THE BUNKBED CONJECTURE IS FALSE

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ABSTRACT. We give an explicit counterexample to the *bunkbed conjecture* introduced by Kasteleyn in 1985. The counterexample is given by a planar graph on 7222 vertices, and is built on the recent work of Hollom (2024).

1. INTRODUCTION

The *bunkbed conjecture* (BBC) is a celebrated open problem in probability introduced by Kasteleyn in 1985, see [BK01, Remark 5]. The conjecture is both natural and intuitively obvious, but has defied repeated proof attempts; it is known only in a few special cases. In this paper we disprove the conjecture without resorting to computer experiments (cf. Section 7).

Let $G = (V, E)$ be a connected graph, possibly infinite and with multiple edges. In *Bernoulli bond percolation*, each edge is *deleted* independently at random with probability $1 - p$, and otherwise *retained* with probability $p \in [0, 1]$. Equivalently, this model gives a random subgraph of G weighted by the number of edges. For $p = \frac{1}{2}$ we obtain a uniform random subgraph of G. See [BR06, Gri99] for standard results and [Dum18, Wer09] for recent overview of percolation.

Let $\mathbb{P}_p[u \leftrightarrow v]$ denote the probability that vertices $u, v \in V$ are connected. It is often of interest to compare these probabilities, as computing them exactly is #P-hard [PB83]. For example, the classical *Harris–Kleitman inequality*, a special case of the *FKG inequality*, implies that $\mathbb{P}_p[u \leftrightarrow v] \leq \mathbb{P}_p[u \leftrightarrow v \,|\, u \leftrightarrow w]$ for all $u, v, w \in V$, see e.g. [AS16, Ch. 6]. Harris used this to prove that the *critical probability* $p_c(G) := \inf\{p : \mathbb{P}_p(G) > 0\}$ satisfies $p_c(\mathbb{Z}^2) \geq \frac{1}{2}$ [Har60], in the first step towards Kesten's remarkable exact value $p_c(\mathbb{Z}^2) = \frac{1}{2}$ [Kes80]. Considerations of percolation monotonicity on $\mathbb{Z}^2$ (see §8.4), led Kasteleyn to the following problem.

Fix a finite connected graph $G = (V, E)$ and a subset $T \subseteq V$. A *bunkbed graph* $\overline{G} = (\overline{V}, \overline{E})$ is a subgraph of the graph product $G \times K_2$ defined as follows. Take two copies of G, which we denote G and $G' = (V', E')$, and add all edges of the form (w, w'), where $w \in T$ and w' is a corresponding vertex in T'; we denote this set of edges by $\overline{T}$. The resulting bunkbed graph has $\overline{V} = V \cup V'$ and $\overline{E} = E \cup E' \cup \overline{T}$.

In the *bunkbed percolation*, the usual bond percolation is performed only on edges in G and G', while all edges in $\overline{T}$ are retained (i.e., not deleted). We use $\mathbb{P}_p^{\mathrm{bb}}[u \leftrightarrow v]$ to denote connecting probability in this case. The vertices in T are called *transversal* and the edges in $\overline{T}$ are called *posts*, to indicate their special status. See e.g. [Lin11, RS16], for these and several other equivalent models of the bunkbed percolation. We refer also to [Gri23, §4.1], [Pak22, §5.5] and [Rud21] for recent overviews and connections to other areas.

Conjecture 1.1 (*bunkbed conjecture*). *Let $G = (V, E)$ be a connected graph, let $T \subseteq V$, and let $0 < p < 1$. Then, for all $u, v \in V$, we have:*

$$\mathbb{P}_p^{\mathrm{bb}}[u \leftrightarrow v] \;\geq\; \mathbb{P}_p^{\mathrm{bb}}[u \leftrightarrow v'].$$

The bunkbed conjecture is known in a number of special cases, including wheels [Lea09], complete graphs [dB16, dB18, HL19], complete bipartite graphs [Ric22], and graphs symmetric w.r.t. the $u \leftrightarrow v$ automorphism [Ric22]. It is also known for one [Lin11, Lemma 2.4] and for two transversal vertices [Lohr18, §6.3], see also [BK01, GZ24+]. Finally, the conjecture was recently proved in the $p \uparrow 1$ limit [HNK23, Hol24a].

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Theorem 1.2. *There is a connected planar graph $G = (V, E)$ with $|V| = 7222$ vertices and $|E| = 14442$ edges, a subset $T \subset V$ with three transversal vertices, and vertices $u, v \in V$, s.t.*

$$\mathbb{P}^{\mathrm{bb}}_{\frac{1}{2}}[u \leftrightarrow v] \, < \, \mathbb{P}^{\mathrm{bb}}_{\frac{1}{2}}[u \leftrightarrow v'] .$$

In particular, the bunkbed conjecture is false.

The result is surprising since analogous inequalities for simple random walks and for the Ising model on bunkbed graphs were proved by Häggström [Häg98, Häg03], cf. §8.5. Recall that three is the smallest number of transversal vertices we can have to disprove the conjecture. On the other hand, the total number of vertices is unlikely to be optimal, see Remark 4.2 and Section 7.

The proof of the theorem is based on an example of Hollom [Hol24b] refuting the 3-uniform hypergraph version of the BBC. Unfortunately, Hollom's example alone cannot disprove the conjecture since it is impossible to find a gadget graph simulating a single 3-hyperedge using bond percolation [GZ24, Thm 1.5].

We give a robust version of Hollom's construction using the approach in [Gla24, GZ24]. The proof of Theorem 1.2 occupies most of the paper. It is self-contained modulo Hollom's result which is small enough to be checked by hand. In Section 6, we extend the theorem to the case when the set of transversal vertices is not fixed but chosen uniformly at random from V, see Theorem 6.1. We conclude with discussion of our computer experiments in Section 7, and final remarks in Section 8.

2. Notation

In percolation, deleted edges are called *closed* while retained edges are called *open*. Note that there are several different models of percolation and variations on the bunkbed conjecture (BBC), see §8.1.

A hypergraph is a collection of subsets of vertices; to simplify the notation we use the same letter to denote both. The hypergraph is called *uniform* if all hyperedges have the same size. A *path* in a hypergraph is a sequence $(v_0 \to v_1 \to \ldots \to v_\ell)$ of vertices, such that v_{i-1}, v_i lie in the same hyperedge, for all $1 \leq i \leq \ell$. We say that two vertices in a hypergraph are *connected* if there is there is a path between them. For further definitions and results on hypergraphs, see e.g. [Ber89, §1.2].

The notion of *hypergraph percolation* is is a natural extension of graph percolation, and goes back to the study of random hypergraphs, see e.g. [SPS85]. In recent years, the study of hypergraph percolation also emerged in probabilistic and statistical physics literature, see e.g. [WZ11] and [BD24], respectively.

3. Hypergraph percolation

3.1. **Hollom's example.** Let H be a finite connected hypergraph on the set V of vertices. We use $\mathbb{P}_p[u \leftrightarrow v]$ to denote probability of connectivity of vertices $u, v \in V$ in the *hypergraph percolation*, where each hyperedge e in H is retained with probability p, or deleted with probability $1 - p$.

Let $T \subseteq V$ be the set of transversal vertices. Denote by $\overline{H}$ be the *bunkbed hypergraph* with levels $H \simeq H'$, and *vertical posts* which are the (usual) edges. Note that $\overline{H}$ has *horizontal* hyperedges and vertical posts.

In [Hol24b], Hollom considers the following natural hypergraph generalization of the *Alternative BBC*, see §8.1. In the *alternative bunkbed hypergraph percolation*, each hyperedge e in H is either deleted while the corresponding hyperedge e' in H' is retained with probability $\frac{1}{2}$, or vice versa: hyperedge e is retained and e' is deleted.

Lemma 3.1 (Hollom [Hol24b, Claim 5.1])**.** *Let H be the hypergraph with six 3-edges as in the Figure 3.1, and let $T = \{u_2, u_7, u_9\}$ is the set of transversal vertices. In the alternative bunkbed hypergraph percolation, we have:*

$$\mathbb{P}^{\mathrm{alt}}[u_1 \leftrightarrow u'_{10}] \, = \, \frac{13}{64} \quad \textit{and} \quad \mathbb{P}^{\mathrm{alt}}[u_1 \leftrightarrow u_{10}] \, = \, \frac{12}{64} \, .$$

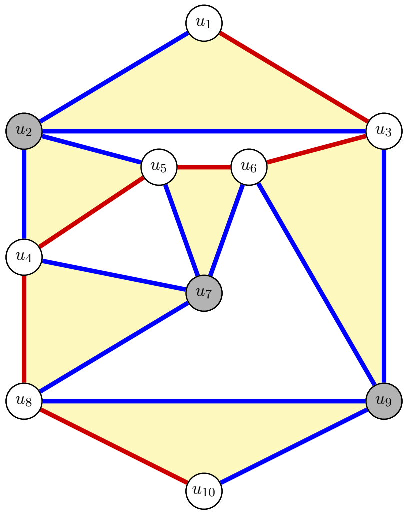

FIGURE 3.1. Hollom's 3-uniform hypergraph H.

We give a robust version of Hollom's construction.

3.2. **Robust hyperedge lemma.** Note that in Hollom's example, each hyperedge has exactly one transversal vertex. We explore this symmetry.

Consider the following *WZ hypergraph percolation model* introduced by Wierman and Ziff in [WZ11] (see also [GZ24]). Let $e = \{a, b, c\}$ be a hyperedge where a is a transversal vertex. In the model, hyperedge e is set to have

- probability p_{abc} to connect all three vertices,
- probability $p_{a|b|c}$ to not connect any of the vertices,
- probability $p_{a|bc}$ to connect two non-transversal vertices, and
- probability $p_{ab|c} = p_{ac|b}$ to connect a transversal to a nontransversal vertex,

and these events are independent on all hyperedges.

Finally, we assume that these five probabilities sum up to 1:

$$p_{abc} + p_{a|b|c} + p_{a|bc} + p_{ab|c} + p_{ac|b} = 1.$$

We say that vertices u and v are *connected*, write $u \leftrightarrow v$, if they are connected in the hypergraph in a way that every two vertices on a hyperedge are connected by the rules above. We use $\mathbb{P}^{\mathrm{wz}}[u \leftrightarrow v]$ to denote connection probabilities in this model, and drop the superscript when the model is clear.

Lemma 3.2. *Let H be Hollom's hypergraph in the Figure 3.1, and let $T = \{u_2, u_7, u_9\}$ be the set of transversal vertices. Consider the WZ hypergraph percolation as described above, where the connection probabilities satisfy*

$$400\, p_{a|bc} \le p_{abc}\, p_{a|b|c} - p_{ab|c}^2. \tag{3.1}$$

Then we have:

$$\mathbb{P}^{\mathrm{wz}}(u_1 \leftrightarrow u_{10}) < \mathbb{P}^{\mathrm{wz}}(u_1 \leftrightarrow u_{10}'). \tag{3.2}$$

It was noted in [Gla24, Cor. 3.6], that the RHS in (3.1) is nonnegative if the hyperedge is simulated by a gadget in Bernoulli edge percolation:

$$p_{ab|c}\, p_{ac|b} = p_{ab|c}^2 \le p_{abc}\, p_{a|b|c}\,. \tag{3.3}$$

This is a consequence of the Harris–Kleitman (HK) inequality. In fact, a slightly stronger inequality always holds (ibid.) Since the LHS in (3.1) is nonnegative, one can view this assumption as strengthening the HK inequality in this case (cf. §8.2).

3.3. **Proof of Lemma 3.2.** Note that each configuration Q of the WZ hypergraph percolation can be viewed as a function $\psi : H \cup H' \to \Upsilon$ from hyperedges in $\overline{H}$ to the set $\Upsilon := \{abc, ab|c, ac|b, a|bc, a|b|c\}$. Here ψ is the probability $\mathbb{P}[Q]$ of the configuration. Clearly, for each hyperedge e in H, there are 25 possibilities for $\psi(e)$ and $\psi(e')$.

To prove the lemma, we employ a combinatorial argument on all configurations that is similar to the involution in the proofs of [Lin11, Lemma 2.3] and [Hol24b, Lemma 2.5]. Denote by $\mathcal{C}$ and $\mathcal{C}'$ the sets of configurations which contain paths $u_1 \leftrightarrow u_{10}$ to $u_1 \leftrightarrow u'_{10}$, respectively. The probability of a configuration Q and of a collection $\mathcal{C}$, are given by

$$\mathbb{P}(Q) := \prod_{e \in H \cup H'} p_{\psi(e)} \qquad \text{and} \qquad \mathbb{P}(\mathcal{C}) := \sum_{Q \in \mathcal{C}} \mathbb{P}(Q).$$

We prove that $\mathbb{P}(\mathcal{C}) < \mathbb{P}(\mathcal{C}')$ by a direct combinatorial argument.

Note the *red path* ρ from u_1 to u_{10} in Figure 3.1, and observe that it goes through every hyperedge exactly once and avoids transversal vertices. Fix the order on the hyperedges of H according to their appearance on the path ρ :

$$(\triangleleft) \qquad (u_1, u_2, u_3),\ (u_3, u_6, u_9),\ (u_5, u_6, u_7),\ (u_2, u_4, u_5),\ (u_4, u_7, u_8),\ (u_8, u_9, u_{10}).$$

Construction: Let $C \in \mathcal{C}$ be a configuration which contains a path $u_1 \to u_{10}$. The construction of the map is split into several cases.

Case 1. Choose $e = (abc) \in H$ to be the first hyperedge w.r.t. the order $(\triangleleft)$, such that

$$(\psi(e), \psi(e')) \in \Lambda^2 \quad \text{and} \quad (\psi(e), \psi(e')) \notin \Xi,$$

where

$$\Lambda := \{abc, ab|c, ac|b, a|b|c\}, \quad \text{and}$$
$$\Xi := \big\{ (abc, a|b|c),\ (a|b|c, abc),\ (ab|c, ac|b),\ (ac|b, ab|c) \big\}.$$

As before, here $a \in T$ denotes a transversal vertex in e. To break the symmetry, we assume that b precedes c along the path ρ.

Note that $\Lambda^2 = \Omega_0 \cup \Omega_1 \cup \Xi$, where

$$\Omega_0 := \left\{ \begin{array}{l} (abc, ac|b),\ (ac|b, abc),\ (a|b|c, ab|c),\ (ab|c, a|b|c), \\ (abc, abc),\ (a|b|c, a|b|c),\ (ab|c, ab|c),\ (ac|b, ac|b) \end{array} \right\},$$

and

$$\Omega_1 := \big\{ (abc, ab|c),\ (ab|c, abc),\ (a|b|c, ac|b),\ (ac|b, a|b|c) \big\}.$$

Consider the following two cases:

(1) Suppose $(\psi(e), \psi(e')) \in \Omega_0$. Exchange the values $\psi(h) \leftrightarrow \psi(h')$ for all hyperedges *after* e or e' along the path ρ.

(2) Suppose $(\psi(e), \psi(e')) \in \Omega_1$. Exchange the values $\psi(e) \leftrightarrow \psi(e')$, and $\psi(h) \leftrightarrow \psi(h')$ for all hyperedges *after* e or e' along the path ρ.

In both cases, denote by C' the resulting configuration, and observe that $\psi(C) = \psi(C')$ since we only rearrange the weights. It remains to show the following:

Claim: $C' \in \mathcal{C}'$.

Proof. Note that $\psi(e) = abc$ implies that vertices b and c are connected in C within e (in the WZ percolation model). It is easy to see that in C', we have a path $(b \to a \to a' \to c')$, i.e. vertices b and c' are connected in C'. The same argument shows that $\psi(e') = abc$ implies vertices b' and c are connected in C'.

A similar argument shows that if vertices a and b are connected in C within e, then they are still connected in C'. If vertices a' and b' are connected in C within e', then a' and b' are connected in C'. Similarly, if vertices a and c are connected in C within e, then a and c' are connected in C'. Finally, if vertices a' and c' are connected in C within e', then a and c are connected in C'.

We have $V = \{u_1, \ldots, u_{10}\}$ and $T = \{u_2, u_7, u_9\}$. Let $L \subseteq (V \smallsetminus T) \cup (V' \smallsetminus T')$ denote the set of nontransversal vertices which lie along the path ρ between u_1 and b, inclusively, along with their copies from another level. Similarly, let $R := (V \smallsetminus T) \cup (V' \smallsetminus T') \smallsetminus L$.

Observe that for every $\gamma : u_1 \to u_{10}$ in C, we now have a path $\gamma' \in C$ constructed as follows. For all $u_i \in R$ on a path γ use vertex u_i' in γ', and vice versa: for all $u_i' \in R$ on a path γ, use vertex u_i'.

Recall that the values of ψ are switched on all hyperedges containing vertices in R except possibly for e, and are *not* switched on all hyperedges containing vertices in L except possibly for e. The result follows from the connectivity observations above. $\square$

<u>*Case 2*</u>. In notation of Case 1, suppose that $(\psi(e), \psi(e')) \in \Xi$. Note that we are unable to make a switch $abc \leftrightarrow a|b|c$ as this would disconnect the path. We make a *probabilistic switch* defined as follows. Denote

$$\varkappa := \frac{p_{ab|c}\, p_{ac|b}}{p_{abc}\, p_{a|b|c}},$$

and note that by (3.1) we have $1 > \varkappa \geq 0$. We make the following switches:

$$(ab|c, ac|b) \to (abc, a|b|c) \quad \text{and} \quad (ac|b, ab|c) \to (a|b|c, abc).$$

Proceeding as before, we conclude that the resulting configuration $C' \in \mathcal{C}'$. However, that note we have $\psi(C') - \psi(C) > 0$, i.e. the map $C \to C'$ is not weight preserving. To correct this, for the reverse map we make switches

$$(abc, a|b|c) \to (ab|c, ac|b) \quad \text{and} \quad (a|b|c, abc) \to (ac|b, ab|c)$$

with probability $\varkappa$. Proceed as in Case 1, and note that this probabilistic map is weight preserving by construction.

<u>*Case 3*</u>. In notation of Case 1, we have the following possible pairs of values of ψ remain to be considered:

$$\begin{aligned} &(abc, a|b|c) \quad \text{and} \quad (a|b|c, abc) \quad \text{with probability } (1-\varkappa)\, p_{abc}\, p_{a|b|c}\,, \\ &(a|bc, *), \quad (*, a|bc) \quad \text{and} \quad (a|bc, a|bc) \quad \text{where } * \in \Lambda. \end{aligned}$$

The remaining part of the proof is a bookkeeping of this case. The idea is that the events in the first line emulate the alternative bunkbed hypergraph percolation on $\overline{H}$, while the events in the second line have probability too small to make the difference by the assumption (3.1).

When conditioned to this case, the WZ hypergraph percolation model has the following probabilities for each pairs of values $(\psi(e), \psi(e'))$:

$$\begin{cases} (abc, a|b|c), & \text{with probability } \frac{1}{Z}\left(p_{abc}p_{a|b|c} - p_{ab|c}p_{ac|b}\right), \\ (a|b|c, abc), & \text{with probability } \frac{1}{Z}\left(p_{abc}p_{a|b|c} - p_{ab|c}p_{ac|b}\right), \\ (a|bc, *), & \text{with probability } \frac{1}{Z}\, p_{a|bc}\, p_* \text{ for } * \in \Lambda, \\ (*, a|bc), & \text{with probability } \frac{1}{Z}\, p_{a|bc}\, p_* \text{ for } * \in \Lambda, \\ (a|bc, a|bc), & \text{with probability } \frac{1}{Z}\, p_{a|bc}^2\,, \end{cases}$$

where

$$Z := 2p_{abc}\, p_{a|b|c} - 2p_{ab|c}\, p_{ac|b} + 2p_{a|bc} - p_{a|bc}^2$$

is the normalizing constant. When referring to this probabilities, we will use $\mathbb{P}^{(3)}$.

Denote by $\mathcal{A}$ the event that for a configuration C we have $\psi(e) \neq \psi(e')$ and $\psi(e), \psi(e') \in \{abc, a|b|c\}$ for all $e \in H$. Using the inequality $(1-x)^a \geq 1 - ax$ and the assumption (3.1) in the lemma, we have:

$$\begin{aligned} \mathbb{P}^{(3)}(\mathcal{A}) &= \left(1 - \frac{2p_{a|bc} - p_{a|bc}^2}{2(p_{abc}\, p_{a|b|c} - p_{ab|c}\, p_{ac|b}) + 2p_{a|bc} - p_{a|bc}^2}\right)^6 \geq \left(1 - \frac{p_{a|bc}}{(p_{abc}\, p_{a|b|c} - p_{ab|c}\, p_{ac|b}) + p_{a|bc}}\right)^6 \\ &\geq 1 - \frac{6p_{a|bc}}{(p_{abc}p_{a|b|c} - p_{ab|c}\, p_{ac|b}) + p_{a|bc}} >_{(3.1)} 1 - \frac{6p_{a|bc}}{401 p_{a|bc}} > \frac{64}{65}\,. \end{aligned}$$

As we mentioned above, if we condition on $\mathcal{A}$, the WZ model turns into alternative bunkbed hypergraph percolation model, so by Hollom's result we have:

$$\mathbb{P}^{(3)}(u_1 \leftrightarrow u_{10} \mid \mathcal{A}) - \mathbb{P}^{(3)}(u_1 \leftrightarrow u'_{10} \mid \mathcal{A}) = \mathbb{P}^{\text{alt}}(u_1 \leftrightarrow u_{10}) - \mathbb{P}^{\text{alt}}(u_1 \leftrightarrow u_{10})$$
$$=_{\text{Lemma 3.1}} \frac{12}{64} - \frac{13}{64} = -\frac{1}{64}.$$

This implies:

$$\mathbb{P}^{(3)}(u_1 \leftrightarrow u_{10}) - \mathbb{P}^{(3)}(u_1 \leftrightarrow u'_{10}) \leq \mathbb{P}^{(3)}(\overline{\mathcal{A}}) + \mathbb{P}^{(3)}(\mathcal{A})\Big(\mathbb{P}^{(3)}(u_1 \leftrightarrow u_{10} \mid \mathcal{A}) - \mathbb{P}^{(3)}(u_1 \leftrightarrow u'_{10} \mid \mathcal{A})\Big)$$
$$< \frac{1}{65} - \frac{1}{64} \cdot \frac{64}{65} = 0.$$

This finishes the analysis of Case 3.

Adding the probabilities. From above, for the cases 1 and 2, respectively, we have:

$$\mathbb{P}^{(1)}(u_1 \leftrightarrow u_{10}) - \mathbb{P}^{(1)}(u_1 \leftrightarrow u'_{10}) = \mathbb{P}^{(2)}(u_1 \leftrightarrow u_{10}) - \mathbb{P}^{(2)}(u_1 \leftrightarrow u'_{10}) = 0.$$

Therefore, we have $\mathbb{P}^{\text{wz}}(u_1 \leftrightarrow u_{10}) - \mathbb{P}^{\text{wz}}(u_1 \leftrightarrow u'_{10}) < 0$, as desired. □

4. Disproof of the Bunkbed Conjecture

4.1. **Hyperedge simulation.** In this section, we construct a graph that simulates a hyperedge in the sense of WZ hypergraph percolation, adhering to the conditions of the Lemma 3.2. We prove the following technical result for the *weighted percolation*.

Lemma 4.1. *Let $n \geq 3$ and $0 < p < 1$. Consider a weighted graph G_n on $(n+1)$ vertices given in Figure 4.1. Denote $b := v_1$ and $c := v_n$. Then $p_{ab|c} = p_{ac|b}$ and*

$$p_{abc}\, p_{a|b|c} - p_{ab|c}\, p_{ac|b} > \big(n \tfrac{1-p}{1+p} - 1\big)\, p_{a|bc}\,, \tag{4.1}$$

where

$$p_{abc} := \mathbb{P}_p[a \leftrightarrow b \leftrightarrow c], \quad p_{a|bc} := \mathbb{P}_p[a \not\leftrightarrow b \leftrightarrow c], \quad p_{ab|c} := \mathbb{P}_p[a \leftrightarrow b \not\leftrightarrow c],$$
$$p_{ac|b} := \mathbb{P}_p[a \leftrightarrow c \not\leftrightarrow b] \quad \text{and} \quad p_{a|b|c} := \mathbb{P}_p[a \not\leftrightarrow b \not\leftrightarrow c \not\leftrightarrow a].$$

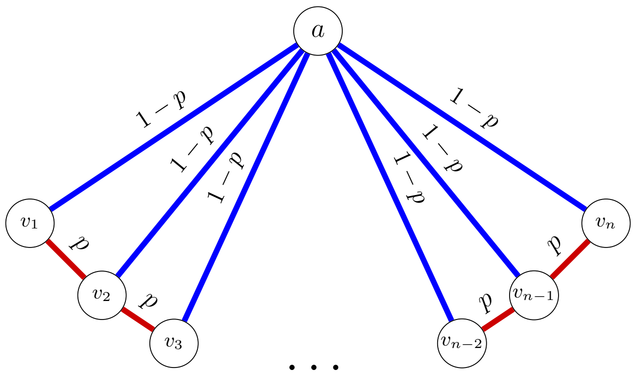

Figure 4.1. Graph G_n with $n+1$ vertices.

We prove the lemma in the next section, see Proposition 5.4.

4.2. **Proof of Theorem 1.2.** In notation of Lemma 3.2, let $p = \frac{1}{2}$ and let $n := 3 \cdot 401 + 1 = 1204$. The resulting graph G_n is planar, has 1205 vertices and 2407 edges.

Take Hollom's hypergraph H from Figure 3.1 and substitute for each 3-hyperedge with a graph G_n from Lemma 4.1, placing it so a is a transversal vertex while $b = v_1$ and $c = v_n$ are the other two vertices. The resulting graph is still planar, has $10 + 6 \cdot 1202 = 7222$ vertices and $6 \cdot 2407 = 14442$ edges.

By Lemma 4.1, the $\frac{1}{2}$-percolation on G_n satisfies conditions of Lemma 3.2. Thus, by Lemma 3.2, we have:

$$\mathbb{P}(u_1 \leftrightarrow u_{10}) < \mathbb{P}(u_1 \leftrightarrow u'_{10}),$$

as desired. □

Remark 4.2. Due to the multiple conditionings and the gadget structure, the difference of probabilities given by the counterexample in Theorem 1.2 is less than 10^{-4331}, out of reach computationally. A computer-assisted computation shows that one can use G_n with $p = \frac{1}{2}$ and $n = 14$, giving a relatively small graph on 82 vertices. However, even in this case, the difference of the probabilities in the BBC is on the order 10^{-47}. This and other computations are collected on author's website, see §8.2.

Since *Weighted BBC* is equivalent to BBC (see §8.1), once can instead take weighted graph G_n with $p = \frac{1}{2n}$ and $n = 402$. This graph is still too large to analyze experimentally. A computer-assisted computation shows that one can use G_n with $p = 0.0349$ and $n = 5$, giving a rather small graph on 28 vertices. However, even in this case, the difference of the probabilities in the Weighted BBC are on the order 10^{-78}.

5. Proof of Lemma 4.1

We prove the lemma as a consequence of elementary calculations.

Lemma 5.1. *We have:*

$$\mathbb{P}_p(a \leftrightarrow v_n) \; = \; \frac{1-p^{2n}}{1+p} \, .$$

Proof. Let $p_n := \mathbb{P}_p(a \leftrightarrow v_n)$ as in the lemma. We establish a recurrence relation for p_n. There are two cases:

(1) The edge (a, v_n) is open. This occurs with probability $1 - p$. In this case, vertices a and v_n are directly connected.

(2) The edge (a, v_n) is closed. This occurs with probability p. In this case, vertex v_n can only connect to a through the edge (v_{n-1}, v_n), which is open with probability p. If this edge is closed, vertex v_n is isolated from a. If it is open, the probability that a and v_{n-1} are in the same connected component is p_{n-1}.

Combining these cases, we obtain the following recurrence relation:

$$p_n \; = \; (1-p) \, + \, p^2 p_{n-1} \, ,$$

with the initial condition $p_0 = 0$. The result follows by induction. □

Lemma 5.2. *We have:*

$$\mathbb{P}_p(a \leftrightarrow v_1 \leftrightarrow v_n) \; = \; \frac{1-p^{2n}}{(1+p)^2} \, + \, \frac{n(1-p)p^{2n-1}}{1+p} \, .$$

Proof. Let $p_n := \mathbb{P}_p(a \leftrightarrow v_1 \leftrightarrow v_n)$ denote the probability as in the lemma. We calculate this probability by analyzing whether edges (a, v_1) and (a, v_n) are open or closed. There are four cases:

(1) Both edges (a, v_1) and (a, v_n) are open, each with probability $1 - p$. Then a is directly connected to both v_1 and v_n. Thus, the probability is $(1 - p)^2$.

(2) Edge (a, v_n) is closed. If the edge (a, v_n) is closed, vertex v_n is connected to the rest of the graph through the edge (v_{n-1}, v_n), which is open with probability p. This reduces the problem to G_{n-1}. Thus, the probability is $p^2 p_{n-1}$.

(3) The edge (a, v_1) is closed. Similarly, if the edge (a, v_1) is closed (with probability p). Thus, the probability is $p^2 p_{n-1}$.

(4) Both edges (a, v_1) and (a, v_n) are closed. If both edges (a, v_1) and (a, v_n) are closed (each with probability p), v_1 must connect to v_2 by the edge (v_1, v_2), and v_n must connect to v_{n-1} by the edge (v_{n-1}, v_n). The problem reduces to finding the probability that a, $\widehat{u}_1 = v_2$, and $\widehat{u}_{n-2} = v_{n-1}$ are in the same connected component in the graph $\widehat{G}_{n-2}$, the restriction of G_n to the vertices $a, v_2, \ldots, v_{n-1}$. Thus, the corresponding probability is $p^4 p_{n-2}$.

Using inclusion-exclusion of these four cases, we obtain the following recurrence relation:

$$p_n \; = \; (1-p)^2 \, + \, 2p^2 p_{n-1} \, - \, p^4 p_{n-2},$$

with initial conditions $p_0 = 0$ and $p_1 = 1 - p$. The result follows by induction. □

Lemma 5.3. *We have:*

$$\mathbb{P}_p(a \nleftrightarrow v_1 \leftrightarrow v_n) \ = \ p^{2n-1}.$$

Proof. If the vertices v_1 and v_n are in the same connected component that does not contain vertex a, they must be connected by the path $\gamma := (v_1 \to v_2 \to \ldots \to v_n)$. The probability that this path is open is p^{n-1}. In addition, any edge (a, v_k) must be closed for all $1 \le k \le n$, as otherwise vertex a is connected to the path γ. The probability that all these edges are closed is p^n. Thus, the probability in the lemma is p^{2n-1}. □

We conclude with the following result which immediately implies Lemma 4.1.

Proposition 5.4. *In notation of Lemma 4.1, we have* $p_{a|bc} = p^{2n-1}$ *and*

$$p_{abc}\, p_{a|b|c} \ - \ p_{ac|b}\, p_{ab|c} \ \ge \ \left(n\, \tfrac{1-p}{1+p} \ - \ 1\right) p^{2n-1}.$$

Proof. The first part is given by Lemma 5.3. For the second part, using Lemmas 5.1, 5.2 and 5.3 and $p_{abc} \le 1$, we have:

$$\begin{aligned}
p_{abc}\, p_{a|b|c} \ - \ p_{ac|b}\, p_{ab|c} \ &= \ p_{abc} \ - \ (p_{abc} + p_{ab|c})(p_{abc} + p_{ac|b}) \ - \ p_{abc}\, p_{a|bc} \\
&= \ \mathbb{P}_p(a \leftrightarrow v_1 \leftrightarrow v_n) \ - \ \mathbb{P}_p(a \leftrightarrow v_1) \cdot \mathbb{P}_p(a \leftrightarrow v_n) \ - \ p_{abc}\, p_{a|bc} \\
&\ge \ \left(\tfrac{1-p^{2n}}{(1+p)^2} \ + \ \tfrac{n\,(1-p)\,p^{2n-1}}{1+p}\right) \ - \ \left(\tfrac{1-p^{2n}}{1+p}\right)^2 - p^{2n-1} \\
&\ge \ \tfrac{p^{2n}(1-p^{2n})}{(1+p)^2} \ + \ \tfrac{n\,(1-p)\,p^{2n-1}}{1+p} \ - \ p^{2n-1} \\
&\ge \ \left(\tfrac{n\,(1-p)}{1+p} - 1\right) p^{2n-1},
\end{aligned}$$

as desired. □

6. Complete BBC

In notation of the Bunkbed Conjecture 1.1, one can ask if a version of the BBC holds for uniform $T \subseteq V$. This is equivalent to $\frac{1}{2}$-percolation on the product graph $G \times K_2$. To distinguish from BBC, we call this *Complete BBC*, see §8.1. Turns out the proof of Theorem 1.2 extends to the proof of Complete BBC, but a counterexample is a little larger:

Theorem 6.1. *There is a connected graph* $G = (V, E)$ *with* $|V|, |E| < 10^6$, *and vertices* $u, v \in V$, *s.t. for the* $\frac{1}{2}$*-percolation on* $G \times K_2$ *we have:*

$$\mathbb{P}_{\frac{1}{2}}[u \leftrightarrow v] \ < \ \mathbb{P}_{\frac{1}{2}}[u \leftrightarrow v'].$$

In particular, the complete bunkbed conjecture is false.

Proof. Recall that Hollom's Model 4.3 in [Hol24b] is the hypergraph version of the Complete BBC. Hollom disproves it in [Hol24b, §5.1] by showing that his 3-hypergraph in Figure 3.1 is minimal in a sense that bunkbed probabilities $\mathbb{P}[u \leftrightarrow v]$ and $\mathbb{P}[u \leftrightarrow v']$ are equal for all subsets $\{u_2, u_7, u_9\} \subset T \subseteq \{u_1, \ldots, u_{10}\}$. He then makes $k = 102$ "clones" of vertices $\{u_2, u_7, u_9\}$ to make sure at least one is always in the percolation cluster with high probability.

We follow the same approach. Note that our counterexample has a similar minimal structure because of the form of the gadget used in its construction. The only path ρ from u_1 to u_{10} avoiding transversal vertices still passes through all nontransversal vertices. From this point on, proceed as in the proof of Theorem 1.2.

In notation of the proof of Lemma 3.2, we have that the only two ways we can have a nonzero probability gap is if at least one of the vertices $\{u_2, u_7, u_9\}$ is not in T, or all vertices along the red path ρ are not in T. Now consider the difference of probabilities $\delta := \mathbb{P}(u_1 \leftrightarrow u_{10}) - \mathbb{P}(u_1 \leftrightarrow u_{10}')$ for the graph G and $T = \{u_2, u_7, u_9\}$. Then for the graph $G = (V, E)$ as above, for a random subset T containing $\{u_2, u_7, u_9\}$, one has $\mathbb{P}(u_1 \leftrightarrow u_{10}) \ - \ \mathbb{P}(u_1 \leftrightarrow u_{10}') \ = \ \delta \cdot 2^{-|E|+3}$.

For each vertex $t \in \{u_2, u_7, u_9\}$, rather than "clone" it, replace it with k additional vertices connected to t. This gadget imitates a single vertex t having a probability of being transversal increased from $\frac{1}{2}$ to $1 - \frac{1}{2}\left(\frac{7}{8}\right)^k$. Let $\mathcal{A}$ be the event that all imitated vertices are transversal. Then $\mathbb{P}(\mathcal{A}) \geq 1 - \frac{3}{2}\left(\frac{7}{8}\right)^k$. We have:

$$\mathbb{P}_{\frac{1}{2}}[u \leftrightarrow v] - \mathbb{P}_{\frac{1}{2}}[u \leftrightarrow v'] \leq 1 - \mathbb{P}(\mathcal{A}) + \mathbb{P}(\mathcal{A}) \cdot \delta \cdot 2^{-|E|+3} \leq \tfrac{3}{2}\left(\tfrac{7}{8}\right)^k + \tfrac{1}{2}\delta 2^{-|E|+3}.$$

This is negative for $\delta 2^{-|E|+3} < -3\left(\frac{7}{8}\right)^k$. It is clear that this holds for k large enough.

We use the computer estimate from Remark 4.2 that $\delta < -10^{-4332}$ to say that this is true for $k \geq 112182$. Therefore, for the graph G' obtained from G by adding $3k = 336546$ vertices and edges, we have

$$\mathbb{P}_{\frac{1}{2}}[u \leftrightarrow v] < \mathbb{P}_{\frac{1}{2}}[u \leftrightarrow v'],$$

as desired. □

7. Experimental testing

Versions of the bunkbed conjecture were repeatedly tested by various researchers, although failed attempts remain largely unreported, see e.g. [Rud21, §3.1]. In this section we describe our own attempt to refute the conjecture using a large scale computer computation.

7.1. **Initial tests.** We started with a series of brute force testing of the Polynomial BBC, see §8.1. We exhaustively tested all connected graphs with at most 8 vertices, and connected graph with at most 15 edges from the `House of Graphs` database, see [CDG23]. In each case, the Polynomial BBC held true. At this point we chose to develop a more refined approach.

7.2. **The algorithm.** Our starting point is the machine learning algorithm by Wagner [Wag21], which we adjusted and modified. Roughly, the algorithm inputs a neural network used in a randomized graph generating algorithm, various constraints and a function to optimize. It outputs new weights for the neural network with the function improved. In his remarkable paper, Wagner describes how he was able to disprove five open problems in graph theory, so we had high hopes that this approach might help to disprove the bunkbed conjecture.

To give a quick outline of our approach, we consider a graph $G = (V, E)$ on $n = |V|$ vertices, with the set of transversal vertices $T \subset V$, and fixed $u, v \notin W$. Flip a fair coin for each edge $e \in E$. Depending on the outcome, either retain e and delete e', or vice versa. Check whether $u \leftrightarrow v$ and $u \leftrightarrow v'$. Repeat this N times to estimate the corresponding probabilities p and p', respectively. Based on these probabilities, use Wagner's algorithm to obtain the next iteration. Repeat M times or until a potential counterexample with $p < p'$ is found.

7.3. **Implementation and results.** We first used Wagner's original code on a laptop computer, but when that proved too slow we made major changes. To speed up the performance and tweak the code, we implemented Wagner's algorithm in Julia.

We then run the code on a shared UCLA Hoffman2 Cluster, which is a Linux compute cluster consisting of 800+ 64-bit nodes and over 26,000 cores, with an aggregate of over 174TB of memory.[1] Each run of the algorithm required about 2 hours. After six runs with different parameters, the results were too similar to continue.

Specifically, we run the algorithm on graphs with $n = 20$ and $n = 30$ vertices, and for 3, 4 and 5 transversal vertices. Although we started with relatively dense graphs, the algorithm converged to relatively sparse graphs with about 100 edges. We used $N = 4000$, pruning the Monte Carlo sampling when the desired probabilities were far apart.

We used $M = 2000$, after which the probabilities p, p' rapidly converged to $\frac{1}{2}$ and became nearly indistinguishable. More precisely, the *probability gap* $p - p'$ became smaller than 0.01 getting close to the Monte Carlo error, i.e. the point when we would need to increase N to avoid false positives. At all stages, we had $p > p'$ suggesting validity of the bunkbed conjecture. At

[1]The system overview is available here: www.hoffman2.idre.ucla.edu/About/System-overview.html

the time of the experiments and prior to this work, we saw no evidence that an experimental approach could ever succeed.

7.4. **Analysis.** Having formally disproved the bunkbed conjecture, it is clear that our computational approach was doomed to failure. For the uniform weights we tested, we could never have reached graphs of size anywhere close to that in Theorem 1.2, of course. Even when the number of vertices is optimized to 82 as suggested in Remark 4.2, the number of edges is still very large while the probability gap in the theorem is on the order of 10^{-47}, thus undetectable in practice.

In hindsight, to reach a small counterexample we should have used the weighted bunkbed percolation rather than the more efficient alternative model, with some edges having a very large weight and some very small weight. Of course, by Remark 4.2, the probability gap in the theorem is still prohibitively small, at least for the graphs we consider.

7.5. **Conclusions.** It seems, the Bunkbed Conjecture has some unique features making it very poorly suited for computer testing. In fact, one reason we stopped our computer experiments is that in our initial rush to testing we failed to contemplate societal implications of working with even moderately large graphs.

Suppose we *did* find a potential counterexample graph with only $m = 100$ edges and the probability gap was large enough to be statistically detectable. Since analyzing all of $2^m \approx 10^{30}$ subgraphs is not feasible, our Monte Carlo simulations could only confirm the desired inequality with high probability. While this probability could be amplified by repeated testing, one could never formally *disprove* the bunkbed conjecture this way, of course.

This raises somewhat uncomfortable questions whether the mathematical community is ready to live with an uncertainty over validity of formal claims that are only known with high probability. It is also unclear whether in this imaginary world the granting agencies would be willing to support costly computational projects to further increase such probabilities (cf. [G+16, Zei93]). Fortunately, our failed computational effort avoided this dystopian reality, and we were able to disprove the bunkbed conjecture by a formal argument.

8. Final remarks

8.1. **Variations on the BBC.** Although the version of the Bunkbed Conjecture 1.1 given in [BK01] is considered the most definitive, over the years several closely related versions has been studied:

(0) *Counting BBC*, where one compares the number of subgraphs connecting vertices u, v and those that do not. This conjecture is a restatement of the BBC in the case $p = \frac{1}{2}$.

(1) *Weighted BBC*, where the edge probabilities $p_e = p_{e'}$ can depend on $e \in E$. This conjecture is equivalent to the BBC by [RS16], since edge probabilities can be approximated by series-parallel graphs.

(2) *Polynomial BBC*, where the edge probabilities above are viewed as variables. In this case the conjecture claims that the difference of polynomials on is a polynomial in nonnegative coefficients. This conjecture is stronger than Weighted BBC as there are polynomials positive on $[0, 1]$ which have negative coefficients such as $(x-y)^2$. Although we did not find a counterexample on graph with at most 8 vertices, is likely that there is a sufficiently small counterexample in this case. Cf. [Ric22], where the difference is a sum of squares.

(3) *Computational BBC*, where one asks if the counting version of the probability gap is in $\#\mathsf{P}$, i.e. has a combinatorial interpretation, see [Pak22, Conj. 5.6]. Clearly, this conjecture implies BBC. We refer to [IP22] for a formal treatment of this problem for general polynomials.

(4) *Alternative BBC*, where fair coin flips determine whether the edge e is deleted and e' retains or vice versa. This conjecture implies BBC [Lin11, Prop. 2.6].

(5) *Complete BBC*, where one takes all $T = V$ and performs the weighted percolation on the full $\overline{G} := G \times K_2$, i.e. on all edges in $\overline{G}$ including the posts. The conjecture in this case is weaker than the BBC, see e.g. [Lin11, Prop. 2.2].

In all but the last case, the corresponding conjecture is refuted by Theorem 1.2. In (5), the corresponding conjecture is refuted by Theorem 6.1.

8.2. **Robustness lemma.** Lemma 3.2 is a finite problem which can be reformulated as follows. By definition, probabilities on both sides of (3.2) are polynomials in 5 variables of degree at most 12, with at most 5^{12} nonzero coefficients. The lemma gives positivity of the difference of these two polynomials on a region of the unit cube $[0,1]^5$ cut out by the quadratic inequality (3.1).

Since our proof of Lemma 3.2 is somewhat cumbersome and uses a case-by-case analysis, we verified the lemma computationally. The results and the code are available on `GitHub`.[2] Of course, the advantage of our combinatorial proof is that it is elementary and amenable for generalizations.

8.3. **Special cases.** Our counterexample makes prior positive results somewhat more valuable. It would be interesting to find other families of graphs on which the Bunkbed Conjecture holds. We are especially interested in the corresponding problem for the Polynomial BBC. Note that we emphasized planarity in Theorem 1.2 since it was speculated in [Lin11] that planarity helps.

8.4. **Percolation in $\mathbb{Z}^d$.** For lattices, the connection probabilities $\mathbb{P}_p[u \leftrightarrow v]$ between vertices are known as the *two-point functions*. For percolation in higher dimensions, these were famously studied by Hara, van der Hofstad and Slade [HHS03], and they are also of interest for other probabilistic models.

Curiously, it is not known whether connection probabilities are monotone as the distance $|u - v|$ increases. This claim would follow from the bunkbed conjecture. This suggests that investigating the BBC for grid-like graphs is still of interest even if the conjecture is false for general planar graphs. Note that the monotonicity is known in the $p \downarrow 0$ limit.

More broadly, the bunkbed conjecture is an example of *reflection positivity* that was famously proved for the Ising model [MMS77]. Reflection positivity is not known for the Bernoulli percolation, and one can view our results as a formal claim that it does not exist.

8.5. **Random cluster model.** It was shown in [Häg03, §3] that the analogue of the BBC holds for the random cluster model with parameter $q = 2$. Our Theorem 1.2 shows that one cannot take $q = 1$. It would be interesting to find the smallest $q > 1$ s.t. the BBC holds for all finite graphs. We note that monotonicity in q is unclear, so e.g. it is not known if BBC holds for all $q \geq 2$.[3]

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[2] *Generating Partitions of Graph Vertices into Connected Components*, description and code at GitHub.com/Kroneckera/bunkbed-counterexample

[3] Alan Sokal suggested to us that BBC should hold for *integer* $q \geq 2$ and fail for *noninteger* $q > 2$ (personal communication, October 2, 2024).

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