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# The bunkbed conjecture is false

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**We give an explicit counterexample to the bunkbed conjecture introduced by Kasteleyn in 1985. The counterexample is given by a planar graph on 7,222 vertices and is built on the recent work of Hollom (2024).**

percolation on graphs | random network | bunkbed conjecture | hypergraph | counterexample

## Significance

The bunkbed conjecture, proposed in 1985, addresses whether certain probabilities in network models, specifically in percolation theory, exhibit a predictable monotonic behavior. While the conjecture has been supported in specific cases, its general validity remained an open problem. This study presents a counterexample, disproving the conjecture and revealing that the expected monotonic behavior does not hold universally. The disproof is based on the careful analysis of a particular hypergraph percolation model. Our results add to the understanding of probabilistic models on graphs and hypergraphs, particularly in percolation theory.

## 1. Introduction

The bunkbed conjecture (BBC) is a celebrated open problem in probability introduced by Kasteleyn in 1985; see Remark 5 in ref. 1. The conjecture is both natural and intuitively obvious but has defied repeated proof attempts; it is known only in a few special cases. In this paper, we disprove the conjecture without resorting to computer experiments (cf., Section 7).

Let $G = (V, E)$ be a connected graph, possibly infinite and with multiple edges. In Bernoulli bond percolation, each edge is deleted independently at random with probability $1 - P$, and otherwise retained with probability $P \in [0, 1]$. Equivalently, this model gives a random subgraph of $G$ weighted by the number of edges. For $P = \frac{1}{2}$ we obtain a uniform random subgraph of $G$. See refs. 2 and 3 for standard results and ref. 4 and 5 for recent overview of percolation.

Let $\mathbb{P}_P[u \leftrightarrow v]$ denote the probability that vertices $u, v \in V$ are connected. It is often of interest to compare these probabilities, as computing them exactly is #P-hard (6). For example, the classical Harris–Kleitman inequality, a special case of the FKG inequality, implies that $\mathbb{P}_P[u \leftrightarrow v] \le \mathbb{P}_P[u \leftrightarrow v \,|\, u \leftrightarrow w]$ for all $u, v, w \in V$; see, e.g., ref. 7, Chapter 6. Harris used this to prove that the critical probability $P_c(G) := \inf\{P : \mathbb{P}_P(G) > 0\}$ satisfies $P_c(\mathbb{Z}^2) \ge \frac{1}{2}$ (8), in the first step toward Kesten's remarkable exact value $P_c(\mathbb{Z}^2) = \frac{1}{2}$ (9), where $\mathbb{P}_P(G)$ denotes the probability that there exists an infinite percolation cluster. Considerations of percolation monotonicity on $\mathbb{Z}^2$ (Section 8.8.4), led Kasteleyn to the following problem.

Fix a finite connected graph $G = (V, E)$ and a subset $T \subseteq V$. A bunkbed graph $\overline{G} = (\overline{V}, \overline{E})$ is a subgraph of the graph product $G \times K_2$ defined as follows. Take two copies of $G$, which we denote $G$ and $G' = (V', E')$, and add all edges of the form $(w, w')$, where $w \in T$ and $w'$ is a corresponding vertex in $T'$; we denote this set of edges by $\overline{T}$. The resulting bunkbed graph has $\overline{V} = V \cup V'$ and $\overline{E} = E \cup E' \cup \overline{T}$.

In the bunkbed percolation, the usual bond percolation is performed only on edges in $G$ and $G'$, while all edges in $\overline{T}$ are retained (i.e., not deleted). We use $\mathbb{P}^{\mathrm{bb}}_P[u \leftrightarrow v]$ to denote the connecting probability in this case. The vertices in $T$ are called transversal and the edges in $\overline{T}$ are called posts, to indicate their special status. See, e.g., refs. 10 and 11, for these and several other equivalent models of the bunkbed percolation. We refer also to refs. 12 (section 4.1), 13 (section 5.5), and 14 for recent overviews and connections to other areas.

**Conjecture 1.1.** [bunkbed conjecture] Let $G = (V, E)$ be a connected graph, let $T \subseteq V$, and let $0 < P < 1$. Then, for all $u, v \in V$, we have

$$\mathbb{P}^{bb}_P[u \leftrightarrow v] \ge \mathbb{P}^{bb}_P[u \leftrightarrow v'].$$

The BBC is known in a number of special cases, including wheels (15), complete graphs (16–18), complete bipartite graphs (19), and graphs symmetric, w.r.t., the $u \leftrightarrow v$ automorphism (19). It is also known for one (10, Lemma 2.4) and for two transversal vertices (20, section 6.3), see also ref. 1. Finally, the conjecture was recently proved in the $P \uparrow 1$ limit (21, 22).

**Theorem 1.2.** *There is a connected planar graph $G = (V, E)$ with $|V| = 7{,}222$ vertices and $|E| = 14{,}442$ edges, a subset $T \subset V$ with three transversal vertices, and vertices $u, v \in V$, such that*

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$$\mathbb{P}^{\mathrm{bb}}_{\frac{1}{2}}[u \leftrightarrow v] \, < \, \mathbb{P}^{\mathrm{bb}}_{\frac{1}{2}}[u \leftrightarrow v'] .$$

*In particular, the BBC is false.*

The result is surprising since analogous inequalities for simple random walks and for the Ising model on bunkbed graphs were proved by Häggström (23, 24), cf. Section 8.8.5. Recall that three is the smallest number of transversal vertices we can have to disprove the conjecture. On the other hand, the total number of vertices is unlikely to be optimal, see Remark 4.2 and Section 7.

The proof of the theorem is based on an example of Hollom (25) refuting the 3-uniform hypergraph version of the BBC. Unfortunately, Hollom's example alone cannot disprove the conjecture since it is impossible to find a gadget graph simulating a single 3-hyperedge using bond percolation (26, Theorem 1.5).

We give a robust version of Hollom's construction using the approach in ref. 26 and 27. The proof of Theorem 1.2 occupies most of the paper. It is self-contained modulo Hollom's result which is small enough to be checked by hand. In Section 6, we extend the theorem to the case when the set of transversal vertices is not fixed but chosen uniformly at random from $V$; see Theorem 6.1. We conclude with discussion of our computer experiments in Section 7, and final remarks in Section 8.

## 2. Notation

In percolation, deleted edges are called closed while retained edges are called open. Note that there are several different models of percolation and variations on the BBC; see Section 8.8.1.

A hypergraph is a collection of subsets of vertices; to simplify the notation, we use the same letter to denote both. The hypergraph is called uniform if all hyperedges have the same size. A path in a hypergraph is a sequence $(v_0 \to v_1 \to \ldots \to v_\ell)$ of vertices, such that $v_{i-1}, v_i$ lie in the same hyperedge, for all $1 \le i \le \ell$. We say that two vertices in a hypergraph are connected if there is a path between them. For further definitions and results on hypergraphs, see, e.g., ref. 28 (section 1.2).

The notion of hypergraph percolation is a natural extension of graph percolation and goes back to the study of random hypergraphs; see, e.g., ref. 29. In recent years, the study of hypergraph percolation also emerged in probabilistic and statistical physics literature, see, e.g., refs. 30 and 31, respectively.

## 3. Hypergraph Percolation

**3.1. Hollom's Example.** Let $H$ be a finite connected hypergraph on the set $V$ of vertices. We use $\mathbb{P}_P[u \leftrightarrow v]$ to denote probability of connectivity of vertices $u, v \in V$ in the hypergraph percolation, where each hyperedge $e$ in $H$ is retained with probability $P$, or deleted with probability $1 - P$.

Let $T \subseteq V$ be the set of transversal vertices. Denote by $\overline{H}$ be the bunkbed hypergraph with levels $H \simeq H'$, and vertical posts which are the (usual) edges. Note that $\overline{H}$ has horizontal hyperedges and vertical posts.

In ref. 25, Hollom considers the following natural hypergraph generalization of the Alternative BBC; see Section 8.8.1. In the alternative bunkbed hypergraph percolation, each hyperedge $e$ in $H$ is either deleted while the corresponding hyperedge $e'$ in $H'$ is retained with probability $\frac{1}{2}$, or vice versa: The hyperedge $e$ is retained and $e'$ is deleted.

**Lemma 3.1.** *[*Hollom (25, Claim 5.1)*] Let $H$ be the hypergraph with six 3-edges as in the Figure 1, and let $T = \{u_2, u_7, u_9\}$ be the set of transversal vertices. In the alternative bunkbed hypergraph percolation, we have*

$$\mathbb{P}^{\mathrm{alt}}[u_1 \leftrightarrow u'_{10}] \, = \, \frac{13}{64} \quad \textit{and} \quad \mathbb{P}^{\mathrm{alt}}[u_1 \leftrightarrow u_{10}] \, = \, \frac{12}{64} \, .$$

We give a robust version of Hollom's construction.

**3.2. Robust Hyperedge Lemma.** Note that in Hollom's example, each hyperedge has exactly one transversal vertex. We explore this structure.

Consider the following WZ hypergraph percolation model introduced by Wierman and Ziff in ref. 30 (see also ref. 26). We define this model only for the graph $\overline{H}$. Let $e = (a, b, c)$ be a hyperedge where $a$ is a transversal vertex. We will fix the order of vertices in each hyperedge precisely in Eq. **5**. In the model, hyperedge $e$ is set to have

- probability $P_{abc}$ to connect all three vertices,
- probability $P_{a|b|c}$ to not connect any of the vertices,
- probability $P_{a|bc}$ to connect two nontransversal vertices, and
- probability $P_{ab|c} = P_{ac|b}$ to connect a transversal to a nontransversal vertex,

and these events are independent on all hyperedges.

Finally, we assume that these five probabilities sum up to 1:

$$P_{abc} \, + \, P_{a|b|c} \, + \, P_{a|bc} \, + \, P_{ab|c} \, + \, P_{ac|b} \, = \, 1 \, .$$

It is easy to see that the hypergraph percolation on $\overline{H}$ is a partial case of the WZ model, where $P_{abc} = P$ and $P_{a|b|c} = 1 - P$.

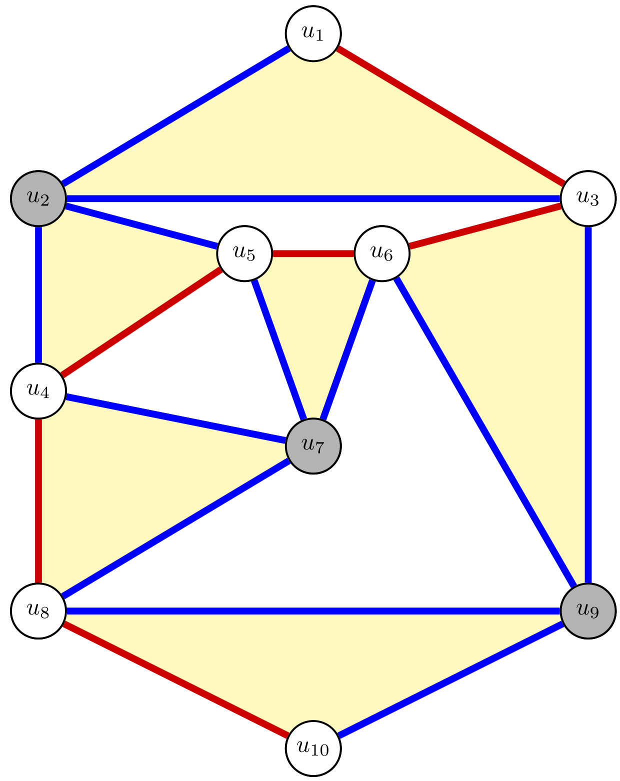


**Fig. 1.** Hollom's 3-uniform hypergraph $H$.

**Definition 3.2.** [Configurations in the WZ model]. A configuration in the WZ model is an assignment of one of the five states $\{abc,\ a|b|c,\ a|bc,\ ab|c,\ ac|b\}$ to each hyperedge in $H \cup H'$. Equivalently, we can represent it by a function

$$\psi : H \cup H' \to \Upsilon \quad \text{where} \quad \Upsilon = \{abc, ab|c, ac|b, a|bc, a|b|c\}.$$

The probability of a configuration $\psi$ is given by

$$\mathbb{P}(\psi) = \prod_{e \in H \cup H'} P_{\psi(e)},$$

where $P_\alpha$ denotes the probability of the state $\alpha \in \Upsilon$.

We say that vertices $u$ and $v$ are connected (written $u \leftrightarrow v$) if they are connected by a path in the bunkbed hypergraph $\overline{H}$ in a way that every two vertices on a hyperedge are connected by the rules above. We use $\mathbb{P}^{\mathrm{wz}}[\, u \leftrightarrow v \,]$ to denote these connection probabilities, omitting the superscript if the model is clear from context.

**Lemma 3.3.** *Let $H$ be Hollom's hypergraph in the Fig. 1, $\overline{H}$ be the bunkbed hypergraph built on it, and let $T = \{u_2, u_7, u_9\}$ be the set of transversal vertices. Consider the WZ hypergraph percolation as described above, where the connection probabilities satisfy*

$$400\, P_{a|bc} \;\le\; P_{abc}\, P_{a|b|c} \;-\; P_{ab|c} P_{ac|b}\,. \tag{1}$$

*Then, we have*

$$\mathbb{P}^{\mathrm{wz}}(u_1 \leftrightarrow u_{10}) \;<\; \mathbb{P}^{\mathrm{wz}}(u_1 \leftrightarrow u_{10}')\,. \tag{2}$$

It was noted in ref. 27, that the RHS in Eq. **1** is nonnegative if the hyperedge is simulated by a gadget in Bernoulli edge percolation:

$$P_{ab|c}\, P_{ac|b} \;=\; P_{ab|c}^2 \;\le\; P_{abc}\, P_{a|b|c}\,. \tag{3}$$

This follows from the Harris–Kleitman (HK) inequality. In fact, a slightly stronger inequality always holds; see ref. 27 (Corollary 3.6). Since the LHS in Eq. **1** is nonnegative, one can view this assumption as strengthening the HK inequality in this case (cf. Section 8.8.2). The hypergraph model also satisfies Eq. **1**, because two of the terms vanish.

**3.3. Refining the State Space.** Let $\mathcal{C}$ be the set of configurations $\psi$ that contain a path $u_1 \leftrightarrow u_{10}$, and let $\mathcal{C}'$ be the set of those containing a path $u_1 \leftrightarrow u_{10}'$. The probabilities of sets $\mathcal{C}$ and $\mathcal{C}'$ are given by

$$\mathbb{P}(\mathcal{C}) \;:=\; \sum_{\psi \in \mathcal{C}} \mathbb{P}(\psi) \quad \text{and} \quad \mathbb{P}(\mathcal{C}') \;:=\; \sum_{\psi \in \mathcal{C}'} \mathbb{P}(\psi).$$

Our goal is to prove $\mathbb{P}(\mathcal{C}) < \mathbb{P}(\mathcal{C}')$ by constructing a suitable map from $\mathcal{C}$ to $\mathcal{C}'$.

Since each hyperedge in $\overline{H}$ has a symmetric counterpart, one can view a configuration not as a function $\psi : \overline{H} \to \Upsilon$, but as a function from $H$ to $\Upsilon^2$. For each hyperedge $e$ in $H$, there are 25 possibilities for a pair $(\psi(e), \psi(e'))$. To handle certain configurations more precisely, we refine the pairs $(abc, a|b|c)$ and $(a|b|c, abc)$ by splitting each into two disjoint subevents:

$$(abc, a|b|c) \mapsto (abc, a|b|c)_+ \cup (abc, a|b|c)_-, \quad (a|b|c, abc) \mapsto (a|b|c, abc)_+ \cup (a|b|c, abc)_-.$$

The probabilities (or weights) of these refined events are given by

$$\mathbb{P}\big[(abc, a|b|c)_+\big] = P_{ab|c} \cdot P_{ac|b}, \quad \mathbb{P}\big[(abc, a|b|c)_-\big] = P_{abc} \cdot P_{a|b|c} - P_{ab|c} \cdot P_{ac|b}, \qquad [4]$$

and similarly for $(a|b|c, abc)_+$ and $(a|b|c, abc)_-$. This refinement increases the total number of possible pairs in $\Upsilon^2$ from 25 to 27, resulting in the extended state space

$$\Upsilon^2_{\text{ext}} := \Upsilon^2 \setminus \big\{(a|b|c, abc), (abc, a|b|c)\big\} \cup \big\{(abc, a|b|c)_+, (abc, a|b|c)_-, (abc, a|b|c)_+, (abc, a|b|c)_-\big\}.$$

In the original model, $\psi(e)$ and $\psi(e')$ were sampled independently from $\Upsilon$. After the refinement, however, we consider a new framework where by configuration we mean a function $\Psi : H \to \Upsilon^2_{\text{ext}}$. In the new model, the state of a pair $(e, e')$ is sampled directly from $\Upsilon^2_{\text{ext}}$ independently with probabilities given by equation Eq. **4** for refined pairs and $\mathbb{P}\big[(\alpha, \beta)\big] = P_\alpha P_\beta$ for nonrefined pairs. Then the states $\psi(e)$ and $\psi(e')$ are defined by $\Psi(e)$.

Building on the refinement of $\Upsilon^2$ to $\Upsilon^2_{\text{ext}}$, we now focus on a particularly important subset of states. Specifically, we consider a smaller set of interest:

$$\Lambda := \{abc, ab|c, ac|b, a|b|c\}.$$

The Cartesian product $\Lambda^2 := \Lambda \times \Lambda$ consists of all ordered pairs of states in $\Lambda$, giving $4 \times 4 = 16$ elements. To incorporate the refined structure introduced earlier, we replace the pairs $(abc, a|b|c)$ and $(a|b|c, abc)$ in $\Lambda^2$ with their "$+$" counterparts.

**Definition 3.4.** [Refined Pair Set $\Lambda^2_+$]. The refined set of pairs $\Lambda^2_+$ is defined as

$$\Lambda^2_+ := \big\{(\alpha, \beta) \in \Lambda^2 : (\alpha, \beta) \notin \{(abc, a|b|c), (a|b|c, abc)\}\big\} \cup \big\{(abc, a|b|c)_+, (a|b|c, abc)_+\big\} \subset \Upsilon^2_{\text{ext}}.$$

**3.4. Involutions on Extended State Spaces.** To proceed with the construction of a map from $\mathcal{C}$ to $\mathcal{C}'$, we define two weight-preserving involutions on the extended state space $\Upsilon^2_{\text{ext}}$.

**Definition 3.5.** [Reflection Involution]. The reflection involution $\rho$ is defined on the extended state space $\Upsilon^2_{\text{ext}}$. For a pair $\Psi(e) \in \Upsilon^2_{\text{ext}}$, it swaps the states of $e$ and $e'$, and is formally given by

$$\rho\big((\alpha, \beta)\big) := (\beta, \alpha).$$

Additionally, for refined states, the reflection involution $\rho$ is defined to preserve the sign.

The reflection involution $\rho$ is weight-preserving because the weight of each configuration is symmetric under the swapping of $\psi(e)$ and $\psi(e')$. While $\rho$ works by simply swapping states, making it straightforward to handle symmetry, the half-reflection involution $\eta$ requires a more detailed approach. It is constructed to modify vertex connections as described in Proposition 3.7, while preserving weights.

Partition the set of pairs $\Lambda^2_+ = \Omega_0 \cup \Omega_1 \cup \Xi$ into the following three subsets:

$$\Omega_0 := \left\{ \begin{array}{l} (abc, ac|b),\ (ac|b, abc),\ (a|b|c, ab|c),\ (ab|c, a|b|c), \\ (abc, abc),\ (a|b|c, a|b|c),\ (ab|c, ab|c),\ (ac|b, ac|b) \end{array} \right\},$$

$$\Omega_1 := \big\{(abc, ab|c), (ab|c, abc), (a|b|c, ac|b), (ac|b, a|b|c)\big\}, \text{ and}$$

$$\Xi := \big\{(abc, a|b|c)_+, (a|b|c, abc)_+, (ab|c, ac|b), (ac|b, ab|c)\big\}.$$

**Definition 3.6.** [Half-Reflection Involution]. The half-reflection involution $\eta$ is defined on the set $\Lambda^2_+$ as follows:

- On $\Omega_0$, the involution $\eta$ is the identity map.
- On $\Omega_1$, the half-reflection coincides with the reflection involution $\rho$ defined earlier.
- On $\Xi$, the half-reflection involution $\eta$ is given by

$$\eta\big((abc, a|b|c)_+\big) := (ab|c, ac|b), \quad \eta\big((ab|c, ac|b)\big) := (abc, a|b|c)_+,$$

$$\eta\big((a|b|c, abc)_+\big) := (ac|b, ab|c), \quad \eta\big((ac|b, ab|c)\big) := (a|b|c, abc)_+.$$

This involution is weight-preserving because it satisfies the following conditions:

- On $\Omega_0$, the involution is constant, making it trivially weight-preserving.
- On $\Omega_1$, the involution coincides with the reflection involution $\rho$, which has already been shown to preserve weights.
- On $\Xi$, the involution swaps pairs in such a way that the probabilities remain balanced. Specifically, since

  $$\mathbb{P}\big((abc, a|b|c)_+\big) = \mathbb{P}\big((a|b|c, abc)_+\big) = P_{ab|c} \cdot P_{ac|b},$$

  swapping $(abc, a|b|c)_+$ with $(ab|c, ac|b)$, and $(a|b|c, abc)_+$ with $(ac|b, ab|c)$, does not alter the total probability.

After defining the half-reflection involution $\eta$, we examine how it modifies connectivity in configurations. The following proposition describes the effect of $\eta$ on states in $\Lambda_+^2$.

**Proposition 3.7.** *Let $\Psi$ be a configuration and $e = (a, b, c) \in H$ such that $\Psi(e) \in \Lambda_+^2$. Let $\Psi'$ be any configuration such that*

$$\Psi'(e) = \eta(\Psi(e)).$$

*Then, the following properties hold:*

1. *If $b$ and $c$ are connected in $\Psi$ within $e$, then $b$ and $c'$ are connected in $\Psi'$. Similarly, if $b'$ and $c'$ are connected in $\Psi$ within $e'$, then $b'$ and $c$ are connected in $\Psi'$.*
2. *If $a$ and $b$ are connected in $\Psi$ within $e$, then they remain connected in $\Psi'$. Similarly, if $a'$ and $b'$ are connected in $\Psi$ within $e'$, then they remain connected in $\Psi'$.*
3. *If $a$ and $c$ are connected in $\Psi$ within $e$, then $a'$ and $c'$ are connected in $\Psi'$. Similarly, if $a'$ and $c'$ are connected in $\Psi$ within $e'$, then $a$ and $c$ are connected in $\Psi'$.*

***Proof:*** We will prove only the first part of each statement. The second part follows directly from the symmetry of $\eta$. In particular, the relation

$$\eta\big(\rho\big(\Psi(e)\big)\big) = \rho\big(\eta\big(\Psi(e)\big)\big)$$

guarantees that the roles of $e$ and $e'$ are interchangeable under $\eta$.

For (i), suppose $b$ and $c$ are connected in $\Psi$ within $e$, which implies $\psi(e) = abc$. In $\Psi'$, we claim there exists a path $b \to a \to a' \to c'$. This holds if:

- The first component of $\eta\big(\Psi(e)\big)$ belongs to $\{abc, ab|c\}$, and
- The second component of $\eta\big(\Psi(e)\big)$ belongs to $\{abc, ac|b\}$,

whenever $\psi(e) = abc$. These conditions follow directly from the definition of $\eta$ and its action on $\Omega_0$, $\Omega_1$, and $\Xi$.

For (ii), assume $a$ and $b$ are connected in $\Psi$ within $e$, which implies $\psi(e) \in \{ab|c, abc\}$. In $\Psi'$, we verify that $a$ and $b$ remain connected within $e$. This requires that the first component of $\eta\big(\Psi(e)\big)$ belongs to $\{ab|c, abc\}$, whenever $\psi(e) \in \{ab|c, abc\}$. Again, this follows from the definition of $\eta$.

For (iii), suppose $a$ and $c$ are connected in $\Psi$ within $e$, which implies $\psi(e) \in \{ac|b, abc\}$. In $\Psi'$, we claim $a'$ and $c'$ are connected within $e'$. This is satisfied if the second component of $\eta\big(\Psi(e)\big)$ belongs to $\{ac|b, abc\}$, whenever $\psi(e) \in \{ac|b, abc\}$. The result follows directly from the definition of $\eta$.

$\square$

**3.5. Proof of Lemma 3.3.** We define the subset of configurations $\mathcal{X}$ as

$$\mathcal{X} := \big\{\Psi : \Psi(e) \in \Lambda_+^2 \text{ for some } e \in H\big\}.$$

Our goal is to construct a weight-preserving involution $\phi : \mathcal{X} \to \mathcal{X}$, which satisfies:

$$\Psi \in \mathcal{C} \Rightarrow \phi(\Psi) \in \mathcal{C}', \quad \text{and} \quad \Psi \in \mathcal{C}' \Rightarrow \phi(\Psi) \in \mathcal{C}.$$

To define $\phi$, we begin by introducing the red path $\rho$ from $u_1$ to $u_{10}$, as shown in Fig. 1. Observe that $\rho$ traverses every hyperedge exactly once and avoids transversal vertices. Fix the order on the hyperedges of $H$ according to their appearance along the path $\rho$:

$$(u_2, u_1, u_3),\ (u_9, u_3, u_6),\ (u_7, u_6, u_5),\ (u_2, u_5, u_4),\ (u_7, u_4, u_8),\ (u_9, u_8, u_{10}). \qquad \textbf{[5]}$$

This notation also establishes a fixed ordering for the vertices within each hyperedge. Specifically, if a hyperedge $e = (a, b, c)$ corresponds to an entry $(u_i, u_j, u_k)$ in the sequence above, then the vertices of $e$ are assigned as $a = u_i$, $b = u_j$, and $c = u_k$, preserving the order within each tuple. In particular, the first vertex $a = u_i$ in each tuple is a transversal vertex.

The map $\phi : \mathcal{X} \to \mathcal{X}$ is defined as follows. Let $e = (a, b, c)$ be the first hyperedge along $\rho$ such that $\Psi(e) \in \Lambda_+^2$. The configuration $\Psi' = \phi(\Psi)$ is constructed according to the following rule:

$$\Psi'(h) = \begin{cases} \Psi(h) & \text{if } h \text{ appears before } e \text{ along } \rho, \\ \eta\big(\Psi(h)\big) & \text{if } h = e, \\ \rho\big(\Psi(h)\big) & \text{if } h \text{ appears after } e \text{ along } \rho. \end{cases}$$

Since both $\eta$ and $\rho$ are weight-preserving by their respective definitions, $\phi$ is also a weight-preserving involution.

Next, we establish that $\phi$ maps configurations in $\mathcal{C}$ to configurations in $\mathcal{C}'$, and vice versa.

**Lemma 3.8.** *Let $\Psi \in \mathcal{X} \cap \mathcal{C}$, and let $\Psi'$ be the configuration obtained by applying the involution $\phi$ to $\Psi$. Then, $\Psi' \in \mathcal{C}'$. Conversely, if $\Psi \in \mathcal{X} \cap \mathcal{C}'$, then $\Psi' \in \mathcal{C}$.*

***Proof:*** We have $V = \{u_1, \ldots, u_{10}\}$ and $T = \{u_2, u_7, u_9\}$. Let $L \subseteq (V \setminus T) \cup (V' \setminus T')$ denote the set of nontransversal vertices that lie along the path $\rho$ between $u_1$ and $b$, inclusively, along with their counterparts from the other level. Similarly, let $R := (V \setminus T) \cup (V' \setminus T') \setminus L$. For any path $\gamma$ in $\Psi$, we construct a corresponding path $\gamma'$ in $\Psi'$ as follows. For all vertices $u_i \in R$ on $\gamma$, replace $u_i$ with its counterpart $u_i'$ in $\gamma'$, and vice versa: For all $u_i' \in R$ on $\gamma$, replace $u_i'$ with $u_i$ in $\gamma'$.

To confirm that $\gamma'$ is a connected path in $\Psi'$, consider two sequential vertices $x_k$ and $x_{k+1}$ in $\gamma$ and their corresponding images $y_k$ and $y_{k+1}$ in $\gamma'$. We analyze the connectivity in the following cases:

- **Transversal edge:** If $x_k x_{k+1}$ is a transversal edge in $\gamma$, then it remains unchanged under $\phi$. The corresponding edge $y_k y_{k+1}$ in $\gamma'$ is also transversal, ensuring connectivity.
- **Hyperedge before** $e$**:** If $x_k$ and $x_{k+1}$ are connected in $\Psi$ through a hyperedge $h$ before $e$ in $\rho$, then $y_k = x_k$, $y_{k+1} = x_{k+1}$, and $\Psi'(h) = \Psi(h)$. The connection is preserved in $\gamma'$.
- **Hyperedge after** $e$**:** If $x_k$ and $x_{k+1}$ are connected in $\Psi$ through a hyperedge $h$ after $e$ in $\rho$, then $y_k = x_k'$, $y_{k+1} = x_{k+1}'$, where the prime indicates the symmetric component (not necessarily in $H'$). Since $\phi$ applies the reflection involution $\rho$ to hyperedges after $e$, we have $\Psi'(h') = \Psi(h)$, ensuring that $y_k$ and $y_{k+1}$ remain connected in $\Psi'$.
- **Hyperedge** $e$**:** If $x_k$ and $x_{k+1}$ are connected in $\Psi$ through the hyperedge $e$, Proposition 3.7 ensures that the connectivity is appropriately modified in $\Psi'$.

Thus, for every sequential pair of vertices $x_k, x_{k+1}$ in $\gamma$, their images $y_k, y_{k+1}$ in $\gamma'$ are connected in $\Psi'$. Moreover, the mapping constructed to transform $\gamma$ into $\gamma'$ always maps $u_1$ to itself and $u_{10}$ to $u_{10}'$. Consequently, if $\gamma$ connects $u_1$ to $u_{10}$, then $\gamma'$ connects $u_1$ to $u_{10}'$, and vice versa. This completes the proof. □

With $\phi$ established as a weight-preserving involution on $\mathcal{X}$ and Lemma 3.8 proving that $\phi$ maps $\mathcal{C} \cap \mathcal{X}$ to $\mathcal{C}' \cap \mathcal{X}$ and vice versa, it follows that:

$$\mathbb{P}(\mathcal{C} \cap \mathcal{X}) = \mathbb{P}(\mathcal{C}' \cap \mathcal{X}).$$

This equivalence allows us to focus on the complementary subset of configurations, $\mathcal{X}^c$, which consists of all $\Psi$ such that for every hyperedge $e \in H$, the pair $\Psi(e)$ belongs to $\Upsilon^2_{\text{ext}} \setminus \Lambda^2_+$. Explicitly, the set $\Upsilon^2_{\text{ext}} \setminus \Lambda^2_+$ is described by the following pairs:

$$\begin{aligned}&(abc, a|b|c)_-, \quad (a|b|c, abc)_- \quad \text{with probability } P_{abc} \cdot P_{a|b|c} - P_{ac|b} \cdot P_{ab|c},\\ &(a|bc, *), \quad (*, a|bc), \quad \text{and} \quad (a|bc, a|bc), \quad \text{where } * \in \Lambda.\end{aligned}$$

In this setting, the WZ hypergraph percolation model conditioned on $\mathcal{X}^c$ has the following probabilities for each remaining possible value of $\Psi(e)$:

$$\begin{cases}(abc, a|b|c)_-, \text{ with probability } \frac{1}{Z}\left(P_{abc}P_{a|b|c} - P_{ab|c}P_{ac|b}\right),\\ (a|b|c, abc)_-, \text{ with probability } \frac{1}{Z}\left(P_{abc}P_{a|b|c} - P_{ab|c}P_{ac|b}\right),\\ (a|bc, *), \text{ with probability } \frac{1}{Z}P_{a|bc} \cdot P_*, \quad \text{for } * \in \Lambda,\\ (*, a|bc), \text{ with probability } \frac{1}{Z}P_{a|bc} \cdot P_*, \quad \text{for } * \in \Lambda,\\ (a|bc, a|bc), \text{ with probability } \frac{1}{Z}P^2_{a|bc},\end{cases}$$

where the normalizing constant is:

$$Z := 2\,P_{abc} \cdot P_{a|b|c} - 2\,P_{ab|c} \cdot P_{ac|b} + 2\,P_{a|bc} - P^2_{a|bc}.$$

We denote the corresponding conditional probabilities by $\mathbb{P}_{\mathcal{X}^c}$. This notation emphasizes the restriction to the subset $\mathcal{X}^c$, making the context of these probabilities explicit.

Denote by $\mathcal{A}$ the subset of events that for all $e \in H$, the value of $\Psi(e)$ belongs to $\{(abc, a|b|c)_-, (a|b|c, abc)_-\}$. Using the inequality $(1-x)^a \geq 1 - ax$ and the assumption Eq. **1** from the Lemma, we compute $\mathbb{P}_{\mathcal{X}^c}(\mathcal{A})$ as follows:

$$\begin{aligned}\mathbb{P}_{\mathcal{X}^c}(\mathcal{A}) &= \left(1 - \frac{2\,P_{a|bc} - P^2_{a|bc}}{2(P_{abc}\,P_{a|b|c} - P_{ab|c}\,P_{ac|b}) + 2\,P_{a|bc} - P^2_{a|bc}}\right)^6\\ &\geq \left(1 - \frac{P_{a|bc}}{(P_{abc}\,P_{a|b|c} - P_{ab|c}\,P_{ac|b}) + P_{a|bc}}\right)^6\\ &\geq 1 - \frac{6\,P_{a|bc}}{(P_{abc}\,P_{a|b|c} - P_{ab|c}\,P_{ac|b}) + P_{a|bc}} >_{Eq.\ \mathbf{1}} 1 - \frac{6\,P_{a|bc}}{401\,P_{a|bc}} > \frac{64}{65}.\end{aligned}$$

Conditioning on $\mathcal{A}$ effectively transforms the WZ model into the alternative bunkbed hypergraph percolation model. By Hollom's result (Lemma 3.1), we have

$$\mathbb{P}_{\mathcal{X}^c}(u_1 \leftrightarrow u_{10} \mid \mathcal{A}) - \mathbb{P}_{\mathcal{X}^c}(u_1 \leftrightarrow u_{10}' \mid \mathcal{A}) = \mathbb{P}^{\text{alt}}(u_1 \leftrightarrow u_{10}) - \mathbb{P}^{\text{alt}}(u_1 \leftrightarrow u_{10}) =_{\text{Lemma 3.1}} \tfrac{12}{64} - \tfrac{13}{64} = -\tfrac{1}{64}.$$

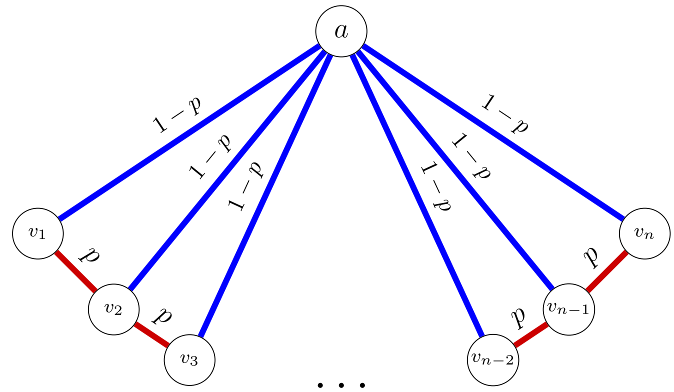


**Fig. 2.** Graph $G_n$ with $n+1$ vertices.

Now, we combine these results:

$$\mathbb{P}_{\mathcal{X}^c}(u_1 \leftrightarrow u_{10}) - \mathbb{P}_{\mathcal{X}^c}(u_1 \leftrightarrow u'_{10}) \le \mathbb{P}_{\mathcal{X}^c}(\overline{\mathcal{A}}) + \mathbb{P}_{\mathcal{X}^c}(\mathcal{A})\Big(\mathbb{P}_{\mathcal{X}^c}(u_1 \leftrightarrow u_{10} \mid \mathcal{A}) - \mathbb{P}_{\mathcal{X}^c}(u_1 \leftrightarrow u'_{10} \mid \mathcal{A})\Big)$$
$$< \tfrac{1}{65} - \tfrac{1}{64} \cdot \tfrac{64}{65} = 0.$$

This completes the proof.

## 4. Disproof of the BBC

**4.1. Hyperedge Simulation.** In this section, we construct a graph that simulates a hyperedge in the sense of WZ hypergraph percolation, adhering to the conditions of the Lemma 3.3. We prove the following technical result for the weighted percolation.

**Lemma 4.1.** *Let $n \ge 3$ and $0 < P < 1$. Consider a weighted graph $G_n$ on $(n+1)$ vertices given in Fig. 2. Denote $b := v_1$ and $c := v_n$. Then $P_{ab|c} = P_{ac|b}$ and*

$$P_{abc}\, P_{a|b|c} - P_{ab|c}\, P_{ac|b} > \left(n\, \tfrac{1-P}{1+P} - 1\right) P_{a|bc}\,, \qquad [1]$$

*where*

$$P_{abc} := \mathbb{P}_P[a \leftrightarrow b \leftrightarrow c], \quad P_{a|bc} := \mathbb{P}_P[a \not\leftrightarrow b \leftrightarrow c], \quad P_{ab|c} := \mathbb{P}_P[a \leftrightarrow b \not\leftrightarrow c],$$
$$P_{ac|b} := \mathbb{P}_P[a \leftrightarrow c \not\leftrightarrow b] \quad \text{and} \quad P_{a|b|c} := \mathbb{P}_P[a \not\leftrightarrow b \not\leftrightarrow c \not\leftrightarrow a].$$

We prove the lemma in the next section; see Proposition 5.4.

**4.2. Proof of Theorem 1.2.** In notation of Lemma 3.3, let $P = \frac{1}{2}$ and let $n := 3 \cdot 401 + 1 = 1{,}204$. The resulting graph $G_n$ is planar, has 1,205 vertices and 2,407 edges.

Take Hollom's hypergraph $H$ from Fig. 1 and substitute for each 3-hyperedge with a graph $G_n$ from Lemma 4.1, placing it so $a$ is a transversal vertex while $b = v_1$ and $c = v_n$ are the other two vertices. The resulting graph is still planar, has $10 + 6 \cdot 1{,}202 = 7{,}222$ vertices and $6 \cdot 2{,}407 = 14{,}442$ edges.

By Lemma 4.1, the $\frac{1}{2}$-percolation on $G_n$ satisfies conditions of Lemma 3.3. Thus, by Lemma 3.3, we have

$$\mathbb{P}(u_1 \leftrightarrow u_{10}) < \mathbb{P}(u_1 \leftrightarrow u'_{10}),$$

as desired. □

***Remark 4.2.*** Due to the multiple conditionings and the gadget structure, the difference of probabilities given by the counterexample in Theorem 1.2 is less than $10^{-4{,}331}$, out of reach computationally. A computer-assisted computation shows that one can use $G_n$ with $P = \frac{1}{2}$ and $n = 14$, giving a relatively small graph on 82 vertices. However, even in this case, the difference of the probabilities in the BBC is on the order $10^{-47}$. This and other computations are collected on the author's website; see Section 8.8.2.

Since Weighted BBC is equivalent to BBC (see Section 8.8.1), one can instead take weighted graph $G_n$ with $p = \frac{1}{2n}$ and $n = 402$. This graph is still too large to analyze experimentally. A computer-assisted computation shows that one can use $G_n$ with $P = 0.0349$ and $n = 5$, giving a rather small graph on 28 vertices. However, even in this case, the differences of the probabilities in the Weighted BBC are on the order $10^{-78}$.

## 5. Proof of Lemma 4.1

We prove the lemma as a consequence of elementary calculations.

**Lemma 5.1.** *We have*

$$\mathbb{P}_P(a \leftrightarrow v_n) \;=\; \frac{1-P^{2n}}{1+P}\,.$$

***Proof:*** Let $P_n := \mathbb{P}_P(a \leftrightarrow v_n)$ as in the lemma. We establish a recurrence relation for $P_n$. There are two cases:
1) The edge $(a, v_n)$ is open. This occurs with probability $1-P$. In this case, vertices $a$ and $v_n$ are directly connected.
2) The edge $(a, v_n)$ is closed. This occurs with probability $p$. In this case, vertex $v_n$ can only connect to $a$ through the edge $(v_{n-1}, v_n)$, which is open with probability $P$. If this edge is closed, vertex $v_n$ is isolated from $a$. If it is open, the probability that $a$ and $v_{n-1}$ are in the same connected component is $P_{n-1}$.

Combining these cases, we obtain the following recurrence relation:

$$P_n \;=\; (1-P) \;+\; P^2\, P_{n-1}\,,$$

with the initial condition $P_0 = 0$. The result follows by induction. □

**Lemma 5.2.** *We have*

$$\mathbb{P}_P(a \leftrightarrow v_1 \leftrightarrow v_n) \;=\; \frac{1-P^{2n}}{(1+P)^2} \;+\; \frac{n(1-P)\,P^{2n-1}}{1+P}\,.$$

***Proof:*** Let $P_n := \mathbb{P}_P(a \leftrightarrow v_1 \leftrightarrow v_n)$ denote the probability as in the lemma. We calculate this probability by analyzing whether edges $(a, v_1)$ and $(a, v_n)$ are open or closed. There are four cases:

1) Both edges $(a, v_1)$ and $(a, v_n)$ are open, each with probability $1-P$. Then $a$ is directly connected to both $v_1$ and $v_n$. Thus, the probability is $(1-P)^2$.
2) Edge $(a, v_n)$ is closed. If the edge $(a, v_n)$ is closed, vertex $v_n$ is connected to the rest of the graph through the edge $(v_{n-1}, v_n)$, which is open with probability $P$. This reduces the problem to $G_{n-1}$. Thus, the probability is $P^2\, P_{n-1}$.
3) The edge $(a, v_1)$ is closed. Similarly, if the edge $(a, v_1)$ is closed (with probability $P$). Thus, the probability is $P^2\, P_{n-1}$.
4) Both edges $(a, v_1)$ and $(a, v_n)$ are closed. If both edges $(a, v_1)$ and $(a, v_n)$ are closed (each with probability $P$), $v_1$ must connect to $v_2$ by the edge $(v_1, v_2)$, and $v_n$ must connect to $v_{n-1}$ by the edge $(v_{n-1}, v_n)$. The problem reduces to finding the probability that $a$, $\widehat{u}_1 = v_2$, and $\widehat{u}_{n-2} = v_{n-1}$ are in the same connected component in the graph $\widehat{G}_{n-2}$, the restriction of $G_n$ to the vertices $a, v_2, \ldots, v_{n-1}$. Thus, the corresponding probability is $P^4\, P_{n-2}$.

Using inclusion–exclusion of these four cases, we obtain the following recurrence relation:

$$P_n \;=\; (1-P)^2 \;+\; 2P^2\, P_{n-1} \;-\; P^4\, P_{n-2},$$

with initial conditions $P_0 = 0$ and $P_1 = 1-P$. The result follows by induction. □

**Lemma 5.3.** *We have*

$$\mathbb{P}_P(a \nleftrightarrow v_1 \leftrightarrow v_n) \;=\; P^{2n-1}.$$

***Proof:*** If the vertices $v_1$ and $v_n$ are in the same connected component that does not contain vertex $a$, they must be connected by the path $\gamma := (v_1 \to v_2 \to \ \ldots\ \to v_n)$. The probability that this path is open is $P^{n-1}$. In addition, any edge $(a, v_k)$ must be closed for all $1 \le k \le n$, as otherwise vertex $a$ is connected to the path $\gamma$. The probability that all these edges are closed is $P^n$. Thus, the probability in the lemma is $P^{2n-1}$. □

We conclude with the following result which immediately implies Lemma 4.1.

**Proposition 5.4.** *In notation of Lemma 4.1, we have* $P_{a|bc} = P^{2n-1}$ *and*

$$P_{abc}\, P_{a|b|c} \;-\; P_{ac|b}\, P_{ab|c} \;\ge\; \left(n\, \tfrac{1-P}{1+P} \;-\; 1\right) P^{2n-1}.$$

***Proof:*** The first part is given by Lemma 5.3. For the second part, using Lemmas 5.1–5.3 and $P_{abc} \le 1$, we have

$$\begin{aligned}
P_{abc}\, P_{a|b|c} \;-\; P_{ac|b}\, P_{ab|c} \;&=\; P_{abc} \;-\; (P_{abc} + P_{ab|c})(P_{abc} + P_{ac|b}) \;-\; P_{abc}\, P_{a|bc} \\
&=\; \mathbb{P}_P(a \leftrightarrow v_1 \leftrightarrow v_n) \;-\; \mathbb{P}_P(a \leftrightarrow v_1) \cdot \mathbb{P}_P(a \leftrightarrow v_n) \;-\; P_{abc}\, P_{a|bc} \\
&\ge\; \left(\tfrac{1-P^{2n}}{(1+P)^2} \;+\; \tfrac{n(1-P)\,P^{2n-1}}{1+P}\right) \;-\; \left(\tfrac{1-P^{2n}}{1+P}\right)^2 - P^{2n-1} \\
&\ge\; \tfrac{P^{2n}(1-P^{2n})}{(1+P)^2} \;+\; \tfrac{n(1-P)\,P^{2n-1}}{1+P} \;-\; P^{2n-1} \\
&\ge\; \left(\tfrac{n(1-P)}{1+P} - 1\right) P^{2n-1},
\end{aligned}$$

as desired. □

## 6. Complete BBC

In notation of the BBC 1.1, one can ask if a version of the BBC holds for uniform $T \subseteq V$. This is equivalent to $\frac{1}{2}$-percolation on the product graph $G \times K_2$. To distinguish from BBC, we call this Complete BBC; see Section 8.8.1. It turns out that the proof of Theorem 1.2 extends to the proof of Complete BBC, but a counterexample is a little larger:

**Theorem 6.1.** *There is a connected graph $G = (V, E)$ with $|V|, |E| < 10^6$, and vertices $u, v \in V$, such that for the $\frac{1}{2}$-percolation on $G \times K_2$ we have*

$$\mathbb{P}_{\frac{1}{2}}[u \leftrightarrow v] \ < \ \mathbb{P}_{\frac{1}{2}}[u \leftrightarrow v'].$$

*In particular, the complete BBC is false.*

***Proof:*** Recall that Hollom's Model 4.3 in ref. 25 is the hypergraph version of the Complete BBC. Hollom disproves it in ref. 25 (section 5.1) by showing that his 3-hypergraph in Fig. 1 is minimal in a sense that bunkbed probabilities $\mathbb{P}[u \leftrightarrow v]$ and $\mathbb{P}[u \leftrightarrow v']$ are equal for all subsets $\{u_2, u_7, u_9\} \subset T \subseteq \{u_1, \ldots, u_{10}\}$. He then makes $k = 102$ "clones" of vertices $\{u_2, u_7, u_9\}$ to make sure at least one is always in the percolation cluster with high probability.

We notice that our counterexample has a similar minimal structure because of the form of the gadget used in its construction. The only path $\rho$ from $u_1$ to $u_{10}$ avoiding transversal vertices still passes through all nontransversal vertices. From this point on, proceed as in the proof of Theorem 1.2. In notation of the proof of Lemma 3.3, we have that the only two ways we can have a nonzero probability gap is if one of the vertices $\{u_2, u_7, u_9\}$ is not in $T$ or all the vertices along the red path $\rho$ are not in $T$.

Now consider the difference of probabilities $\delta := \mathbb{P}(u_1 \leftrightarrow u_{10}) - \mathbb{P}(u_1 \leftrightarrow u_{10}')$ for the graph $G$ and $T = \{u_2, u_7, u_9\}$. Then for the graph $G$, where $T$ is a random subset containing $\{u_2, u_7, u_9\}$ one has $\mathbb{P}(u_1 \leftrightarrow u_{10}) - \mathbb{P}(u_1 \leftrightarrow u_{10}') = \delta \cdot 2^{-|G|+3}$.

For each of the vertices $t \in \{u_2, u_7, u_9\}$ replace it with the gadget—add $k$ additional vertices $w_{t,i}$ for $i \in [k]$ and connect then to $t$. This gadget imitates a single vertex $t$ having a probability of being transversal increased from $\frac{1}{2}$ to $1 - \frac{1}{2}\left(\frac{7}{8}\right)^k$. Let $\mathcal{A}$ be the event that all imitated vertices are transversal. Then $\mathbb{P}(\mathcal{A}) \geq 1 - \frac{3}{2}\left(\frac{7}{8}\right)^k$. We have

$$\mathbb{P}_{\frac{1}{2}}[u \leftrightarrow v] - \mathbb{P}_{\frac{1}{2}}[u \leftrightarrow v'] \leq 1 - \mathbb{P}(\mathcal{A}) + \mathbb{P}(\mathcal{A}) \cdot \delta \cdot 2^{-|G|+3} \leq \frac{3}{2}\left(\frac{7}{8}\right)^k + \frac{1}{2}\delta \cdot 2^{-|G|+3}.$$

This is negative if $\delta \cdot 2^{-|G|+3} < -3\left(\frac{7}{8}\right)^k$. It is obvious such $k$ exists. We use the computer estimate from Remark 4.2 that $\delta < -10^{-4,332}$ to say that this is true for $k \geq 112{,}182$. Therefore, for the graph $G'$ obtained from $G$ by adding $3k = 336{,}546$ vertices and edges, we have

$$\mathbb{P}_{\frac{1}{2}}[u \leftrightarrow v] \ < \ \mathbb{P}_{\frac{1}{2}}[u \leftrightarrow v'],$$

as desired. □

## 7. Experimental Testing

Versions of the BBC were repeatedly tested by various researchers, although failed attempts remain largely unreported, see, e.g., ref. 14 (section 3.1). In this section, we describe our own attempt to refute the conjecture using a large scale computer computation.

**7.1. Initial Tests.** We started with a series of brute force tests of the Polynomial BBC; see Section 8.8.1. We exhaustively tested all connected graphs with at most 8 vertices, and connected graph with at most 15 edges from the House of Graphs database; see ref. 32. In each case, the Polynomial BBC held true. At this point, we chose to develop a more refined approach.

**7.2. The Algorithm.** Our starting point is the machine learning algorithm by Wagner (33), which we adjusted and modified. Roughly, the algorithm inputs a neural network used in a randomized graph generating algorithm, various constraints, and a function to optimize. It outputs new weights for the neural network with the function improved. In his remarkable paper, Wagner describes how he was able to disprove five open problems in graph theory, so we had high hopes that this approach might help to disprove the BBC.

To give a quick outline of our approach, we consider a graph $G = (V, E)$ on $n = |V|$ vertices, with the set of transversal vertices $T \subset V$, and fixed $u, v \notin W$. Flip a fair coin for each edge $e \in E$. Depending on the outcome, either retain $e$ and delete $e'$, or vice versa. Check whether $u \leftrightarrow v$ and $u \leftrightarrow v'$. Repeat this $N$ times to estimate the corresponding probabilities $P$ and $P'$, respectively. Based on these probabilities, use Wagner's algorithm to obtain the next iteration. Repeat $M$ times or until a potential counterexample with $P < P'$ is found.

**7.3. Implementation and Results.** We first used Wagner's original code on a laptop computer, but when that proved too slow we made major changes. To speed up the performance and tweak the code, we implemented Wagner's algorithm in Julia.

We then ran the code on a shared UCLA Hoffman2 Cluster, which is a Linux compute cluster consisting of 800+ 64-bit nodes and over 26,000 cores, with an aggregate of over 174 TB of memory.* Each run of the algorithm required about 2 h. After six runs with different parameters, the results were too similar to continue.

*The system overview is available here: www.hoffman2.idre.ucla.edu/About/System-overview.html.

Specifically, we ran the algorithm on graphs with $n = 20$ and $n = 30$ vertices, and for 3, 4, and 5 transversal vertices. Although we started with relatively dense graphs, the algorithm converged to relatively sparse graphs with about 100 edges. We used $N = 4{,}000$, pruning the Monte Carlo sampling when the desired probabilities were far apart.

We used $M = 2{,}000$, after which the probabilities $p, p'$ rapidly converged to $\frac{1}{2}$ and became nearly indistinguishable. More precisely, the probability gap $P - P'$ became smaller than 0.01 getting close to the Monte Carlo error, i.e. the point when we would need to increase $N$ to avoid false positives. At all stages, we had $P > P'$ suggesting validity of the BBC. At the time of the experiments and prior to this work, we saw no evidence that an experimental approach could ever succeed.

**7.4. Analysis.** Having formally disproved the BBC, it is clear that our computational approach was misguided. For the uniform weights we tested, we could never have reached graphs of size anywhere close to that in Theorem 1.2, of course. Even when the number of vertices is optimized to 82 as suggested in Remark 4.2, the number of edges is still very large while the probability gap in the theorem is on the order of $10^{-47}$, thus undetectable in practice.

In hindsight, to reach a small counterexample we should have used the weighted bunkbed percolation rather than the more efficient alternative model, with some edges having a very large weight and some very small weight. Of course, by Remark 4.2, the probability gap in the theorem is still prohibitively small, at least for the graphs we consider.

**7.5. Conclusions.** It seems, the BBC has some unique features making it very poorly suited for computer testing. In fact, one reason we stopped our computer experiments is that in our initial rush to testing we failed to contemplate societal implications of working with even moderately large graphs.

Suppose we did find a potential counterexample graph with only $m = 100$ edges and the probability gap was large enough to be statistically detectable. Since analyzing all of $2^m \approx 10^{30}$ subgraphs is not feasible, our Monte Carlo simulations could only confirm the desired inequality with high probability. While this probability could be amplified by repeated testing, one could never formally disprove the BBC this way, of course.

This raises somewhat uncomfortable questions whether the mathematical community is ready to live with an uncertainty over validity of formal claims that are only known with high probability. It is also unclear whether in this imaginary world the granting agencies would be willing to support costly computational projects to further increase such probabilities (cf. refs. 34 and 35). Fortunately, our failed computational effort avoided this dystopian reality, and we were able to disprove the bunkbed conjecture by a formal argument.

## 8. Final Remarks

**8.1. Variations on the BBC.** Although the version of the BBC 1.1 given in ref. 1 is considered the most definitive, over the years several closely related versions has been studied:

1) Counting BBC, where one compares the number of subgraphs connecting vertices $u, v$ and those that do not. This conjecture is a restatement of the BBC in the case $P = \frac{1}{2}$.
2) Weighted BBC, where the edge probabilities $P_e = P_{e'}$ can depend on $e \in E$. This conjecture is equivalent to the BBC by Rudzinski and Smyth (11), since edge probabilities can be approximated by series-parallel graphs.
3) Polynomial BBC, where the edge probabilities above are viewed as variables. In this case, the conjecture claims that the difference of polynomials corresponding to $\mathbb{P}[u \leftrightarrow v]$ and $\mathbb{P}[u \leftrightarrow v']$ is a polynomial with nonnegative coefficients. This conjecture is stronger than Weighted BBC as there are polynomials positive on $[0, 1]$ which have negative coefficients such as $(x - y)^2$. Although we did not find a counterexample on a graph with at most eight vertices, it is likely that there is a sufficiently small counterexample in this case. Cf. ref. 19, where the difference is a sum of squares.
4) Computational BBC, where one asks whether the counting version of the probability gap is in #P, i.e., has a combinatorial interpretation, see ref. 13 (Conj. 5.6). Clearly, this conjecture implies BBC. We refer to ref. 36 for a formal treatment of this problem for general polynomials.
5) Alternative BBC, where fair coin flips determine whether the edge $e$ is deleted and $e'$ retains or vice versa. This conjecture implies BBC (10, Prop. 2.6).
6) Complete BBC, where one takes all $T = V$ and performs the weighted percolation on the full $\overline{G} := G \times K_2$, i.e. on all edges in $\overline{G}$ including the posts. The conjecture in this case is weaker than the BBC; see, e.g., ref. 10 (Prop. 2.2).

In all but the last case, the corresponding conjecture is refuted by Theorem 1.2. In Complete BBC, the corresponding conjecture is refuted by Theorem 6.1 by a more involved counterexample (based on a more involved counterexample by Hollom).

**8.2. Robustness Lemma.** Lemma 3.3 is a finite problem which can be reformulated as follows. By definition, probabilities on both sides of Eq. **2** are polynomials in five variables of degree at most 12, with at most $5^{12}$ nonzero coefficients. The Lemma gives positivity of the difference of these two polynomials on a region of the unit cube $[0, 1]^5$ cut out by the quadratic inequality Eq. **1**.

Since our proof of Lemma 3.3 is somewhat cumbersome and uses a case-by-case analysis, we verified the lemma computationally. The results and the code are available on GitHub.[†] Of course, the advantage of our combinatorial proof is that it is elementary and amenable for generalizations.

[†] Generating Partitions of Graph Vertices into Connected Components, description and code at https://github.com/Kroneckera/bunkbed-counterexample.

**8.3. Special Cases.** Our counterexample makes prior positive results somewhat more valuable. It would be interesting to find other families of graphs on which the BBC holds. We are especially interested in the corresponding problem for the Polynomial BBC. Note that we emphasized planarity in Theorem 1.2 since it was speculated in ref. 10 that planarity helps.

**8.4. Percolation in $\mathbb{Z}^d$.** For lattices, the connection probabilities $\mathbb{P}_p[u \leftrightarrow v]$ between vertices are known as the two-point functions. For percolation in higher dimensions, these were famously studied by Hara et al. (37), and they are also of interest for other probabilistic models.

Curiously, it is not known whether connection probabilities are monotone as the distance $|u - v|$ increases. This claim would follow from the bunkbed conjecture. This suggests that investigating the BBC for grid-like graphs is still of interest even if the conjecture is false for general planar graphs. Note that the monotonicity is known in the $P \downarrow 0$ limit.

**8.5. Random Cluster Model.** It was shown in ref. 24 (section 3) that the analogue of the BBC holds for the random cluster model with parameter $q = 2$. Our Theorem 1.2 shows that one cannot take $q = 1$. It would be interesting to find the smallest $q > 1$ such that the BBC holds for all finite graphs. We note that monotonicity in $q$ is unclear, so, e.g., it is not known if BBC holds for all $q \geq 2$.

**Data, Materials, and Software Availability.** There are no data underlying this work.

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