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Kirszbraun-type theorems for graphs. (English summary)

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In this paper, the authors introduce a new notion of G-*Kirszbraun graphs*, where G is a vertex-transitive graph. The idea is to discretize the classical Kirszbraun theorem in metric geometry [M. Kirszbraun, Fund. Math. **22** (1934), 77–108, doi:10.4064/fm-22-1-77-108] (see also [Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis. Vol. 1*, Amer. Math. Soc. Colloq. Publ., 48, Amer. Math. Soc., Providence, RI, 2000 (Subsection 1.2); MR1727673]). The authors' main goal is to explain the variational principle for the height functions of tilings introduced by the third author in [*Tiling by bars*, Ph.D. thesis, Brown Univ., 2014] and further developed in [I. Pak, A. Sheffer and M. Tassy, Discrete Comput. Geom. **56** (2016), no. 2, 377–394; MR3530972; G. Menz and M. Tassy, "A variational principle for a non-integrable model", preprint, arXiv:1610.08103]. Their second goal is to clarify the connection to the Helly theorem, a foundational result in convex and discrete geometry [E. Helly, Jahresber. Dtsch. Math.-Ver. **32** (1923), 175–176] (see also [L. W. Danzer, B. Grünbaum and V. L. Klee Jr., in *Proc. Sympos. Pure Math., Vol. VII*, 101–180, Amer. Math. Soc., Providence, RI, 1963; MR0157289; J. Matoušek, *Lectures on discrete geometry*, Grad. Texts in Math., 212, Springer, New York, 2002; MR1899299]). Roughly, the authors show that $\mathbb{Z}^d$-Kirszbraun graphs are somewhat rare and are exactly the graphs that satisfy Helly's property with certain parameters.

Let ℓ_2 denote the usual Euclidean metric on $\mathbb{R}^n$ for all n. Given a metric space X and a subset A, we write $A \subset X$ to mean that the subset A is endowed with the restricted metric from X. Let $A \subset (\mathbb{R}^n, \ell_2)$. A function $f\colon A \to \mathbb{R}^n$ is called a α-*Lipschitz function* (or a *Lipschitz function of order* α) if $\alpha > 0$, and for some constant L and all $x, y \in A$, we have

$$\ell_2(f(x), f(y)) \le L\ell_2(x,y)^\alpha.$$

The Kirszbraun theorem says that for all $A \subset (\mathbb{R}^n, \ell_2)$, and all Lipschitz functions $f\colon A \to (\mathbb{R}^n, \ell_2)$, there is an extension to a Lipschitz function on $\mathbb{R}^n$ with the same Lipschitz constant.

Recall now the Helly theorem: Suppose a collection of convex sets $B_1, B_2, \ldots, B_k$ satisfies the property that every $(n+1)$-subcollection has a nonempty intersection; then $\bigcap B_i \neq \varnothing$. F. A. Valentine in [Amer. J. Math. **67** (1945), 83–93; MR0011702] famously showed how the Helly theorem can be used to obtain the Kirszbraun theorem.

Given metric spaces X and Y, we say that Y is X-*Kirszbraun* if for all $A \subset X$, every 1-Lipschitz map $f\colon A \to Y$ has a 1-Lipschitz extension from A to X. In this notation, the Kirszbraun theorem says that $(\mathbb{R}^n, \ell_2)$ is $(\mathbb{R}^n, \ell_2)$-Kirszbraun.

Let $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$, $n > m$. The metric space X is said to have the (n,m)-Helly property if, for every collection of closed balls $B_1, B_2, \ldots, B_n$ of radius ≥ 1, whenever every m-subcollection has a nonempty intersection, we have $\bigcap_{i=1}^n B_i \neq \varnothing$. Since balls in $\mathbb{R}^n$ with the Euclidean metric are convex, the Helly theorem can be restated to say that $(\mathbb{R}^n, \ell_2)$ is $(\infty, n+1)$-Helly.

Given a graph H, we endow the set of vertices (also denoted by H) with the path metric. By $\mathbb{Z}^d$ we mean the Cayley graph of the group $\mathbb{Z}^d$ with respect to standard generators. The following is the main result of the paper under review.

Theorem 1.1 (Main theorem). A graph H is $\mathbb{Z}^d$-Kirszbraun if and only if H is $(2d, 2)$-Helly.

In the paper under review, the authors also present extensions and applications of the main theorem. They include a continuous analogue of the main theorem (Theorem 4.1), the extension to larger integral Lipschitz constants (Theorem 4.3), and the bipartite extension useful for domino tilings (Theorem 4.6).

A metric space $(X, \mathbf{m})$ is *geodesically complete* if, for all $x, y \in X$, there exists a continuous function $f\colon [0,1] \to X$ such that

$$\mathbf{m}(x, f(t)) = t\mathbf{m}(x, y) \quad \text{and} \quad \mathbf{m}(f(t), y) = (1-t)\mathbf{m}(x, y).$$

Theorem 4.1. Let Y be a metric space such that every closed ball in Y is compact. Then Y is $(\mathbb{R}^d, \ell_1)$-Kirszbraun if and only if Y is geodesically complete and $(2d, 2)$-Helly.

Theorem 4.3. Let $t \in \mathbb{N}$ and H be a connected graph. Then every t-Lipschitz map $f\colon A \to H$, $A \subset \mathbb{Z}^d$, has a t-Lipschitz extension to $\mathbb{Z}^d$ if and only if

$$\text{for all balls } B_1, B_2, \ldots, B_{2d} \text{ of radii multiples of } t \text{ mutually intersect} \Longrightarrow \bigcap B_i \neq \varnothing.$$

Theorem 4.6. A graph H is bipartite $\mathbb{Z}^d$-Kirszbraun if and only if H is bipartite $(2d, 2)$-Helly.

In addition, the authors also discuss computational aspects of the $\mathbb{Z}^d$-Kirszbraun property, motivated entirely by applications to tilings. *Safet Penjić*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.