). Then the events $x \leftrightarrow_{ab} x'$, $y \leftrightarrow_{ac} y'$, $z \leftrightarrow_{ad} z'$ are pairwise independent and mutually positively dependent:

$$\mathbb{P}(x \leftrightarrow_{ab} x', y \leftrightarrow_{ac} y', z \leftrightarrow_{ad} z')$$
$$\geq \mathbb{P}(x \leftrightarrow_{ab} x')\, \mathbb{P}(y \leftrightarrow_{ac} y')\, \mathbb{P}(z \leftrightarrow_{ad} z'),$$

thus giving a variation on (1). In general, the probabilities in the RHS are distinct. When G is a lattice and $x' = y' = z' = \infty$, the probabilities on the RHS are equal; denote them by ρ. We then have $\mathbb{P}(x \leftrightarrow_{ab} \infty, y \leftrightarrow_{ac} \infty) = \rho^2$ but

$$P_G := \mathbb{P}(x \leftrightarrow_{ab} \infty,\, y \leftrightarrow_{ac} \infty,\, z \leftrightarrow_{ad} \infty) \geq \rho^3.$$

When G is a square lattice, we have $p_c = \frac{1}{2}$ and $\rho = 0$, so the result is trivial [2, 24]. However, when G is a triangular lattice we have $p_c = 0.3473... < \frac{1}{2}$ and $\rho > 0$ [26]. Note also that even proving $P_G > 0$ is nontrivial in this case and unattainable by any other means. In particular, using color a for each of the three $\frac{1}{2}$-percolations is not enough to show $P_G > 0$, since $p_c > \frac{1}{4}$ and so $\mathbb{P}(x \leftrightarrow_a \infty) = 0$.

Positive dependence in colored percolation. We are now ready to formalize the approach above to state the result in full generality.

Let $f : E \to \{a, b, c, d\}$ be a uniform random coloring of the edges of G, where each edge is colored uniformly and independently. As before, denote by E_s, $s \in \{a, b, c, d\}$, a subset of edges of the corresponding color. Similarly, for every two distinct colors $s, t \in \{a, b, c, d\}$, let $E_{st} := E_s \cup E_t$. One can think of E_{st} as either a $\frac{1}{2}$-percolation or a uniformly random subset of edges of G, so that $G_{st} = (V, E_{st})$ is a uniform random subgraph of G.

Theorem 1. *Let $\mathcal{U}, \mathcal{V}, \mathcal{W}$ be closed upward graph properties. Denote by $\mathcal{U}_{ab}$, $\mathcal{V}_{ac}$ and $\mathcal{W}_{bc}$ the corresponding properties of G_{ab}, G_{ac} and G_{bc}, respectively. Then the events $\mathcal{U}_{ab}$, $\mathcal{V}_{ac}$ and $\mathcal{W}_{bc}$ are pairwise independent, but have negative mutual dependence:*

$$\mathbb{P}(\mathcal{U}_{ab} \cap \mathcal{V}_{ac} \cap \mathcal{W}_{bc}) \le \mathbb{P}(\mathcal{U}_{ab})\, \mathbb{P}(\mathcal{V}_{ac})\, \mathbb{P}(\mathcal{W}_{bc}), \quad (3)$$

where the probability is over uniform random colorings $f : E \to \{a, b, c, d\}$. Similarly, events $\mathcal{U}_{ab}$, $\mathcal{V}_{ac}$ and $\mathcal{W}_{ad}$ are pairwise independent, but have positive mutual dependence:

$$\mathbb{P}(\mathcal{U}_{ab} \cap \mathcal{V}_{ac} \cap \mathcal{W}_{ad}) \ge \mathbb{P}(\mathcal{U}_{ab})\, \mathbb{P}(\mathcal{V}_{ac})\, \mathbb{P}(\mathcal{W}_{ad}), \quad (4)$$

where $\mathcal{W}_{ad}$ is the property of E_{ad}. Additionally, for $\mathcal{U}, \mathcal{V}, \mathcal{W}$ closed downward graph properties, the inequalities in both (3) *and* (4) *are reversed.*

Since all E_{st} are $\frac{1}{2}$-percolations, we can rewrite the RHS of both (3) and (4) as a more symmetric product:

$$\mathbb{P}_{\frac{1}{2}}(\mathcal{U})\, \mathbb{P}_{\frac{1}{2}}(\mathcal{V})\, \mathbb{P}_{\frac{1}{2}}(\mathcal{W}). \quad (5)$$

The proof of the theorem is given in the appendix. After a quick argument proving pairwise independence, we now proceed to a number of applications of the theorem to many percolation examples.

Why pairwise independence? Let E_1 and E_2 be two independent $\frac{1}{2}$-percolations, and let $E_3 = E_1 \oplus E_2$ to be the new $\frac{1}{2}$-percolation where every edge is open if it is open in exactly one of E_1, E_2. Consider the coloring

$$f(e) := \begin{cases} a & \text{if } \ e \in E_1 \cap E_2 \\ b & \text{if } \ e \in E_1, e \notin E_2 \\ c & \text{if } \ e \in E_2, e \notin E_1 \\ d & \text{if } \ e \notin E_1, e \notin E_2 \end{cases}$$

Then $E_{ab} = E_1$, $E_{ac} = E_2$, $E_{bc} = E_3$, which implies the pairwise independence. This observation is motivational and generalizes to $k \ge 2$ mutually independent $\frac{1}{2}$-percolations (see the appendix).

Crossing probabilities in a rectangle. Let $G = (V, E)$ be a $n \times (n + 1)$ rectangle as in Figure 1. Consider a uniform random edge coloring $f : E \to \{a, b, c, d\}$. Note that E_{ab}, E_{ac} and E_{ad} are pairwise independent bond $\frac{1}{2}$-percolations with free boundary conditions (BC). Let $\mathcal{U} = \{12 \leftrightarrow 34\}$ be the connectivity property of the opposite sides of G, and recall that $\mathbb{P}_{\frac{1}{2}}(\mathcal{U}_{ab}) = \frac{1}{2}$, see e.g. [24]. Then (4) gives:

$$\mathbb{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) \ \ge \ \mathbb{P}_{\frac{1}{2}}(\mathcal{U})^3 \ = \ \tfrac{1}{8}, \qquad (6)$$

for all $n \ge 1$. On the other hand, by the pairwise independence we have:

$$\mathbb{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) \le \mathbb{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac}\big) = \mathbb{P}_{\frac{1}{2}}(\mathcal{U})^2 = \tfrac{1}{4}.$$

Note that as a function of p the crossing probability in a rhombus under p-percolation has a sharp threshold [24], so the trivial lower bound is unhelpful:

$$\mathbb{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) \ge \mathbb{P}(\mathcal{U}_a) = \mathbb{P}_{\frac{1}{4}}(\mathcal{U}) \xrightarrow[n \to \infty]{} 0$$

For $n = 30$, the sampling of $N = 4 \cdot 10^7$ trials gives an approximation $\mathbb{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) = 0.125098 \pm 0.000052$. We conjecture that this probability is $\frac{1}{8}$ in the limit $n \to \infty$.

Crossing probabilities in a rhombus. Let $G = (V, E)$ be a m-rhombus on the triangular lattice, see Figure 1. Consider a uniform random vertex coloring $f : V \to \{a, b, c, d\}$. Note that V_{ab}, V_{ac} and V_{ad} are pairwise independent site $\frac{1}{2}$-percolations with free BC. Let $\mathcal{U} = \{12 \leftrightarrow 34\}$ and $\mathcal{U}' = \{14 \leftrightarrow 23\}$ be connectivity properties of the opposite sides of G. Recall that $\mathbb{P}_{\frac{1}{2}}(\mathcal{U}_{ab}) + \mathbb{P}_{\frac{1}{2}}(\mathcal{U}'_{cd}) = 1$ by a topological argument, so $\mathbb{P}_{\frac{1}{2}}(\mathcal{U}) = \mathbb{P}_{\frac{1}{2}}(\mathcal{U}') = \frac{1}{2}$ by the symmetry. Then (3) and (4) give:

$$\begin{aligned} \mathbb{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{bc}\big) \ &\le \ \mathbb{P}_{\frac{1}{2}}(\mathcal{U})^3 \ = \ \tfrac{1}{8}, \\ \mathbb{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) \ &\ge \ \mathbb{P}_{\frac{1}{2}}(\mathcal{U})^3 \ = \ \tfrac{1}{8}, \end{aligned} \qquad (7)$$

for all $m \ge 1$. We conjecture that

$$\mathbb{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{bc}\big) \text{ and } \mathbb{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) \to \tfrac{1}{8}$$

as $m \to \infty$. If this holds, we also have other similar limits, e.g.

$$\begin{aligned} &\mathbb{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}'_{bc}\big) \\ &\qquad = \mathbb{P}_{\frac{1}{2}}(\mathcal{U})^2 - \mathbb{P}\big(\mathcal{U}_{ab} \cap \mathcal{U}_{ac} \cap \mathcal{U}_{ad}\big) \ \to \ \tfrac{1}{8}. \end{aligned}$$

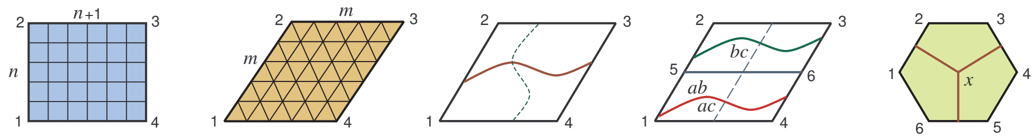

FIG. 1: Crossing probabilities in a rectangle, rhombus and a hexagon.

This is in contrast with limits such as $\mathbb{P}\big(\mathcal{U}_{ab}\cap\mathcal{U}_{bc}\cap\mathcal{U}_{cd}\big)$ which can be computed using *Watts' formula* [27] (see also [28, 29]).

Crossing probabilities in a hexagon. Consider a regular hexagon $G=(V,E)$ on the triangular lattice with side lengths ℓ, see Figure 1. Consider a site $\frac{1}{2}$-percolations with free BC as above. Let $\mathcal{U}:=\big\{\exists x\in V : x\leftrightarrow 12, x\leftrightarrow 34, x\leftrightarrow 56\big\}$ be the joint connectivity property of the percolation graph. It was computed by Simmons [30] (see also [31]), that $\mathbb{P}_{\frac{1}{2}}(\mathcal{U})=0.2556897...$ in the limit $\ell\to\infty$. Consider a uniform random vertex coloring $f: V\to\{a,b,c,d\}$. Then (4) gives:

$$\mathbb{P}_{\frac{1}{2}}(\mathcal{U})^2 = 0.0653772... \geq \mathbb{P}\big(\mathcal{U}_{ab}\cap\mathcal{U}_{ac}\cap\mathcal{U}_{ad}\big) \geq \mathbb{P}_{\frac{1}{2}}(\mathcal{U})^3 = 0.0167162...$$

in the limit $\ell\to\infty$. Similarly, the inequality (3) gives:

$$\mathbb{P}\big(\mathcal{U}_{ab}\cap\mathcal{U}_{ac}\cap\mathcal{U}_{bc}\big) \leq \mathbb{P}_{\frac{1}{2}}(\mathcal{U})^3 = 0.0167162...$$

in the limit $\ell\to\infty$. For $\ell=30$, the sampling of $N=64000$ trials gives $\mathbb{P}\big(\mathcal{U}_{ab}\cap\mathcal{U}_{ac}\cap\mathcal{U}_{ad}\big)=0.0172\pm0.0005$ and $\mathbb{P}\big(\mathcal{U}_{ab}\cap\mathcal{U}_{ac}\cap\mathcal{U}_{bc}\big)=0.0166\pm0.0005$. We conjecture that both probabilities are $\mathbb{P}_{\frac{1}{2}}(\mathcal{U})^3=0.0167162...$ in the limit $\ell\to\infty$.

New critical probability. Recall the setting we discussed earlier. Let $G=(V,E)$ be an infinite connected graph. Consider a uniform random coloring $f: E\to\{a,b,c,d\}$. For a vertex $x\in V$, consider

$$P(x) := \mathbb{P}\big(x\leftrightarrow_{ab}\infty, x\leftrightarrow_{ac}\infty, x\leftrightarrow_{ad}\infty\big), \quad (8)$$

where $x\leftrightarrow_{st}\infty$ means that x belongs to an infinite cluster of st-colored edges. Now (4) gives:

$$\mathbb{P}_{\frac{1}{2}}\big(x\leftrightarrow\infty\big)^2 \;\geq\; P(x) \;\geq\; \mathbb{P}_{\frac{1}{2}}\big(x\leftrightarrow\infty\big)^3. \quad (9)$$

Suppose now that G is a lattice with critical probability $p_c<\frac{1}{2}$. For $\alpha\in[0,\frac{1}{4}]$, consider a random 5-coloring $f: E\to\{a,b,c,d,\diamond\}$, where the probabilities of colors a,b,c,d are α, and the probability of $\diamond$ is $(1-4\alpha)$. Then E_{ab}, E_{ac} and E_{ad} are pairwise independent 2α-percolations. Denote by $P_\alpha(x)$ the probability given by (8) in this deformation. Define the following *critical probability* for the colored percolation:

$$\alpha_c := \sup\big\{\alpha : P_\alpha(x)=0\big\}.$$

Now (9) implies that $\alpha_c\leq\frac{1}{2}p_c$ while the examples above suggest that $\alpha_c=\frac{1}{2}p_c$. The numerical experiments also seem to confirm this. We tested the colored bond and site percolations on a triangular lattice with $p_c=2\sin\frac{\pi}{18}=0.3473...$ and $p_c=\frac{1}{2}$, respectively [26]. Similarly, we tested the colored bond and site percolations on a cubic lattice $G=\mathbb{Z}^3$ with $p_c=0.2488...$ and $p_c=0.3116...$, respectively (see e.g. [32]). The results are given in Figure 2.

Conclusions. The subject of positive dependence for colored percolation is largely unexplored and can be viewed as a special case of algebraic inequalities for cumulants of positive functions. The latter has been actively studied (see [21, 22] for recent references), but the type of inequalities we consider are new.

In full generality, our results extend the Harris–Kleitman inequality (2) to multiple pairwise independent events. This allows us to give lower and upper bounds on the mutual dependence of these events, which are asymptotically tight for the (conjectured) crossing probabilities of the colored percolation on lattices, exhibiting the same phenomenon as the majority property.

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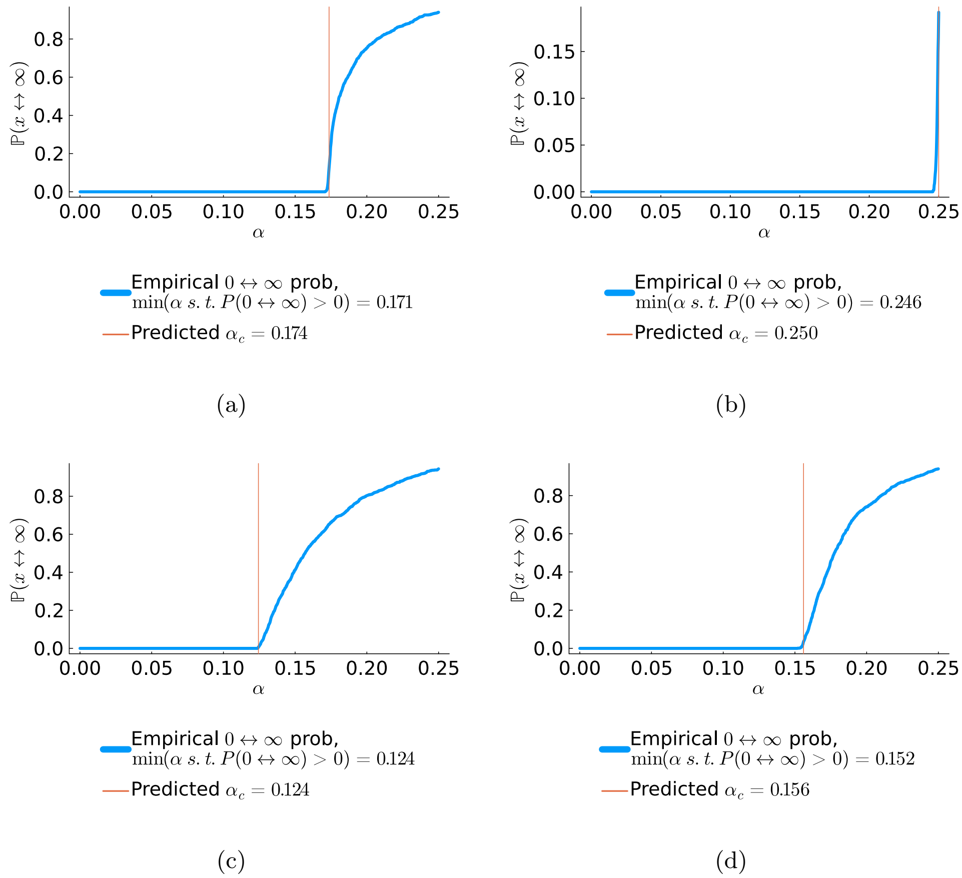

FIG. 2: Colored bond/site percolations in triangular and cubic lattices. a) $P_\alpha(x)$ versus α for bond percolation on triangular lattice with hexagon side length 500, using 1000 trials; b) $P_\alpha(x)$ versus α for site percolation on triangular lattice with hexagon side length 500, using 1000 trials; c) $P_\alpha(x)$ versus α for bond percolation on cubic lattice with cube side length 100, using 1000 trials; d) $P_\alpha(x)$ versus α for site percolation on cubic lattice with cube side length 100, using 1000 trials.

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[1] S. R. Broadbent and J. M. Hammersley, Proc. Cambridge Philos. Soc. **53**, 629 (1957).

[2] G. Grimmett, *Percolation*, 2nd ed. (Springer, Berlin, 1999) p. 444.

[3] D. Stauffer, A. Coniglio, and M. Adam, Gelation and critical phenomena, in *Polymer networks* (Springer, Berlin, 2005) pp. 103–158.

[4] M. Sahimi, *Applications of percolation theory*, 2nd ed. (Springer, Cham, 2023) p. 679.

[5] N. Araújo, P. Grassberger, B. Kahng, K. J. Schrenk, and R. M. Ziff, Eur. Phys J. Special Topics **223**, 2307 (2014).

[6] H. Duminil-Copin, in *Proc. ICM*, Vol. 4 (World Sci., Hackensack, NJ, 2018) pp. 2829–2856.

[7] G. Grimmett, in *Lecture Notes in Math.*, Vol. 2313 (Springer, Cham, 2023) pp. 603–614.

[8] R. Morris, European J. Combin. **66**, 250 (2017).

[9] S. Kundu and S. S. Manna, Phys. Rev. E **95**, (2017).

[10] R. Zallen, Phys. Rev. B **16**, 1426 (1977).

[11] T. E. Harris, Proc. Cambridge Philos. Soc. **56**, 13 (1960).

[12] D. J. Kleitman, J. Combin. Theory **1**, 153 (1966).

[13] S. Chatterjee, *Large deviations for random graphs* (Springer, Cham, 2017) p. 167.

[14] S. Janson, T. Łuczak, and A. Rucinski, *Random graphs* (Wiley, New York, 2000) p. 333.

[15] P. C. Fishburn, Discrete Appl. Math. **39**, 173 (1992).

[16] L. A. Shepp, Ann. Probab. **10**, 824 (1982).

[17] S. H. Chan and I. Pak, Pacific J. Math. **323**, 223 (2023).

[18] H. Duminil-Copin and S. Smirnov, in *Probability and statistical physics in two and more dimensions* (AMS, Providence, RI, 2012) pp. 213–276.
[19] C. M. Newman, Comm. Math. Phys. **74**, 119 (1980).
[20] W. Werner, *Percolation et modèle d'Ising* (Soc. Math. de France, Paris, 2009) p. 161.
[21] N. Gladkov (2023), arXiv:2305.02653.
[22] E. H. Lieb and S. Sahi, J. Math. Phys. **63**, (2022), 11 pp.
[23] S. Sahi, Combinatorica **28**, 209 (2008).
[24] B. Bollobás and O. Riordan, *Percolation* (Cambridge Univ. Press, New York, 2006) p. 323.
[25] H. Kesten, Comm. Math. Phys. **74**, 41 (1980).
[26] M. F. Sykes and J. W. Essam, J. Mathematical Phys. **5**, 1117 (1964).
[27] G. M. T. Watts, J. Phys. A **29**, L363 (1996).
[28] J. Dubédat, Probab. Theory Related Fields **134**, 453 (2006).
[29] S. Sheffield and D. B. Wilson, Ann. Probab. **39**, 1844 (2011).
[30] J. J. H. Simmons, J. Phys. A **46**, (2013), 30 pp.
[31] S. M. Flores, R. M. Ziff, and J. J. H. Simmons, J. Phys. A **48**, (2015), 17 pp.
[32] D. Stauffer and A. Aharony, *Introduction to percolation theory*, 2nd ed. (Taylor & Francis, London, 1994) p. 192.
[33] N. Alon and J. H. Spencer, *The probabilistic method*, 4th ed. (John Wiley, Hoboken, NJ, 2016) p. 375.
[34] N. Gladkov, *Ph.D. thesis*, Ph.D. thesis, UCLA, in preparation.

Proof of Theorem 1. Since E_{ab} and E_{ac} are independent $\frac{1}{2}$-percolations, this implies that events $\mathcal{U}_{ab}$ and $\mathcal{V}_{ac}$ are also independent. This proves the pairwise independence part.

We prove (3) by induction on the number of edges in E. For $E = \varnothing$, the inequality is trivial. Fix an edge $e \in E$. Consider the probability space of colorings of $E - e$. For an event $\mathcal{X}_{ab} \subseteq 2^E$, denote by $\mathcal{X}_{ab}^+$ the subset of $\mathcal{X}_{ab}$ such that $f(e) \in \{a, b\}$. Similarly, denote by $\mathcal{X}_{ab}^-$ the subset of $\mathcal{X}_{ab}$ such that $f(e) \in \{c, d\}$.

By the symmetry, we have:

$$\mathbb{P}\big(\mathcal{X}_{ab} : f(e) = a\big) = \mathbb{P}(\mathcal{X}_{ab} : f(e) = b) = 2\mathbb{P}_{\frac{1}{2}}(\mathcal{X}^+),$$
$$\mathbb{P}\big(\mathcal{X}_{ab} : f(e) = c\big) = \mathbb{P}(\mathcal{X}_{ab} : f(e) = d) = 2\mathbb{P}_{\frac{1}{2}}(\mathcal{X}^-).$$

Clearly, $\mathbb{P}_{\frac{1}{2}}(\mathcal{X}) = \mathbb{P}_{\frac{1}{2}}(\mathcal{X}^-) + \mathbb{P}_{\frac{1}{2}}(\mathcal{X}^+)$. When $\mathcal{X}$ is closed upward, we also have $\mathbb{P}_{\frac{1}{2}}(\mathcal{X}^-) \leq \mathbb{P}_{\frac{1}{2}}(\mathcal{X}^+)$. We use this notation for $\mathcal{X} \in \{\mathcal{U}, \mathcal{V}, \mathcal{W}\}$ and all pairs of colors.

Considering all possible colors of e and using the induction hypothesis, we have:

$$\begin{aligned}
\mathbb{P}(\mathcal{U}_{ab} \cap \mathcal{V}_{ac} \cap \mathcal{W}_{bc}) \;&=\; \mathbb{P}\big(\mathcal{U}_{ab}^+ \cap \mathcal{V}_{ac}^+ \cap \mathcal{W}_{bc}^-\big) + \mathbb{P}\big(\mathcal{U}_{ab}^+ \cap \mathcal{V}_{ac}^- \cap \mathcal{W}_{bc}^+\big) \\
&\quad + \mathbb{P}\big(\mathcal{U}_{ab}^- \cap \mathcal{V}_{ac}^+ \cap \mathcal{W}_{bc}^+\big) + \mathbb{P}\big(\mathcal{U}_{ab}^- \cap \mathcal{V}_{ac}^- \cap \mathcal{W}_{bc}^-\big) \\
&\leq\; 2\Big(\mathbb{P}\big(\mathcal{U}_{ab}^+\big)\mathbb{P}\big(\mathcal{V}_{ac}^+\big)\mathbb{P}\big(\mathcal{W}_{bc}^-\big) + \mathbb{P}\big(\mathcal{U}_{ab}^+\big)\mathbb{P}\big(\mathcal{V}_{ac}^-\big)\mathbb{P}\big(\mathcal{W}_{bc}^+\big) \\
&\quad + \mathbb{P}\big(\mathcal{U}_{ab}^-\big)\mathbb{P}\big(\mathcal{V}_{ac}^+\big)\mathbb{P}\big(\mathcal{W}_{bc}^+\big) + \mathbb{P}\big(\mathcal{U}_{ab}^-\big)\mathbb{P}\big(\mathcal{V}_{ac}^-\big)\mathbb{P}\big(\mathcal{W}_{bc}^-\big)\Big).
\end{aligned}$$

Simplifying the notation as above, the RHS is equal to:

$$\begin{aligned}
&2\Big(\mathbb{P}_{\frac{1}{2}}(\mathcal{U}^+)\mathbb{P}_{\frac{1}{2}}(\mathcal{V}^+)\mathbb{P}_{\frac{1}{2}}(\mathcal{W}^-) + \mathbb{P}_{\frac{1}{2}}(\mathcal{U}^+)\mathbb{P}_{\frac{1}{2}}(\mathcal{V}^-)\mathbb{P}_{\frac{1}{2}}(\mathcal{W}^+) \\
&\qquad + \mathbb{P}_{\frac{1}{2}}(\mathcal{U}^-)\mathbb{P}_{\frac{1}{2}}(\mathcal{V}^+)\mathbb{P}_{\frac{1}{2}}(\mathcal{W}^+) + \mathbb{P}_{\frac{1}{2}}(\mathcal{U}^-)\mathbb{P}_{\frac{1}{2}}(\mathcal{V}^-)\mathbb{P}_{\frac{1}{2}}(\mathcal{W}^-)\Big) \\
&\;=\; \big(\mathbb{P}_{\frac{1}{2}}(\mathcal{U}^+) + \mathbb{P}_{\frac{1}{2}}(\mathcal{U}^-)\big)\big(\mathbb{P}_{\frac{1}{2}}(\mathcal{V}^+) + \mathbb{P}_{\frac{1}{2}}(\mathcal{V}^-)\big)\big(\mathbb{P}_{\frac{1}{2}}(\mathcal{W}^+) + \mathbb{P}_{\frac{1}{2}}(\mathcal{W}^-)\big) \\
&\qquad - \big(\mathbb{P}_{\frac{1}{2}}(\mathcal{U}^+) - \mathbb{P}_{\frac{1}{2}}(\mathcal{U}^-)\big)\big(\mathbb{P}_{\frac{1}{2}}(\mathcal{V}^+) - \mathbb{P}_{\frac{1}{2}}(\mathcal{V}^-)\big)\big(\mathbb{P}_{\frac{1}{2}}(\mathcal{W}^+) - \mathbb{P}_{\frac{1}{2}}(\mathcal{W}^-)\big) \\
&\;\leq\; \mathbb{P}_{\frac{1}{2}}(\mathcal{U})\mathbb{P}_{\frac{1}{2}}(\mathcal{V})\mathbb{P}_{\frac{1}{2}}(\mathcal{W}),
\end{aligned}$$

as desired. The proof of (4) goes along the same lines. Finally, the closed downward version follows the inclusion exclusion argument earlier in the paper. □

Variations and generalizations. First, note that we never use the graph structure, and the theorem can be viewed as a result about abstract set systems, cf. [12, 33]. Second, the pairwise independent $\frac{1}{2}$-percolation argument that we discussed after the theorem can be generalized in several ways. Notably, it can be extended to the p-percolation for all $0 \le p \le 1$, but the resulting coupling of percolations then require seven colors and have somewhat inelegant probabilities [34].

Next, the theorem can be extended to a larger number of events. Start by taking $k-1$ independent $\frac{1}{2}$-percolations $E_1, \ldots, E_{k-1}$ on the same graph. Define a new $\frac{1}{2}$-percolation

$$E_k := \bigoplus_{i=1}^{k-1} E_i \mod 2,$$

where the edge e is present if and only if it is present in an odd number of E_i's. Observe that every $k-1$ of $E_1, \ldots, E_k$ are mutually independent.

Then, for every closed downward properties $\mathcal{X}_1, \ldots, \mathcal{X}_k$ we have:

$$\mathbb{P}\big(\mathcal{X}_1 \cap \cdots \cap \mathcal{X}_k\big) \ \ge \ \mathbb{P}(\mathcal{X}_1) \cdots \mathbb{P}(\mathcal{X}_k). \qquad (10)$$

Once again, the proof follows verbatim the proof of the theorem. Note that for $k = 2$, we have $E_1 = E_2$ and (10) is the Harris–Kleitman inequality (2). For $k = 3$, the inequality (10) gives (4).

Finally, one can easily obtain a colored version with $m = 2^{k-1}$ colors. For example, for $k = 4$, take a uniform random edge coloring $f : E \to \{1, \ldots, 8\}$. Consider four pairwise independent $\frac{1}{2}$-percolations E_{1234}, E_{1256}, E_{1357} and E_{1467} with natural labeling. Note that every three of these are mutually independent. Then, for closed downward properties $\mathcal{U}, \mathcal{V}, \mathcal{W}$ and $\mathcal{X}$, the inequality (10) gives:

$$\begin{aligned}&\mathbb{P}(\mathcal{U}_{1234} \cap \mathcal{V}_{1256} \cap \mathcal{W}_{1357} \cap \mathcal{X}_{1467})\\ &\quad \ge \ \mathbb{P}(\mathcal{U}_{1234})\,\mathbb{P}(\mathcal{V}_{1256})\,\mathbb{P}(\mathcal{W}_{1357})\,\mathbb{P}(\mathcal{X}_{1467}).\end{aligned}$$