LINEAR EXTENSIONS AND CONTINUED FRACTIONS

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Abstract. We introduce several new constructions of finite posets with the number of linear extensions given by generalized continued fractions. We apply our results to the problem of the minimum number of elements needed for a poset with a given number of linear extensions.

1. Introduction

1.1. **Foreword.** Continued fractions go back to antiquity [Bre91] and are surprisingly versatile. They appear across mathematics, from number theory [BPSZ14, RS92] to analysis [JT80, Khi97], from cluster algebras [ÇS18] to discrete geometry [Kar13] to signal processing [Sau21]. In combinatorics, they famously enumerate partitions [AB05], lattice paths [Fla80] (see also [FS09, GJ83, PSZ23]), permutations [Eli17, SZ22], and perfect matchings [Vie85] (see also [Sch19, Spi21]).

Curiously, the applications go in both directions: the asymptotics of combinatorial sequences can be derived from analytic properties of continued functions, while combinatorial interpretations imply positivity properties. This paper explores connections between linear extensions of finite posets and continued fractions, and their asymptotic applications to counting.

1.2. **Linear extensions.** Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Denote $[n] := \{1, \ldots, n\}$. A *linear extension* of P is a bijection $f : X \to [n]$, such that $f(x) < f(y)$ for all $x \prec y$. Let $\mathcal{E}(P)$ be the set of linear extensions of P, and denote $e(P) := |\,\mathcal{E}(P)|$. Clearly, $1 \le e(P) \le n!$ See [CP23c] for a detailed recent survey.

Denote by $\mu(n)$ the minimum number of elements in a poset with n linear extensions. See [OEIS, A160371] for the numerical data (see also [OEIS, A281723]). For example, $\mu(5) = 4$ since $e(Z_4) = 5$, where Z_4 is a *zigzag poset* on 4 elements (with an N-shaped comparability graph).

The asymptotics of $\{\mu(n)\}$ remains an important open problem. Clearly, $\mu(n) \le n$ since for the parallel sum or chains we have: $e(C_{n-1} + C_1) = n$. In a different direction, $\mu(n) = \Omega(\log n / \log\log n)$ since $e(P) \le n!$ The first nontrivial upper bound $\mu(n) = O(\sqrt{n})$ was found by Tenner [Ten09]. Most recently, this bound was greatly improved:

Theorem 1.1 (Kravitz–Sah [KS21, Thm 1.1]). *We have:* $\mu(n) = O(\log n \log\log n)$.

The authors use a simple but surprising connection to continued fractions, the starting point of this paper (see below). They state the following:

Conjecture 1.2 ([KS21, Conj. 7.3]). *We have:* $\mu(n) = O(\log n)$.

In this paper, we are mostly interested in the combinatorial aspects of the connection between linear extensions and continued fractions, suggesting new technical tools towards the conjecture.

1.3. **Simple continued fractions.** Let $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbb{P} := \{1, 2, \ldots\}$. A *simple continued fraction* (CF) is defined as follows:

$$[b_0, b_1, b_2, \ldots, b_m] \;:=\; b_0 \,+\, \cfrac{1}{b_1 \,+\, \cfrac{1}{b_2 \,+\, \cfrac{1}{\ddots \,+\, \frac{1}{b_m}}}}\,, \tag{1.1}$$

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where integers $b_0 \geq 0$, $b_1, \ldots, b_{m-1} \geq 1$, and $b_m \geq 2$ for $m \geq 1$. Integers b_i are called *quotients*. The sum of these quotients $\mathrm{S}(b_0, \ldots, b_m) := b_0 + \ldots + b_m$ is called the *weight* of $[b_0, \ldots, b_m]$. Recall that for every $\alpha \in \mathbb{Q}_{\geq 0}$ there is a unique simple continued fraction $[b_0, b_1, b_2, \ldots, b_m] = \alpha$, and we write $\mathrm{s}(\alpha) := \mathrm{S}(b_0, b_1, b_2, \ldots, b_m)$ in this case. Note that $\mathrm{s}(\alpha) = \mathrm{s}\big(\alpha^{-1}\big)$.

In the terminology of [YK75] (see also [Knu98, §4.5.3]), the weight $\mathrm{s}\big(\frac{c}{d}\big)$ is the number of steps of the *subtraction algorithm*, the original (classical) version of the Euclidean algorithm for finding the greatest common divisor that uses only subtractions instead of divisions. The following result is the key to the proof of Theorem 1.1.

Theorem 1.3 (Larcher [Lar86], see also [KS21, Thm 1.2])**.** *For every integer $d \geq 1$, there exists an integer $1 \leq c < d$, $\gcd(c, d) = 1$, such that*

$$\mathrm{s}\big(\tfrac{c}{d}\big) \;\leq\; C\,\frac{d}{\phi(d)}\,\log d\,\log\log d, \tag{1.2}$$

where $\phi(n)$ is Euler's totient function, and $C > 0$ is a universal constant.

See §5.3 for more on the theorem. Now, Kravitz and Sah observed that Conjecture 1.2 follows from the following conjectural extension of Theorem 1.3.

Conjecture 1.4 ([KS21, Conj. 7.2])**.** *For every prime d, there is an integer $1 \leq c < d$, such that*

$$\mathrm{s}\big(\tfrac{c}{d}\big) \;\leq\; C\,\log d, \tag{1.3}$$

where $C > 0$ is a universal constant.

Note that in a CF (1.1) for $\frac{c}{d}$, the number of quotients is $m = O(\log d)$. Thus, Conjecture 1.4 follows from the celebrated *Zaremba's conjecture* (see also §5.5):

Conjecture 1.5 (Zaremba [Zar72, p. 76])**.** *For every integer $d \geq 1$, there is an integer $1 \leq c < d$, such that $c/d = [0, b_1, \ldots, b_m]$ and $b_1, \ldots, b_m \leq A$, where $A > 0$ is a universal constant.*

1.4. **From continued fractions to linear extensions.** In poset $P = (X, \prec)$, an *antichain* is a subset of pairwise independent elements. The *width* of a poset is the size of the maximal antichain. An element $x \in X$ is *minimal*, if for every $y \in X$ we have either $x \preccurlyeq y$ or $x \,\|\, y$. Denote by $\min(P)$ the set of all minimal elements in P.

Theorem 1.6 (see [KS21, Prop. 4.1])**.** *For all integers $1 \leq c < d$ with $\gcd(c, d) = 1$, there is a poset $P = (X, \prec)$ of width two, such that $|X| = \mathrm{s}\big(\frac{c}{d}\big)$, $e(P) = d$ and $e(P - x) = c$ for some minimal element $x \in \min(P)$.*

The proof of the theorem uses two simple transformations of posets $(P, x) \to (P', x')$ and (P'', x''), such that for $e(P) = d$, $e(P - x) = c$ the new posets satisfy $e(P') = e(P'') = c + d$, $e(P' - x') = c$, $e(P'' - x'') = d - c$. In Section 3 we modify and generalize this construction.

Before we proceed to generalizations, consider

$$\mathcal{T}(k) \;:=\; \big\{e(P) \;:\; P = (X, \prec),\, |X| \leq k\big\},$$

so that $\mu(n) = \min\{k \;:\; n \in \mathcal{T}(k)\}$. Open Problems 7.5 and 7.6 in [KS21] ask about the asymptotics of $|\mathcal{T}(k)|$, and of the largest $L = L_c(k)$ such that $\big|\mathcal{T}(k) \cap \{1, \ldots, L\}\big| > cL$. We have the following direct application of Theorem 1.6 (not noticed in [KS21]), which gives partial answers to both open problems:

Corollary 1.7. *We have: $|\mathcal{T}(k)| = \exp \Omega(k)$. Moreover, there is a constant $c > 1$, such that*

$$\frac{1}{c^k}\,\big|\,\mathcal{T}(k) \cap \big\{1, 2, \ldots, \lfloor c^k \rfloor\big\}\,\big| \;\to\; 1 \quad \textit{as} \quad k \to \infty. \tag{1.4}$$

Proof. Recall the following remarkable result of Bourgain and Kontorovich [BK14] (see also §5.5), giving an asymptotic version of Zaremba's Conjecture 1.5: $\gamma(n) \to 1$ as $n \to \infty$, where $\gamma(n)$ denotes the proportion of $d \in \{1, \ldots, n\}$, such that c/d has all quotients ≤ 50 for some $1 \leq c < d$, $\gcd(c, d) = 1$. Since $\mathrm{s}\big(\frac{c}{d}\big) = O(\log n)$ for such fractions, by Theorem 1.6 we obtain the result. $\square$

1.5. **Relative version.** Let $P = (X, \prec)$ and let $x \in X$. Following [CP23b], consider the *relative number of linear extensions*:

$$\rho(P, x) := \frac{e(P)}{e(P - x)}.$$

It follows from Theorem 1.6, that every rational number $\alpha \geq 1$ is equal to $\rho(P, x)$ for some poset P and element $x \in X$.

For $n \geq m \geq 1$, let $\nu(m, n)$ denote the minimal number of elements in a poset $P = (X, \prec)$, such that $\rho(P, x) = \frac{n}{m}$ for some $x \in X$. The following upper bound can be viewed as a relative version of Theorem 1.1.

Theorem 1.8. *For all $n \geq 3m$, we have:*

$$\nu(m, n) \leq \frac{n}{m} + O(\log n \log\log n). \tag{1.5}$$

In [CP23b, Prop. 8.8], we showed an asymptotically matching lower bound:

$$\nu(m, n) \geq \frac{n}{m}. \tag{1.6}$$

The key part of the proof is the following tail estimate for the weight of random continued fractions:

Theorem 1.9 (Rukavishnikova [Ruk11])**.** *There is a universal constant $C > 0$, such that*

$$\frac{1}{n}\ \#\left\{\left|\mathrm{s}\left(\tfrac{m}{n}\right) - \tfrac{12}{\pi^2}\log n \log\log n\right| > (\log n)(\log\log n)^{2/3}\right\} < \frac{C}{(\log\log n)^{1/3}} \tag{1.7}$$

Here we are stating a special case of the main theorem in [Ruk11] which suffices for our purposes.

1.6. **Generalized continued fractions.** Let $m \geq 0$, $a_1, \ldots, a_m \in \mathbb{P}$, $b_0, \ldots, b_m \in \mathbb{P}$. A *generalized continued fraction* (GCF) is defined as

$$[a_1, \ldots, a_m; b_0, \ldots, b_m] := b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \frac{a_m}{b_m}}}}. \tag{1.8}$$

Note that when $a_1 = \ldots = a_m = 1$ we get a simple continued fraction. We define the *weight* of GCFs as follows:

$$\mathrm{G}(a_1, \ldots, a_m; b_0, \ldots, b_m) := (b_0 + \ldots + b_m) - (a_1 + \ldots + a_m) + m,$$

and note that $\mathrm{G}(1, \ldots, 1; b_0, \ldots, b_m) = \mathrm{S}(b_0, \ldots, b_m)$. Observe that a rational number can have many presentations as a GCF, some of which can have weight smaller than the weight of the corresponding CFs. For example,

$$\frac{20}{7} = 2 + \frac{1}{1 + \frac{1}{6}} = 2 + \frac{2}{2 + \frac{1}{3}},$$

so $\mathrm{s}\left(\frac{20}{7}\right) = \mathrm{S}(2, 1, 6) = 9$ and $\mathrm{G}(2, 1; 2, 2, 3) = 6$.

A generalized continued fraction (1.8) is called *balanced* if

$$b_i \geq a_i + a_{i+1} - 1 \quad \text{for all} \quad 0 \leq i \leq m, \tag{1.9}$$

where by convention we assume that $a_0 = a_{m+1} = 1$. Clearly, every simple continued fraction of $\alpha \in \mathbb{Q}_{\geq 1}$ is balanced. The following is the GCF analogue of Theorem 1.6.

Theorem 1.10. *Let $m \geq 0$, $a_1, \ldots, a_m \in \mathbb{P}$, $b_0, \ldots, b_m \in \mathbb{P}$ be integers satisfying (1.9). Then there exists a poset $P = (X, \prec)$ of width at most three, and a minimal element $x \in \min(P)$, such that $|X| = \mathrm{G}(a_1, \ldots, a_m; b_0, \ldots, b_m)$, and*

$$[a_1, \ldots, a_m; b_0, \ldots, b_m] \, = \, \rho(P, x),$$

where $[a_1, \ldots, a_m; b_0, \ldots, b_m]$ is a balanced GCF defined in (1.8).

For $\alpha \in \mathbb{Q}_{\geq 1}$, define

$$\mathrm{g}(\alpha) \, := \, \min \big\{ \mathrm{S}(a_1, \ldots, a_m; b_0, \ldots, b_m) \, : \, [a_1, \ldots, a_m; b_0, \ldots, b_m] = \alpha \big\},$$

where the minimum is over all balanced GCF (1.9) such that all partial fractions $\frac{C_i}{D_i}$ are reduced, i.e. $\gcd(C_i, D_i) = 1$ for all $1 \leq i \leq m$ (see the definition in §2.2). For example, if $a_1 = \ldots = a_m = r$ for some integer $r \geq 1$, and integers $b_1, \ldots, b_m$ are coprime to r, then all partial fractions $\frac{C_i}{D_i}$ are reduced. In particular, this condition automatically holds for all simple CFs. From above, we have $\mathrm{g}(\alpha) \leq \mathrm{s}(\alpha)$. Thus, the following conjecture is a natural weakening of Conjecture 1.4.

Conjecture 1.11. *For every prime d, there is an integer $1 \leq c < d$, such that*

$$\mathrm{g}\big(\tfrac{d}{c}\big) \, \leq \, C \, \log d, \tag{1.10}$$

where $C > 0$ is a universal constant.

From Theorem 1.10, we have:

Proposition 1.12. *Conjecture 1.11 implies Conjecture 1.2.*

1.7. **Rational GCFs.** We call a continued fraction of the form (1.8) *rational* if $a_i \in \mathbb{Q}_{\geq 1}$. A rational generalized continued fraction (RGCF) is called *balanced* if it is of the form

$$b_0 + \alpha_1 + \cfrac{\alpha_1}{\mathrm{s}(\alpha_1) - 1 + b_1 + \alpha_2 + \cfrac{\alpha_2}{\mathrm{s}(\alpha_2) - 1 + b_2 + \alpha_3 + \cfrac{\alpha_3}{\ddots + \frac{\alpha_m}{\mathrm{s}(\alpha_m) - 1 + b_m}}}}\,, \tag{1.11}$$

where $\alpha_1, \ldots, \alpha_m \in \mathbb{Q}_{\geq 1}$ and $b_0, \ldots, b_m \in \mathbb{N}$ s.t. $b_m \geq 1$. We use $[\alpha_1, \ldots, \alpha_m; b_0, \ldots, b_m]$ to denote this RGCF.

Note that for $\alpha_1, \ldots, \alpha_m \in \mathbb{P}$, this is a balanced GCF, since the inequalities (1.9) are automatically satisfied. Denote by

$$\mathrm{R}(\alpha_1, \ldots, \alpha_m; b_0, \ldots, b_m) \, := \, b_0 + \ldots + b_m \, + \, \mathrm{s}(\alpha_1) + \ldots + \mathrm{s}(\alpha_m)$$

the *weight* of (1.11). For example, take $m = 1$, $\alpha_1 = \frac{3}{2}$, $b_0 = 1$, $b_1 = 3$. Then

$$\mathrm{s}\big(\tfrac{3}{2}\big) \, = \, 3, \quad \big[\tfrac{3}{2}\,; 1, 3\big] \, = \, 1 + \tfrac{3}{2} + \frac{\frac{3}{2}}{\mathrm{s}\big(\frac{3}{2}\big) - 1 + 3} \, = \, \tfrac{14}{5} \quad \text{and} \quad \mathrm{R}\big(\tfrac{3}{2}\,; 1, 3\big) \, = \, 1 + 3 + \mathrm{s}\big(\tfrac{3}{2}\big) \, = \, 7.$$

The following result is a variation of Theorem 1.10 to RGCF:

Theorem 1.13. *Let $m \geq 0$, $\alpha_1, \ldots, \alpha_m \in \mathbb{Q}_{\geq 1}$ and $b_0, \ldots, b_m \in \mathbb{P}$. Then there exists a poset $P = (X, \prec)$ of width at most three, and a minimal element $x \in \min(P)$, such that $|X| = \mathrm{R}(\alpha_1, \ldots, \alpha_m; b_0, \ldots, b_m)$, and*

$$[\alpha_1, \ldots, \alpha_m; b_0, \ldots, b_m] \, = \, \rho(P, x),$$

where $[\alpha_1, \ldots, \alpha_m; b_0, \ldots, b_m]$ is a balanced RGCF defined in (1.11).

For $\beta \in \mathbb{Q}_{\geq 1}$, define

$$\mathrm{r}(\beta) := \min\left\{ \mathrm{R}(\alpha_1, \ldots, a_m; b_0, \ldots, b_m) \, : \, [\alpha_1, \ldots, \alpha_m; b_0, \ldots, b_m] = \beta \right\},$$

where the minimum is over all RGCF (1.11) such that all partial fractions $\frac{C_i}{D_i}$ are reduced (see the definition in §2.2). From above, $\mathrm{r}(\alpha) \leq \mathrm{s}(\alpha)$. Thus, the following conjecture is a natural weakening of both Conjecture 1.4 and Conjecture 1.11.

Conjecture 1.14. *For every prime d, there is an integer $1 \leq c < d$, such that*

$$\mathrm{r}\big(\tfrac{d}{c}\big) \,\leq\, C \, \log d, \tag{1.12}$$

where $C > 0$ is a universal constant.

To motivate the conjecture, note that $\mathrm{r}(\beta)$ can be much smaller than $\mathrm{s}(\beta)$. Take, for example, $m = 1$, $\alpha_1 = \frac{13}{7}$ and $\beta = \frac{173}{56}$. We have:

$$\alpha_1 \,=\, 1 + \frac{1}{1 + \frac{1}{6}}\,, \quad \mathrm{s}(\alpha_1) \,=\, 8\,, \quad \beta \,=\, 3 + \frac{1}{11 + \frac{1}{5}} \,=\, 1 + \alpha_1 + \frac{\alpha_1}{s(\alpha_1) - 1 + 1} \,=\, [\alpha_1; 1, 1]\,.$$

Thus, $\mathrm{s}(\beta) = 19$ while $\mathrm{r}(\beta) \leq \mathrm{R}(\alpha_1; 1, 1) = 10$ in this case. Again, by Theorem 1.13 we have:

Proposition 1.15. *Conjecture 1.14 implies Conjecture 1.2.*

1.8. **Paper structure.** We recall poset theoretic definitions and notation in Section 2. Recursive constructions of posets are studied in Section 3. We present the proofs in Section 4. We conclude with final remarks and open problems in Section 5.

2. Basic definitions and notation

2.1. **Posets.** For a poset $P = (X, \prec)$ and a subset $Y \subset X$, denote by $P_Y = (Y, \prec)$ a *subposet* of P. We use $(P - z)$ to denote a subposet P_{X-z}, where $z \in X$. Element $x \in X$ is *minimal* in P, if there exists no element $y \in X - x$ such that $y \prec x$. Denote by $\min(P)$ the set of minimal elements in P.

In a poset $P = (X, \prec)$, elements $x, y \in X$ are called *incomparable* if $x \not\prec y$ and $y \not\prec x$. We write $x \parallel y$ in this case. An *antichain* is a subset $A \subset X$ of pairwise incomparable elements. The *width* of poset $P = (X, \prec)$, denoted $\mathrm{width}(P)$, is the size of a maximal antichain. A *chain* is a subset $C \subset X$ of pairwise comparable elements. Denote by A_n and C_n the antichain and the chain with n elements, respectively.

A *dual poset* is a poset $P^* = (X, \prec^*)$, where $x \prec^* y$ if and only if $y \prec x$. A *parallel sum* $P \oplus Q$ of posets $P = (X, \prec)$ and $Q = (Y, \prec')$ is a poset $(X \cup Y, \prec^\diamond)$, where the relation $\prec^\diamond$ coincides with $\prec$ and $\prec'$ on X and Y, and $x \| y$ for all $x \in X$, $y \in Y$. A *linear sum* $P \circledcirc Q$ of posets $P = (X, \prec)$ and $Q = (Y, \prec')$ is a poset $(X \cup Y, \prec^\diamond)$, where the relation $\prec^\diamond$ coincides with $\prec$ and $\prec'$ on X and Y, and $x \prec^\diamond y$ for all $x \in X$, $y \in Y$.

Note that $e(P^*) = e(P)$, $e(P \circledcirc Q) = e(P)\, e(Q)$ and $e(P \oplus Q) = \binom{m+n}{m}\, e(P)\, e(Q)$, where $|X| = m$ and $|Y| = n$. We refer to [Sta12, Ch. 3] for an accessible introduction, and to surveys [BW00, CP23c, Tro95] for further definitions and standard results.

2.2. **Continued fractions.** Consider a GCF $[a_1, \ldots, a_m; b_0, \ldots, b_m]$ given by (1.8). Recursively define $C_i := C_i(a_1, \ldots, a_m; b_0, \ldots, b_m)$ and $D_i := D_i(a_1, \ldots, a_m; b_0, \ldots, b_m)$, $0 \leq i \leq m$, as follows:

$$\begin{aligned} C_m \,&:=\, b_m\,, \qquad & D_m \,&:=\, 1, \\ D_i \,&:=\, C_{i+1}\,, \qquad & C_i \,&:=\, b_i\, D_i \,+\, a_{i+1}\, D_{i+1}. \end{aligned}$$

It is well known and easy to see by induction that

$$[a_{i+1}, a_{i+2}, \ldots, a_m; b_i, b_{i+1}, \ldots, b_m] \,=\, \frac{C_i}{D_i}\,.$$

These are called *partial fractions*. Note that for simple CFs we have $\gcd(C_i, D_i) = 1$, but this does not always hold for GCFs.

Similarly, consider a RGCF $[\alpha_1, \ldots, \alpha_m; b_0, \ldots, b_m]$ given by (1.11). Let $\alpha_i = c_i/d_i$ where $\gcd(c_i, d_i) = 1$, $1 \leq i \leq m$. Recursively define $C_i := C_i(\alpha_1, \ldots, \alpha_m; b_0, \ldots, b_m)$ and $D_i := D_i(\alpha_1, \ldots, \alpha_m; b_0, \ldots, b_m)$ as follows:

$$\begin{aligned} C_m \ &:= \ b_m \,, \qquad D_m \ := \ 1, \\ D_i \ &:= \ d_{i+1} \bigl(C_{i+1} \, + \, (\mathrm{s}(\alpha_{i+1}) - 1) \, D_{i+1}\bigr), \\ C_i \ &:= \ b_i \, D_i \, + \, c_{i+1} \bigl(C_{i+1} \, + \, \mathrm{s}(\alpha_{i+1}) \, D_{i+1}\bigr). \end{aligned}$$

The ratios $\frac{C_i}{D_i}$ are called *partial fractions* in this case.

3. Recursive constructions

3.1. **Hybrid sums.** Let $P = (X, \prec)$ and $Q = (Y, \prec')$ be posets on $|X| = m$ and $|Y| = n$ elements. Fix $x \in \min(P)$. The *hybrid sum* $Q \oslash_x P$ is the poset $R = (X \cup Y, \prec^\diamond)$ given by the relations

$$\begin{aligned} u \ \prec^\diamond \ u' \quad &\text{for every} \quad u \prec u', \ \ u, u' \in X, \\ v \ \prec^\diamond \ v' \quad &\text{for every} \quad v \prec' v', \ \ v, v' \in Y, \\ v \ \prec^\diamond \ u \quad &\text{for every} \quad u \in X - x, \ \ v \in Y. \end{aligned}$$

Note that x is incomparable to Y in R, and thus $x \in \min(R)$.

We have:

$$e(Q \oslash_x P) \ = \ e(Q) \, e(P) \, + \, e(Q \oplus x) \, e(P - x) \, - \, e(Q) \, e(P - x).$$

Indeed, the term $e(Q) \, e(P)$ counts linear extensions $f \in \mathcal{E}(R)$ for which $f(x) \geq n+1$. Similarly, the term $e(Q \oplus x) \, e(P - x)$ counts $f \in \mathcal{E}(R)$ for which $f(x) \leq n+1$. Finally, the term $e(Q) \, e(P-x)$ counts $f \in \mathcal{E}(R)$ for which $f(x) = n+1$. Because $e(Q \oplus x) = (n+1) \cdot e(Q)$, we then have

$$e(Q \oslash_x P) \ = \ e(Q) \, e(P) \ + \ n \cdot e(Q) \, e(P - x). \tag{3.1}$$

It then follows that for all $y \in \min(Q)$, we have

$$e(R - y) \ = \ e\bigl((Q - y) \ \oslash_x P\bigr) \ = \ e(Q - y) \, e(P) \ + \ (n - 1) \cdot e(Q - y) \, e(P - x). \tag{3.2}$$

Since $(Q \oslash_x P) - x \ = \ Q \oslash (P - x)$, we also have:

$$e\bigl((Q \oslash_x P) - x\bigr) \ = \ e(Q) \, e(P - x). \tag{3.3}$$

Finally, note that

$$\mathrm{width}(Q \oslash_x P) \ \leq \ \max\bigl\{ \mathrm{width}(P) - 1, \, \mathrm{width}(Q) \bigr\} \ + \ 1. \tag{3.4}$$

Remark 3.1. Hybrid sum is a special case of the *quasi-series composition* defined similarly in [HJ85] for general subsets of minimal elements. Also, when $Y = \{y\}$, we have $R = \{y\} \oplus_{y,x} P$, where $\oplus_{y,x}$ is the direct sum operation defined in [KS21, §2].

3.2. **Properties of hybrid sums.** We now use hybrid sums to construct posets for which the numbers of linear extensions satisfy recurrence relations emulating continued fractions.

Lemma 3.2. *Let $P = (X, \prec)$ and $Q = (Y, \prec')$ be posets on $m = |X|$ and $n = |Y|$ elements, and let $x \in \min(P)$, $y \in \min(Q)$. Then there exists a poset $R = (Z, \prec^\diamond)$ and $z \in \min(R)$, such that*

$$\begin{aligned} e(R) \ &= \ e(Q) \bigl(e(P) \ + \ n \cdot e(P - x)\bigr), \\ e(R - z) \ &= \ e(Q - y) \bigl(e(P) \ + \ (n - 1) \cdot e(P - x)\bigr), \\ |Z| \ &= \ m \, + \, n, \\ \mathrm{width}(R) \ &\leq \ \max\bigl\{ \mathrm{width}(P), \, \mathrm{width}(Q) + 1 \bigr\}. \end{aligned}$$

Additionally, we have:

$$\rho(R,z) \ = \ \rho(Q,y)\left(1 \, + \, \frac{1}{n-1 \, + \, \rho(P,x)}\right).$$

Proof. Let $R := Q \oslash_x P$, and let $z := y$. The first four conclusions follows from (3.1), (3.2), (3.4). We conclude that

$$\begin{aligned}
\rho(R,z) \ &= \ \frac{e(Q)}{e(Q-y)} \cdot \frac{e(P) \, + \, n \cdot e(P-x)}{e(P) \, + \, (n-1) \cdot e(P-x)} \\
&= \ \rho(Q,y) \left(1 \, + \, \frac{e(P-x)}{e(P) \, + \, (n-1) \cdot e(P-x)}\right) \\
&= \ \rho(Q,y) \left(1 \, + \, \frac{1}{\rho(P,x) \, + \, (n-1)}\right),
\end{aligned}$$

as desired. □

Lemma 3.3. *Let $P = (X, \prec)$ be a poset on $|X| = m$ elements, let $x \in \min(P)$, and let $b \geq 0$. Then there exists a poset $R = (Z, \prec^\diamond)$ and $z \in \min(R)$, such that*

$$\begin{aligned}
e(R) \ &= \ e(P) \, + \, b \cdot e(P-x), \\
e(R-z) \ &= \ e(P-x), \\
|Z| \ &= \ m \, + \, b, \\
\operatorname{width}(R) \ &\leq \ \max\big\{\operatorname{width}(P), 2\big\}.
\end{aligned}$$

Additionally, we have:

$$\rho(R,z) \ = \ b \, + \, \rho(P,x).$$

Proof. Let $Q := C_b$ be a chain of b elements, so $e(Q) = 1$. Let $R := Q \oslash_x P$, and let $z := x$. Then the first four conclusions follows from (3.1), (3.3), and (3.4). We conclude that

$$\rho(R,z) \ = \ \frac{e(P) \, + \, b \cdot e(P-x)}{e(P-x)} \ = \ \rho(P,x) \, + \, b,$$

as desired. □

By combining the two lemmas above, we get the following:

Lemma 3.4. *Let $P = (X, \prec)$ and $Q = (Y, \prec')$ be posets on $m = |X|$ and $n = |Y|$ elements, and let $x \in \min(P)$, $y \in \min(Q)$. Fix $b \geq 0$. Then there exists a poset $R = (Z, \prec^\diamond)$ and $z \in \min(R)$, such that*

$$\begin{aligned}
e(R-z) \ &= \ e(Q-y)\,\big(e(P) \ + \ (n-1)\, e(P-x)\big), \\
e(R) \ &= \ b \cdot e(R-z) \ + \ e(Q)\,\big[e(P) \, + \, n \cdot e(P-x)\big], \\
|Z| \ &= \ m \, + \, n \, + \, b, \\
\operatorname{width}(R) \ &\leq \ \max\big\{\operatorname{width}(P), \operatorname{width}(Q)+1, 2\big\}.
\end{aligned}$$

Additionally, we have:

$$\rho(R,z) \ = \ b \ + \ \rho(Q,y)\left(1 + \frac{1}{n-1 \, + \, \rho(P,x)}\right).$$

Proof. This follows from first applying Lemma 3.2 then applying Lemma 3.3. □

Lemma 3.5. *Let $P=(X,\prec)$ be a poset on $|X|=m$ elements, let $x\in\min(P)$, and let $b\geq a\geq 0$. Then there exists a poset $R=(Z,\prec^\diamond)$ and $z\in\min(R)$, such that*

$$\begin{aligned} e(R-z) \ &= \ e(P) \ + \ (a-1)\cdot e(P-x), \\ e(R) \ &= \ (b-a)\cdot e(R-z) \ + \ a\cdot \big[e(P)+a\cdot e(P-x)\big] \ = \ b\cdot e(R-z)+a\cdot e(P-x), \\ |Z| \ &= \ m+b, \\ \operatorname{width}(R) \ &\leq \ \max\big\{\operatorname{width}(P),3\big\}. \end{aligned}$$

Additionally, we have:

$$\rho(R,z) \ = \ b \ + \ \frac{a}{a-1+\rho(P,x)}\,.$$

Proof. Let $Q=(Y,\prec')\coloneqq y\oplus C_{a-1}$ be the parallel sum of an element y with a chain of $a-1$ elements. Note that

$$e(Q) \ = \ a, \quad e(Q-y) \ = \ 1, \quad |Y| \ = \ a, \quad \text{and} \quad \operatorname{width}(Q) \ = \ 2.$$

The lemma now follow from by substituting $b\leftarrow(b-a)$ into Lemma 3.4. □

3.3. **A flip-flop construction.** We will need the following variation on the hybrid sum construction to prove Theorem 1.8.

Lemma 3.6. *Let $P=(X,\prec)$ and $Q=(Y,\prec')$ be posets on $m=|X|$ and $n=|Y|$ elements, and let $x\in\min(P)$, $y\in\min(Q)$. Then there exists a poset $R=(Z,\prec^\diamond)$ and $z\in Z$, such that*

$$\begin{aligned} e(R) \ &= \ e(P)\,e(Q-y) \ + \ e(P-x)\,e(Q), \\ e(R-y) \ &= \ e(P-x)\,e(Q-y), \\ |Z| \ &= \ m+n, \\ \operatorname{width}(R) \ &\leq \ \operatorname{width}(P) \ + \ \operatorname{width}(Q). \end{aligned}$$

Additionally, we have:

$$\rho(R) \ = \ \rho(P) \ + \ \rho(Q).$$

We warn the reader that the element z is not necessarily a minimal element of R, so this construction cannot be easily iterated.

Proof. Let $R=(Z,\prec^\diamond)$ be a poset defined as follows. Let

$$Z \ \coloneqq \ (X-x)\,\cup\,(Y-y)\,\cup\{u,v\},$$

where u,v are new elements. Let the partial order $\prec^\diamond$ be defined by

$$\begin{array}{rll} p \prec^\diamond p' & \text{for every} & p,p'\in X-x \quad \text{s.t.} \quad p\succ p', \\ q \prec^\diamond q' & \text{for every} & q,q'\in Y-y \quad \text{s.t.} \quad q\prec' q', \\ p \prec^\diamond u & \text{for every} & p\in X-x \quad \text{s.t.} \quad x\prec p, \\ u \prec^\diamond q & \text{for every} & q\in Y-y \quad \text{s.t.} \quad y\prec' q, \\ p \prec^\diamond v \prec^\diamond q & \text{for every} & p\in X-x, \quad q\in Y-y, \\ & \text{and} & u\,||_{\prec^\diamond}\, v. \end{array}$$

We have then:

$$e(R) \ = \ e(P)\cdot e(Q-y) \ + \ e(P-x)\cdot e(Q).$$

Indeed, the factor $e(P)\cdot e(Q-y)$ counts linear extensions $f\in\mathcal{E}(R)$ for which $f(u)<f(v)$, while the factor $e(P-x)\,e(Q)$ counts linear extensions $f\in\mathcal{E}(R)$ for which $f(u)>f(v)$. Also note that

$$e(R-u) \ = \ e(P-x)\cdot e(Q-y),$$

because $(R-u)$ is isomorphic to the linear sum $(P-x) \circledcirc \{v\} \circledcirc (Q-y)$. Finally, note that

$$\begin{aligned} \text{width}(R) &\leq \text{width}(P) + \text{width}(Q), \\ |Z| &= |X| + |Y|, \end{aligned}$$

by construction. This completes the proof. □

4. Proofs

4.1. **Proof of Theorem 1.10.** We prove the claim by induction on m. First, let $m=0$. Recall the notation in §2.2. Note that condition (1.9) becomes $b_0 \geq 1$, which holds by the assumption. Let $P = (X, \prec) := \{x\} \oplus C_{b_0-1}$. Then we have:

$$e(P) = b_0 = C_0(b_0) \quad \text{and} \quad e(P-x) = 1 = D_0(b_0).$$

We also have $|X| = b_0$ and $\text{width}(P) = 2$, as desired.

Suppose now that the claim holds for $m-1$. Let $b_1' := b_1 - a_1 + 1$. The balanced assumptions (1.9) gives $b_1' \geq a_2$. Thus, by the inductive assumption, there exist $P' = (X', \prec')$ and $x' \in \min(P)'$, such that

$$\begin{aligned} e(P'-x') &= D_0(a_2, \ldots, a_m; b_1', b_2, \ldots, b_m) = D_1(a_1, \ldots, a_m; b_0, b_1, \ldots, b_m), \\ e(P') &= C_0(a_2, \ldots, a_m; b_1', b_2, \ldots, b_m) \\ &= C_0(a_2, \ldots, a_m; b_1, b_2, \ldots, b_m) - (a_1 - 1) \cdot D_0(a_2, \ldots, a_m; b_1, b_2, \ldots, b_m) \\ &= C_1(a_1, \ldots, a_m; b_0, \ldots, b_m) - (a_1 - 1) \cdot D_1(a_1, \ldots, a_m; b_0, \ldots, b_m). \end{aligned}$$

Now, apply Lemma 3.5 to P' with $b \leftarrow b_0$ and $a \leftarrow a_1$. We obtain a poset $P = (X, \prec)$ on $|X| = n$ elements, and $x \in \min(P)$, such that

$$\begin{aligned} e(P-x) &= e(P') + (a_1 - 1) \cdot e(P'-x') = C_1(a_1, \ldots, a_m; b_0, \ldots, b_m) \\ &= D_0(a_1, \ldots, a_m; b_0, \ldots, b_m), \\ e(P) &= b_0 \cdot e(P-x) + a_1 \cdot e(P'-x') \\ &= b_0 \cdot D_0(a_1, \ldots, a_m; b_0, \ldots, b_m) + a_1 \cdot D_1(a_1, \ldots, a_m; b_0, \ldots, b_m) \\ &= C_0(a_1, \ldots, a_m; b_0, \ldots, b_m). \end{aligned}$$

We also have

$$\begin{aligned} n &= b_0 + |X'| = b_0 + b_1' + \sum_{i=2}^{m} b_i - \sum_{i=2}^{m} a_i + m - 1 \\ &= \sum_{i=0}^{m} b_i - \sum_{i=1}^{m} a_i + m = \mathrm{G}(a_1, \ldots, a_m; b_0, \ldots, b_m), \end{aligned}$$

and $\text{width}(P) \leq \max\big\{\text{width}(P'), 3\big\} \leq 3$. Finally, we have:

$$\rho(P, x) = b_0 + \frac{a_1}{a_1 - 1 + \rho(P', x')} = [a_1, \ldots, a_m; b_0, \ldots, b_m].$$

This completes the proof. □

4.2. **Proof of Theorem 1.13.** We prove the claim by induction on m. For $m=0$, let $P = (X, \prec) := x \oplus C_{b_0-1}$. We have:

$$e(P) = b_0 = C_0(b_0) \quad \text{and} \quad e(P-x) = 1 = D_0(b_0).$$

We also have $|X| = b_0$ and $\text{width}(P) = 2$, which proves the case $m=0$.

We now suppose the claim is already proved for $(m-1)$. By the induction assumption, there exists a poset $P' = (X', \prec')$ and element $x' \in \min(P')$, such that

$$\begin{aligned} e(P'-x') \ &= \ D_0(\alpha_2,\ldots,\alpha_m;b_1,b_2,\ldots,b_m) \ = \ D_1(\alpha_1,\ldots,\alpha_m;b_0,b_1,\ldots,b_m), \\ e(P') \ &= \ C_0(\alpha_2,\ldots,\alpha_m;b_1,b_2,\ldots,b_m) \ = \ C_1(\alpha_1,\ldots,\alpha_m;b_0,\ldots,b_m). \end{aligned}$$

Applying Theorem 1.6 to α_1, there exists a poset $Q = (Y, \prec')$ and $y \in \min(Q)$ such that

$$e(Q) \ = \ c_1, \quad e(Q-y) \ = \ d_1, \quad |Y| \ = \ \mathrm{s}(\alpha_1) \quad \text{and} \quad \mathrm{width}(Q) \ \leq \ 2.$$

Now, apply Lemma 3.4 to posets P', Q, and element b_0. We obtain a poset $P = (X, \prec)$ and $x \in \min(P)$, such that

$$\begin{aligned} e(P-x) \ &= \ e(Q-y)\left[e(P') \ + \ (|Y|-1)\, e(P'-x)\right], \\ &= \ d_1 \cdot \left[C_1(\alpha_1,\ldots,\alpha_m;b_0,\ldots,b_m) \ + \ \big(\mathrm{s}(\alpha_1)-1\big)\cdot D_1(\alpha_1,\ldots,\alpha_m;b_0,b_1,\ldots,b_m)\right] \\ &= \ D_0(\alpha_1,\ldots,\alpha_m;b_0,\ldots,b_m). \end{aligned}$$

We also have

$$\begin{aligned} e(P) \ &= \ b_0 \cdot e(P-x) \ + \ e(Q)\left[e(P') \ + \ |Y| \cdot e(P'-x')\right] \\ &= \ b_0 \cdot D_0(\alpha_1,\ldots,\alpha_m;b_0,\ldots,b_m) \\ &\qquad + \ c_1 \cdot \left[C_1(\alpha_1,\ldots,\alpha_m;b_0,\ldots,b_m) \ + \ \mathrm{s}(\alpha_1)\cdot D_1(\alpha_1,\ldots,\alpha_m;b_0,b_1,\ldots,b_m)\right] \\ &= \ C_0(\alpha_1,\ldots,\alpha_m;b_0,\ldots,b_m) \end{aligned}$$

and

$$\mathrm{width}(P) \ \leq \ \max\left\{\mathrm{width}(P'), \mathrm{width}(Q)+1, 2\right\} \ \leq \ 3.$$

Finally, we have

$$\begin{aligned} |X| \ &= \ |X'| + |Y| + b_0 \\ &= \ (b_1 + \ldots + b_m) + \mathrm{s}(\alpha_2) + \ldots + \mathrm{s}(\alpha_m)) + \mathrm{s}(\alpha_1) + b_0 \\ &= \ \mathrm{R}(\alpha_1,\ldots,\alpha_m;b_0,\ldots,b_m). \end{aligned}$$

This completes the proof. □

4.3. **Proof of Propositions 1.12 and 1.15.** For Proposition 1.12, recall from the introduction that the Conjecture 1.11 implies Conjecture 1.2 for prime d. Indeed, by Theorem 1.6 for a GCF $[a_1,\ldots,a_m;b_0,\ldots,b_m] = \frac{d}{c}$, we obtain a poset $P = (X, \prec)$ and $x \in X$ such that $|X| = \mathrm{g}\big(\frac{d}{c}\big) \leq C \log d$ and $\frac{e(P)}{e(P-x)} = \frac{d}{c}$. By the reduced condition on the definition of g, it then follows that $e(P) = d$, as desired.

To show that the first part of Conjecture 1.11 suffices, let $p_1^{m_1} \ldots p_\ell^{m_\ell}$ be the prime factorizations of d. For each prime p_i, let $P_i = (X_i, \prec_i)$ be the corresponding poset with $e(P_i) = p_i$ and $|X_i| \leq C \log p_i$. Define

$$P \ := \ \underbrace{P_1 \circledcirc \cdots \circledcirc P_1}_{m_1 \text{ times}} \circledcirc \ \cdots \ \circledcirc \underbrace{P_\ell \circledcirc \cdots \circledcirc P_\ell}_{m_\ell \text{ times}}$$

be the linear sum of posets P_i. We have:

$$e(P) \ = \ \prod_{i=1}^{\ell} e(P_i)^{m_i} \ = \ d,$$

and

$$|X| \ = \ \sum_{i=1}^{\ell} m_i \, |X_i| \ \leq \ C \sum_{i=1}^{\ell} m_i \, \log p_i \ = \ C \log d.$$

This completes the proof of Proposition 1.12. The proof of Proposition 1.15 follows verbatim. □

4.4. **Proof of Theorem 1.8.** First, observe that there exists a constant $C > 0$, such that for all coprime integers $a, b \le n$ which satisfy $C < b \le a \le 2b$, there exists a positive integer $\ell := \ell(a, b)$ such that $1 \le \ell < b$, and

$$\mathrm{s}\big(\tfrac{\ell}{b}\big) \,\le\, 2\, \log b\, \log\log b \quad \text{ and } \quad \mathrm{s}\big(\tfrac{a-\ell}{b}\big) \,\le\, 2\, \log b\, \log\log b. \tag{4.1}$$

Indeed, by Theorem 1.9 and using $\frac{12}{\pi^2} < 2$, for random $\ell \in \{1, \ldots, b\}$, the probability that each inequality fails $\to 0$ as $b \to \infty$. Taking C large enough so that each probability is $< \frac{1}{2}$ proves the claim.

Let a, b be given by

$$a \,:=\, m + n - \Big\lfloor \frac{n}{m} \Big\rfloor\, m, \quad b \,:=\, m,$$

so that $b \le a \le 2b$ and $b \le n$. From above, there exists $1 \le \ell \le b$, such that (4.1) holds. Let

$$\alpha \,:=\, 1 + \frac{\ell}{b} \quad \text{and} \quad \beta \,:=\, \Big\lfloor \frac{n}{m} \Big\rfloor - 2 + \frac{a-\ell}{b}\,.$$

It follows from the construction that $\alpha + \beta = \frac{n}{m}$. Since $\frac{n}{m} \ge 3$, we have $\alpha, \beta \ge 1$.

Applying Theorem 1.10 to simple continued fractions, we obtain a poset $P = (X, \prec)$ and element $x \in \min(P)$, such that

$$\rho(P, x) \,=\, \alpha \quad \text{and} \quad |X| \,=\, 1 + \mathrm{s}\big(\tfrac{\ell}{b}\big).$$

Similarly, we obtain a poset $Q = (Y, \prec')$ and element $y \in \min(Q)$, such that

$$\rho(Q, y) \,=\, \beta \quad \text{and} \quad |Y| \,=\, \big\lfloor \tfrac{n}{m} \big\rfloor - 2 + \mathrm{s}\big(\tfrac{a-\ell}{b}\big).$$

By Lemma 3.6, there exists a poset $R = (Z, \prec^{\diamond})$ and element $z \in Z$, such that

$$\rho(R, z) \,=\, \rho(P, x) + \rho(Q, y) \,=\, \frac{n}{m}\,,$$

and

$$|Z| \,=\, |X| + |Y| \,=\, \Big\lfloor \frac{n}{m} \Big\rfloor - 1 + \mathrm{s}\big(\tfrac{\ell}{b}\big) + \mathrm{s}\big(\tfrac{a-\ell}{b}\big) \,\le\, \frac{n}{m} + O(\log n\, \log\log n).$$

This completes the proof. □

5. Final remarks and open problems

5.1. The nature of connections between counting combinatorial objects and continued fractions described in §1.1 is clear and easy to explain: when objects are decomposed into smaller objects, they often have simple recurrences of the type described in §2.2. Fundamentally, this is the same reason why the generating functions are so powerful in combinatorial enumeration, see e.g. [GJ83, Sta12]. And yet, every time such a connection is found it is an unexpected delight, stemming both from the sheer elegance of continued fractions as well as the power of technical tools developed for them. While we tend to be swayed by the latter arguments, we appreciate the former sentiments.

5.2. The upper bound in Larcher's Theorem 1.3 was sharpened by Rukavishnikova [Ruk11] to $O(\log d \log\log d)$. Since $\frac{d}{\phi(d)}$ can be as large as $C \log\log d$, see e.g. [HW08, Thm 328], this is a significant asymptotic improvement. This result was further sharpened by Aistleitner, Borda and Hauke [ABH22, Cor. 2], who proved that for all $d \ge 3$ there exist $1 \le c < d$, such that

$$\mathrm{s}\big(\tfrac{c}{d}\big) \,\le\, \tfrac{12}{\pi^2} \log d \, \log\log d \,+\, O\big(\log d\big). \tag{5.1}$$

Note that we are using only prime d for our applications, which it why we postponed this recent result. We note in passing that the authors of [KS21] stated Conjecture 1.4 in the generality of all d; while plausible this remains out of reach with the existing technology. They were unaware of the earlier work and rediscovered Theorem 1.3.[1]

[1]Personal communication.

5.3. The asymptotics in the upper bound (5.1) cannot be easily improved by probabilistic arguments. This follows from a version on the tail estimates (1.7) given in [Ruk06]. A stronger result was proved in [ABH22, Thm 1], which implies that for all $c < 0$, $n \geq 3$, and $\varepsilon = \varepsilon(n) > (\log n)^c$, for the $(1-\varepsilon)$ fraction of $m \in \{1, \ldots, n\}$ with $\gcd(m,n) = 1$, we have:

$$\left| \mathrm{s}\left(\tfrac{m}{n}\right) - \tfrac{12}{\pi^2} \log n \log\log n \right| = O\left(\tfrac{\log n}{\varepsilon}\right). \tag{5.2}$$

Of course, this does not preclude the outlying small values predicted by Zaremba's conjecture. In fact, as was pointed out in [ABH22], the distribution of $\mathrm{s}\left(\frac{m}{n}\right)$ is heavy-tailed and has a large mean:

$$\frac{1}{\phi(n)} \sum_{m} s\left(\tfrac{m}{n}\right) = \tfrac{6}{\pi^2} (\log n)^2 + O\big((\log n)(\log\log n)^2\big), \tag{5.3}$$

where the summation is over all $m \in \{1, \ldots, n\}$ such that $\gcd(m,n) = 1$. This was proved independently in [Lie83, Pan82, YK75].

5.4. It was pointed out by Kravitz and Sah (see Remark 5.31 in [CP23a]), that the numerator c in Theorem 1.6 can be found in probabilistic polynomial time $\mathrm{poly}(\log d)$. Tail estimates (5.2) give a simpler (and faster) probabilistic algorithm: pick a random c, check if $\gcd(c,d) = 1$, compute a simple CF (1.1), repeat if $\mathrm{s}\left(\frac{c}{d}\right) > 2 \log d \log\log d$. It is an interesting open problem if this can be done deterministically. More broadly, is there a deterministic polynomial time construction of a poset with exactly n linear extension? So far, the only deterministic construction we know if by Tenner [Ten09], which is exponential in $(\log n)$.

5.5. Zaremba's Conjecture 1.5 is often stated with $A = 5$ or even $A = 4$ for all sufficiently large integers. It is known to hold for integers of the form $2^m 3^n$, for other families of powers of small primes and sufficiently large powers of all primes, see [Shu23]. We refer to [BPSZ14, §6.2] for an elegant presentation of the 2^m case. Of course, the Kravitz–Sah Conjecture 1.2 is trivial in this case. Note that the constant 50 in the Bourgain–Kontorovich theorem that was used in the proof of Corollary 1.7, has been improved to 5 in [Hua15]. See [Kan21] for further extensions, and [Shk21, §7] for an overview.

5.6. It would be interesting to find an elementary proof of the first part of Corollary 1.7. The result is especially surprising given that the bound is obtained on a relatively small family of posets of width two. On the other hand, we know of no nontrivial bound for the much larger family of height two posets (cf. [Sou23]).

5.7. In [CP23a, Conj. 5.17], we conjecture that all but finitely many integers are the numbers of linear extensions of posets of height two. We also observe (Prop. 5.18, ibid.), that this would imply Conjecture 1.2 with a sharp $\Theta\left(\frac{\log n}{\log\log n}\right)$ asymptotics.

5.8. The idea of Theorem 1.8 comes from the approach in [CP23a], where we studied relative versions of several counting functions (domino tilings, spanning trees, etc.) The proof of Theorem 1.8 is based on the approach in [CP23b, §8.2]. It would be interesting to see if the condition $n \geq 3m$ can be weakened to $n \geq (1+\varepsilon)m$ or even dropped. Additionally, by analogy with the Kravitz–Sah Conjecture 1.2, we conjecture that (1.5) can be improved to

$$\nu(m,n) \leq \tfrac{n}{m} + O(\log n). \tag{5.4}$$

In a different direction, one can ask about the smallest size poset with $e(P) = n$ and $e(P-x) = m$, since the construction in the proof can result in an integer multiple of both.

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