LOWER BOUNDS FOR CONTINGENCY TABLES VIA LORENTZIAN POLYNOMIALS

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Abstract. We present a new lower bound on the number of *contingency tables*, improving upon and extending previous lower bounds by Barvinok [Bar09, Bar16] and Gurvits [Gur15]. As an application, we obtain new lower bounds on the volumes of *flow* and *transportation polytopes*. Our proofs are based on recent results on *Lorentzian polynomials*.

1. Introduction

Contingency tables are fundamental objects across the sciences. In statistics, they are employed to study dependence structure between two or more variables, see e.g. [Eve92, FLL17, Kat14]. They play an important role in combinatorics and graph theory since they are in bijection with bipartite multigraphs with given degrees, see e.g. [Bar09, DG95]. In discrete geometry and combinatorial optimization, they are frequently studied as integer points in transportation polytopes [DK14]. They also appear in a variety of other contexts, from algebraic and enumerative combinatorics [Pak00, PP20] to commutative algebra [DS98] and topology [KS20].

Motivated by these connections and applications, a great deal of effort is made to approximate and to estimate the number of contingency tables, both theoretically and computationally. For that, a variety of tools have been developed in different areas, such as the traditional and probabilistic divide-and-conquer [DZ15+, GM77], the asymptotic analysis [BH12, CM10, GM08], the MCMC algorithms [C+06, DKM97], approximation algorithms [AH20, B+10], and integer programming [BDV04, D09a].

In this paper we present a new lower bound (Theorem 2.1) on the number of *contingency tables with cell-bounded entries*, a setting which includes a *permanent*. This bound improves upon previous lower bounds, holds for all marginals, is fast to compute, and behaves well in many examples. We begin with an important special case.

Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ be integer vectors. A *contingency table* with marginals $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is an $m \times n$ matrix $A = (a_{ij})$, such that $a_{ij} \in \mathbb{N}$,

$$\sum_{i=1}^{m} a_{ij} \,=\, \beta_j \ \text{ for all } \ 1 \le j \le n\,, \quad \text{and} \quad \sum_{j=1}^{n} a_{ij} \,=\, \alpha_i \ \text{ for all } \ 1 \le i \le m.$$

Denote by $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ the number of contingency tables with marginals $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Theorem 1.1 (= Corollary 2.2)**.** *For every $\boldsymbol{\alpha}, \boldsymbol{\beta}$ as above, we have:*

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}} \,\ge\, \mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \,\ge\, \left[\frac{1}{e^{m+n-1}} \prod_{i=2}^{m} \frac{1}{\alpha_i + 1} \prod_{j=1}^{n} \frac{1}{\beta_j + 1} \right] \mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}\,,$$

where

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}} \,:=\, \inf_{\substack{x_i \in (0,1) \\ 1 \le i \le m}} \ \inf_{\substack{y_j \in (0,1) \\ 1 \le j \le n}} \left[\prod_{i=1}^{m} x_i^{\alpha_i} \prod_{j=1}^{n} y_j^{\beta_j} \right]^{-1} \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1 - x_i y_j}\,.$$

Here the upper bound is elementary and follows from the definition of contingency tables. The lower bound is a special case of our Main Theorem 2.1, which is an improvement over Barvinok's lower

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bound [Bar09] (Theorem 2.4). Note that Main Theorem 2.1 gives a stronger bound when α_i, β_j are bounded, and generalizes to the case of cell-bounded contingency tables, i.e. to all $A = (a_{ij})$, such that $a_{ij} \le k_{ij}$. When all $k_{ij} \in \{0,1\}$, the corresponding contingency tables are in bijection with perfect matchings in a bipartite graph with adjacency matrix $K = (k_{ij})$, and our results recover Gurvits's recent lower bound in [Gur15] (Theorem 2.9).

There are several ways to understand the theorem. First, it is a general theoretical result in line with other results on contingency tables and graph counting, and can be used in both asymptotic enumeration and analysis of graph algorithms (see e.g. [Bar16, Wor18] and references therein). Second, the bound in the theorem can be effectively computed, and thus gives a fast approximation algorithm for a problem of computing $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$, see §11.2.

Third, letting $\alpha_i, \beta_j \to \infty$ at the same rate in Theorem 1.1, allows us to obtain new lower bounds for the volumes of *transportation polytopes*. We explore this connection in Section 8. Finally, the theorem can be used to obtain exact bounds in many special cases of interest in the statistics literature, and then make explicit comparisons with other bounds; we report our numerical experiments in Section 10.

Paper structure. We begin with a lengthy Section 2 stating our new results on the number of contingency tables and discussing prior work on the subject. We continue presenting new results in Section 3, this time in probabilistic framework of contingency tables with random constraints. In Sections 4 and 5 we discuss some known results on Lorentzian and related classes of polynomials, and then derive new capacity bounds for coefficients. In Section 6 we prove the main results of the paper.

In the second part of the paper, we give applications of our results and explore connections to earlier work. First, in Section 7, we prove that our bounds are sharper than the two earlier bounds by Barvinok, at least asymptotically. In the next two Sections 8 and 9, we apply our results to computing lower bounds for the volume of flow and transportation polytopes, and to the special case of uniform marginals. We conclude with numerical examples in Section 10, and final remarks in Section 11.

We should mention that the paper is very far from being self-contained, as we repeatedly use available tools with very little preparation, but with the exact references to the literature. This is why we upfronted the results in Sections 2 and 3, to ease access to our theorems. We repeat the pattern in Sections 8 and 9.

2. Main results and prior work

The main results in this paper are lower bounds on $\mathrm{CT}_K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for general marginals $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, and cell-bounded entries given by matrix K. In this section we present the results and compare them with similar lower bounds due to Barvinok and Gurvits. Along with the actual bounds, we also give asymptotic bounds by naively applying Stirling's approximation wherever possible. Since Stirling's approximation is off by at most a factor of $\frac{e}{\sqrt{2\pi}}$, the asymptotic values given are decent approximations for all asymptotic regimes. For two multivariate functions $F(\boldsymbol{a})$ and $G(\boldsymbol{a})$, $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, we write $F \gtrsim G$ if $F(\boldsymbol{a}) \ge C \cdot G(\boldsymbol{a})$ for a universal constant $C > 0$ and $\min\{a_i\} \to \infty$.

2.1. **Definitions.** Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$ be integer vectors of *marginals*, such that

$$\sum_{i=1}^{m} \alpha_i \, = \, \sum_{j=1}^{n} \beta_j \, = \, N.$$

Let $K = (k_{ij})$ be an $m \times n$ matrix with entries in $\mathbb{N} \cup \{+\infty\}$. We use $K = \infty$ to denote an all ∞ matrix, and $K = \mathbf{1}$ to denote the all-one matrix.

Matrix $A = (a_{ij})$ is called a *K-contingency table* with marginals $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, if A is a contingency table with cell-bounded entries $0 \le a_{ij} \le k_{ij}$. When $K = \infty$ we obtain the usual (unrestricted) contingency tables. When $K = \mathbf{1}$, matrix A is called a *binary contingency table.*

As in the introduction, let $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ denote the number of contingency tables with marginals $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, and let $\mathrm{CT}_K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ denote the number of K-contingency tables with the same marginals. When all $k_{ij} \in \{0,1\}$, we call such matrix $K = (k_{ij})$ *graphical*, and write $\mathrm{BCT}_K(\alpha, \beta)$ for the number of binary contingency tables. In particular, when all $k_{ij} = 1$, we may further write $\mathrm{BCT}(\alpha, \beta)$ for the number of such tables. Finally, when all $k_{ij} \in \{0, +\infty\}$, we call such K *multigraphical.*

Consider the generating polynomial $P_K(\boldsymbol{x},\boldsymbol{y}) \in \mathbb{N}[\boldsymbol{x},\boldsymbol{y}]$ for all contingency tables w.r.t. their marginals:

$$P_K(\boldsymbol{x},\boldsymbol{y}) \,:=\, \prod_{i=1}^{m}\prod_{j=1}^{n}\sum_{\ell=0}^{k_{ij}} x_i^{\ell} y_j^{\ell} \,=\, \sum_{\boldsymbol{\alpha},\,\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{y}^{\boldsymbol{\beta}}\,\mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta}),$$

where $\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{y}^{\boldsymbol{\beta}} = x_1^{\alpha_1}\cdots x_m^{\alpha_m}\,y_1^{\beta_1}\cdots y_n^{\beta_n}$. In a special case, for (usual) and binary contingency tables the generating polynomials are given by

$$P_\infty(\boldsymbol{x},\boldsymbol{y}) \,:=\, \prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-x_iy_j} \quad \text{and} \quad P_{\mathbf{1}}(\boldsymbol{x},\boldsymbol{y}) \,:=\, \prod_{i=1}^{m}\prod_{j=1}^{n}(1+x_iy_j)\,.$$

Let $F \in \mathbb{R}_+[\boldsymbol{x},\boldsymbol{y}]$ be a polynomial in $(m+n)$ variables. Define the *capacity* of F as

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(F) \,:=\, \inf_{\boldsymbol{x},\,\boldsymbol{y}>0} \boldsymbol{x}^{-\boldsymbol{\alpha}}\boldsymbol{y}^{-\boldsymbol{\beta}}\,F(\boldsymbol{x},\boldsymbol{y})\,.$$

We consider capacity of polynomials P_K and other polynomials throughout the paper.

2.2. **Main theorem.** We are now ready to state the main result of the paper.

Theorem 2.1 (Main theorem)**.** *Let $\boldsymbol{\alpha}\in\mathbb{N}^m$, $\boldsymbol{\beta}\in\mathbb{N}^n$, such that $\sum_i \alpha_i = \sum_j \beta_j$. Let $K=(k_{ij})$ be an $m\times n$ matrix with all entries $k_{ij}\in\mathbb{N}\cup\{+\infty\}$. Then:*

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_K) \,\ge\, \mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta}) \,\ge\, \left[\prod_{i=2}^{m}\frac{a_i^{a_i}}{(a_i+1)^{a_i+1}}\,\prod_{j=1}^{n}\frac{b_j^{b_j}}{(b_j+1)^{b_j+1}}\right]\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_K),$$

where $a_i := \min\{\alpha_i, \lambda_i-\alpha_i\}$, $b_j := \min\{\beta_j,\gamma_j-\beta_j\}$, $\lambda_i := \sum_j k_{ij}$ and $\gamma_j := \sum_i k_{ij}$, for all $1\le i\le m$, $1\le j\le n$.

Note that the first product starting with $i=2$ is not a typo, but a feature of the bound. Given a choice, the lower bound is the largest when a_1 is chosen to be maximal. Let us mention that the capacity constant in the theorem can be computed in polynomial time (see §11.2).

Before moving on, let us emphasize the unrestricted nature of the matrix K. In previously known bounds, entries in K were either restricted, or certain values of K were allowed by weighting variables in certain ways. For us though, the value of K is inconsequential to our proof method, and so it shows up as a parameter in our bounds in a straightforward way.

2.3. **Two special cases.** First, in the unrestricted case $K=\infty$, we obtain the following result:

Corollary 2.2 (= Theorem 1.1)**.** *Let $\boldsymbol{\alpha}\in\mathbb{N}^m$, $\boldsymbol{\beta}\in\mathbb{N}^n$, such that $\sum_i\alpha_i=\sum_j\beta_j=N$. Then:*

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}} \,\ge\, \mathrm{CT}(\boldsymbol{\alpha},\boldsymbol{\beta}) \,\ge\, \left[\frac{1}{e^{m+n-1}}\prod_{i=2}^{m}\frac{1}{\alpha_i+1}\prod_{j=1}^{n}\frac{1}{\beta_j+1}\right]\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}\,.$$

In particular, we have

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}} \,\ge\, \mathrm{CT}(\boldsymbol{\alpha},\boldsymbol{\beta}) \,\ge\, e^{-4N}\cdot\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}\,.$$

Second, in the case that α_i,β_j are bounded for $i,j\ge 2$, the approximation ratio we obtain is independent of the dominant marginal values, α_1 and β_1. Note that the number of terms in the polynomial P_K in the following result *does* depend on the dominant marginal values, even though the approximation ratio does not.

Theorem 2.3. *Let $\boldsymbol{\alpha}\in\mathbb{N}^m$, $\boldsymbol{\beta}\in\mathbb{N}^n$, such that $\sum_i\alpha_i=\sum_j\beta_j$, and $\alpha_i,\beta_j\le c$ for all $i,j\ge 2$. Then:*

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_K) \,\ge\, \mathrm{CT}(\boldsymbol{\alpha},\boldsymbol{\beta}) \,\ge\, \frac{1}{(m+n-1)\,(e(c+1))^{m+n-1}}\cdot\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_K)$$

where $K=(k_{ij})$, and $k_{ij}=\min(\alpha_i,\beta_j)$ for all i,j. Note that this is the entrywise minimal value of K such that $\mathrm{CT}(\boldsymbol{\alpha},\boldsymbol{\beta})=\mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta})$.

2.4. **Barvinok's first bound.** For the case of general contingency tables with a multigraphical matrix $K = (k_{ij})$, Barvinok gives the following bounds:

Theorem 2.4 (Barvinok [Bar09, Thm 1.3])**.** *Let* $\boldsymbol{\alpha} \in \mathbb{N}^m$, $\boldsymbol{\beta} \in \mathbb{N}^n$, *such that* $N = \sum_i \alpha_i = \sum_j \beta_j$. *Let* $K = (k_{ij})$ *be an* $m \times n$ *multigraphical matrix, i.e.* $k_{ij} \in \{0, +\infty\}$, *for all* $1 \le i \le m$ *and* $1 \le j \le n$. *Then, for* $m + n \ge 10$, *we have:*

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_K) \,\ge\, \mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta}) \,\ge\, C_{\mathrm{Barv}}(K,\boldsymbol{\alpha},\boldsymbol{\beta})\; \mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_K),$$

where

$$C_{\mathrm{Barv}}(K,\boldsymbol{\alpha},\boldsymbol{\beta}) \,=\, \frac{\Gamma(\frac{m+n}{2})}{2\,e^5\,\pi^{\frac{m+n-2}{2}}\,mn(N+mn)} \left(\frac{2}{(mn)^2(N+1)(N+mn)}\right)^{m+n-1}$$
$$\times\; \frac{N!\,(N+mn)!\,(mn)^{mn}}{N^N(N+mn)^{N+mn}\,(mn)!}\,\prod_{i=1}^m \frac{\alpha_i^{\alpha_i}}{\alpha_i!}\,\prod_{j=1}^n \frac{\beta_j^{\beta_j}}{\beta_j!}\,.$$

We also give a simplified version of Barvinok's bound in the following.

Theorem 2.5. *The value of* $C_{\mathrm{Barv}}(K,\boldsymbol{\alpha},\boldsymbol{\beta})$ *in the previous theorem is such that asymptotically we have*

$$R^{n+m-1}\,S \,\gtrsim\, C_{\mathrm{Barv}}(K,\boldsymbol{\alpha},\boldsymbol{\beta}) \gtrsim R^{n+m}\,S,$$

where $\min\{\alpha_i,\beta_j\} \to \infty$, $\min\{m,n\} \to \infty$,

$$R \,=\, \frac{(m+n)^{\frac{1}{2}}}{\pi\,e^{\frac{1}{2}}\,(mn)^2\,(N+1)\,(N+mn)}, \qquad \textit{and} \qquad S \,=\, \left(\prod_{i=1}^m \frac{1}{\alpha_i}\,\prod_{j=1}^n \frac{1}{\beta_j}\right)^{\frac{1}{2}}.$$

In particular, we have

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_K) \ge \mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta}) \gtrsim N^{-7(m+n)} \cdot \mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_K).$$

Note that the asymptotics of the bound ratios $N^{-7(m+n)}$ in Theorem 2.5 and e^{-4N} in Theorem 2.5 are not directly comparable. In §7.1 we compare the lower bounds directly and show that our bound in Theorem 2.1 is sharper. See also Section 10 for numerical examples.

Remark 2.6 (Shapiro's upper bound)**.** In [Sha10] (see also [Bar17]), the Shapiro improves upon Barvinok's first upper bound by adding a capacity-based *correction term*. It is best presented in the dual form of the proof of Lemma 9.2, and states:

$$\mathrm{CT}(\boldsymbol{\alpha},\boldsymbol{\beta}) \,\le\, \left[\min_{\tau\in K_{mn}}\,\prod_{(i,j)\in\tau}\frac{1}{1+z_{ij}}\right]\,\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}\,,$$

where $Z = (z_{ij})$ is the typical matrix defining capacity in the proof of the lemma, the minimum is over spanning trees τ in the complete bipartite graph K_{mn}, and the product is over all edges in τ. Since we concentrate on the lower bounds, we omit the general $K = (k_{ij})$ case. We only use this bound in Section 10 for numerical comparisons.

2.5. **Binary contingency tables.** In an important special case of binary contingency tables, Barvinok gives the following bounds.

Theorem 2.7 (Barvinok [Bar10a, Thm 5])**.** *Let* $\boldsymbol{\alpha} \in \mathbb{N}^m$, $\boldsymbol{\beta} \in \mathbb{N}^n$, *such that* $\sum_i \alpha_i = \sum_j \beta_j$. *Let* $K = (k_{ij})$ *be an* $m \times n$ *graphical matrix, i.e.* $k_{ij} \in \{0,1\}$ *for all* i, j. *Then:*

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_K) \,\ge\, \mathrm{BCT}_K(\boldsymbol{\alpha},\boldsymbol{\beta}) \,\ge\, \left[\frac{(mn)!}{(mn)^{mn}}\,\prod_{i=1}^m \frac{(n-\alpha_i)^{n-\alpha_i}}{(n-\alpha_i)!}\,\prod_{j=1}^n \frac{\beta_j^{\beta_j}}{\beta_j!}\right]\,\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_K).$$

For the usual (unrestricted) binary contingency tables, this gives:

Corollary 2.8. *Let* $\boldsymbol{\alpha} \in \mathbb{N}^m$, $\boldsymbol{\beta} \in \mathbb{N}^n$, *such that* $\sum_i \alpha_i = \sum_j \beta_j$. *Let* $K = \mathbf{1}$ *be an* $m \times n$ *all-ones matrix. Then:*

$$\operatorname{Cap}_{\boldsymbol{\alpha}\boldsymbol{\beta}}(P_K) \,\geq\, \operatorname{BCT}(\boldsymbol{\alpha},\boldsymbol{\beta}) \gtrsim \left[\frac{mn}{(2\pi)^{m+n-1}} \prod_{i=1}^{m} \frac{1}{n-\alpha_i} \prod_{j=1}^{n} \frac{1}{\beta_j}\right]^{\frac{1}{2}} \operatorname{Cap}_{\boldsymbol{\alpha}\boldsymbol{\beta}}(P_K),$$

where $\min\{\alpha_i, \beta_j\} \to \infty$.

Gurvits [Gur15] was able to improve upon Barvinok's bound in the following result by proving a better constant for all graphical $m \times n$ matrices K. For the sake of simplicity, he only explicitly provides a bound in the case of $\boldsymbol{\alpha} = \boldsymbol{\beta}$ are uniform. He also gives other bounds for non-uniform $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, in the form that are similar to Theorem 2.7.

Theorem 2.9 (Gurvits [Gur15, Thm 5.1])**.** *Let* $\boldsymbol{\alpha} \in \mathbb{N}^m$, $\boldsymbol{\beta} \in \mathbb{N}^n$, *such that* $\sum_i \alpha_i = \sum_j \beta_j$. *Let* $K = (k_{ij})$ *be an* $m \times n$ *graphical matrix, i.e.* $k_{ij} \in \{0,1\}$ *for all* i, j. *Then:*

$$\operatorname{Cap}_{\boldsymbol{\alpha}\boldsymbol{\beta}}(P_K) \,\geq\, \operatorname{BCT}_K(\boldsymbol{\alpha},\boldsymbol{\beta}) \,\geq\, \left[\prod_{i=2}^{m} \binom{\lambda_i}{\alpha_i} \frac{\alpha_i^{\alpha_i}(\lambda_i-\alpha_i)^{\lambda_i-\alpha_i}}{\lambda_i^{\lambda_i}} \prod_{j=1}^{n} \binom{\gamma_j}{\beta_j} \frac{\beta_j^{\beta_j}(\gamma_j-\beta_j)^{\gamma_j-\beta_j}}{\gamma_j^{\gamma_j}}\right] \operatorname{Cap}_{\boldsymbol{\alpha}\boldsymbol{\beta}}(P_K),$$

where $\lambda_i = \sum_j k_{ij}$ *and* $\gamma_j = \sum_i k_{ij}$, *for all* i *and* j.

See §11.3 for a combinatorial interpretation. In the usual (unrestricted) binary contingency tables, Gurvits's theorem gives:

Corollary 2.10. *Let* $\boldsymbol{\alpha} \in \mathbb{N}^m$, $\boldsymbol{\beta} \in \mathbb{N}^n$, *such that* $\sum_i \alpha_i = \sum_j \beta_j$. *Let* $K = \mathbf{1}$ *be an* $m \times n$ *all-ones matrix. Then:*

$$\operatorname{Cap}_{\boldsymbol{\alpha}\boldsymbol{\beta}}(P_K) \,\geq\, \operatorname{BCT}(\boldsymbol{\alpha},\boldsymbol{\beta}) \gtrsim \left[\frac{1}{(2\pi)^{m+n-1}} \prod_{i=2}^{m} \frac{n}{\alpha_i(n-\alpha_i)} \prod_{j=1}^{n} \frac{m}{\beta_j(m-\beta_j)}\right]^{\frac{1}{2}} \operatorname{Cap}_{\boldsymbol{\alpha}\boldsymbol{\beta}}(P_K),$$

where $\min\{\alpha_i, \beta_j\} \to \infty$.

In §6.4, we give a more straightforward proof of Theorem 2.9 using the same technique as in the proof of our Main Theorem 2.1. Note that Gurvits's technique in [Gur15] cannot be applied to general contingency tables.

2.6. **Barvinok's second bound.** In [Bar16], Barvinok gives another upper and lower bound for $\operatorname{CT}_K(\alpha,\beta)$, similar to the form of Theorem 2.4, except the polynomial P_K is replaced by

$$H_N(\boldsymbol{x},\boldsymbol{y}) \,:=\, h_N(\boldsymbol{x}\cdot\boldsymbol{y}) \,=\, h_N(\ldots, x_i y_j, \ldots)\,,$$

where $h_N(\ldots)$ is the complete homogeneous polynomial in mn variables. Barvinok observes that the coefficients of $H_N(\boldsymbol{x},\boldsymbol{y})$ are precisely $\operatorname{CT}(\boldsymbol{\alpha},\boldsymbol{\beta})$ for all $\sum_i \alpha_i = \sum_j \beta_j = N$. Using this, he obtains:

Theorem 2.11 (Barvinok [Bar16, Thm 8.4.2])**.** *Let* $\boldsymbol{\alpha} \in \mathbb{N}^m$ *and* $\boldsymbol{\beta} \in \mathbb{N}^n$, *such that* $\sum_i \alpha_i = \sum_j \beta_j = N$. *Then:*

$$\operatorname{Cap}_{\boldsymbol{\alpha}\boldsymbol{\beta}}(H_N) \,\geq\, \operatorname{CT}(\boldsymbol{\alpha},\boldsymbol{\beta}) \,\geq\, C_{\mathrm{H}}(\boldsymbol{\alpha},\boldsymbol{\beta}) \cdot \operatorname{Cap}_{\boldsymbol{\alpha}\boldsymbol{\beta}}(H_N),$$

where

$$C_{\mathrm{H}}(\boldsymbol{\alpha},\boldsymbol{\beta}) \,=\, \binom{N+m-1}{m-1}^{-1} \binom{N+n-1}{n-1}^{-1} \frac{N!}{N^N} \max\left\{\prod_{i=1}^{m} \frac{\alpha_i^{\alpha_i}}{\alpha_i!}, \prod_{j=1}^{n} \frac{\beta_j^{\beta_j}}{\beta_j!}\right\}.$$

Note that Barvinok gives bounds for general K with entries in $\{0,+\infty\}$, but we suppress this generalization here for the sake of simplicity. In §7.2, we compare our bound with this second Barvinok's bound. Namely, we prove that the lower bound in Theorem 2.1 is sharper than that in Theorem 2.7. See also Section 10 for numerical examples, and §11.4 for the independence heuristic partly motivating this bound.

3. Random contingency tables

3.1. **The setup.** In this section, we give new lower bounds on the probability that a random contingency table has certain marginals when the entries are drawn from various random variables. These results parallel similar results given in [Bar12]; e.g., Theorem 6.3 (2). In what follows, we let $\mu_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ denote the probability that a random contingency table has marginals $\boldsymbol{\alpha},\boldsymbol{\beta}$.

We will also consider the capacity of a different family of polynomials in what follows. Specifically given choices of $\boldsymbol{\alpha},\boldsymbol{\beta}$, a choice of $K = (k_{ij})$ with finite entries, and a choice of $s \in [0,1]$, we consider $\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(Q_{K,s})$, where

$$Q_{K,s}(\boldsymbol{x},\boldsymbol{y}) := \prod_{i=1}^{m}\prod_{j=1}^{n}\big(s x_i y_j + (1-s)\big)^{k_{ij}}.$$

We now state our bounds on $\mu_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ for two specific cases: when entries are binomial-distributed, and when entries are Poisson-distributed.

3.2. **Binomial-distributed entries.** Our first result regarding random contingency tables gives bounds in the case that the entries of the table are binomial random variables with parameter $s \in [0,1]$. Specifically given a K with finite entries, a random contingency table A is sampled by sampling a_{ij} from the set $\{0,1,\ldots,k_{ij}\}$ with probability given by

$$\Pr[a_{ij} = \ell] = \binom{k_{ij}}{\ell} s^{\ell}(1-s)^{k_{ij}-\ell}.$$

We now bound $\mu_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ in this case as follows.

Theorem 3.1. *Let $\boldsymbol{\alpha} \in \mathbb{N}^m$, $\boldsymbol{\beta} \in \mathbb{N}^n$, and let $K = (k_{ij})$ be an $m \times n$ matrix with finite entries $k_{ij} \in \mathbb{N}$. Let $A = (a_{ij})$ be an $m \times n$ random matrix where each entry a_{ij} is an independent binomial random variable on $\{0,1,\ldots,k_{ij}\}$ with parameter $s \in [0,1]$. The probability $\mu_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ that A has marginals $(\boldsymbol{\alpha},\boldsymbol{\beta})$ is bounded by*

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(Q_{K,s}) \geq \mu_{\boldsymbol{\alpha},\boldsymbol{\beta}} \geq \left[\prod_{i=2}^{m}\binom{\lambda_i}{\alpha_i}\frac{\alpha_i^{\alpha_i}(\lambda_i-\alpha_i)^{\lambda_i-\alpha_i}}{\lambda_i^{\lambda_i}}\prod_{j=1}^{n}\binom{\gamma_j}{\beta_j}\frac{\beta_j^{\beta_j}(\gamma_j-\beta_j)^{\gamma_j-\beta_j}}{\gamma_j^{\gamma_j}}\right]\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(Q_{K,s}),$$

where $\lambda_i = \sum_j k_{ij}$ and $\gamma_j = \sum_i k_{ij}$.

Note that this is the same constant as is given in the case of counting binary contingency tables in Theorem 2.9.

3.3. **Capacity via typical matrices.** In Theorem 3.1, we can replace the expression for $\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(Q_{K,s})$ by a more ostensibly combinatorial optimization problem. This is very similar to the idea of maximizing an entropy-like function found in [BH12], and in particular in Lemma 5.3 (2) of [Bar12]. In those papers, the optimal input is referred to as the "typical matrix" with row sums $\boldsymbol{\alpha}$ and column sums $\boldsymbol{\beta}$. In the binomial entries case, we have the following result.

Theorem 3.2. *In notation of Theorem 3.1, we have:*

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(Q_{K,s}) = \sup_{0\leq M\leq K,\, M\in\mathcal{T}_{\alpha,\beta}}\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{k_{ij}^{k_{ij}}\, s^{m_{ij}}(1-s)^{k_{ij}-m_{ij}}}{m_{ij}^{m_{ij}}(k_{ij}-m_{ij})^{k_{ij}-m_{ij}}},$$

where the sup *is over all real matrices $M = (m_{ij})$ for which $0 \leq M \leq K$ entrywise, and M is in the transportation polytope $\mathcal{T}_{\alpha,\beta}$ of nonnegative real matrices with row sums $\boldsymbol{\alpha}$ and colums sums $\boldsymbol{\beta}$, see §8.1.*

Note that, as above, we are able to incorporate the matrix K into this alternate expression.

3.4. **Poisson-distributed entries.** Our next result gives bounds in the case that the entries of the table are Poisson random variables with rate parameter $s > 0$. Specifically, a random contingency table A is sampled by sampling a_{ij} from the set $\{0, 1, 2, 3, \ldots\}$ with probability given by

$$\Pr[a_{ij} = \ell] \,=\, \frac{s^\ell e^{-s}}{\ell!}\,.$$

For this case, we obtain explicit bounds on the probabilities.

Theorem 3.3. *Let $\boldsymbol{\alpha} \in \mathbb{N}^m$, $\boldsymbol{\beta} \in \mathbb{N}^n$, such that $\sum_i \alpha_i = \sum_j \beta_j = N$. Let A be an $m \times n$ random matrix where each entry a_{ij} is an independent Poisson random variable on $\{0, 1, 2, 3, \ldots\}$ with rate parameter $s > 0$. The probability $\mu_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ that A has marginals $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is bounded by*

$$\frac{(sN)^N}{e^{smn-N}} \prod_{i=1}^m \frac{1}{\alpha_i^{\alpha_i}} \prod_{j=1}^n \frac{1}{\beta_j^{\beta_j}} \,\geq\, \mu_{\boldsymbol{\alpha},\boldsymbol{\beta}} \,\geq\, \frac{(sN)^N}{e^{smn+N}} \prod_{i=1}^m \frac{1}{\alpha_i!} \prod_{j=1}^n \frac{1}{\beta_j!}\,.$$

The proof is based on the fact that the value of $\mathrm{Cap}_{\boldsymbol{\alpha}\boldsymbol{\beta}}$ can be explicitly calculated in this case, see §6.6.

4. Real stable and denormalized Lorentzian polynomials

4.1. **Notation.** We use $\mathbb{C}$ and $\mathbb{R}$ to denote the complex and real numbers, $\mathbb{R}_+ = \{x \geq 0\}$, $\mathbb{R}_{>0} = \{x > 0\}$, and $\mathbb{N} = \{0, 1, \ldots\}$. We also use $[n] = \{1, \ldots, n\}$. We denote by ∂_i the partial derivative $\frac{\partial}{\partial x_i}$.

We use some standard vector shorthand. For vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we let most standard operations be entrywise; e.g. $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ and $\boldsymbol{\alpha} \pm \boldsymbol{\beta}$. We also define $\boldsymbol{\alpha}! := \prod_i \alpha_i!$, $\binom{\boldsymbol{\alpha}}{\boldsymbol{\beta}} := \prod_i \binom{\alpha_i}{\beta_i}$, and $\boldsymbol{\alpha}^{\boldsymbol{\beta}} := \prod_i \alpha_i^{\beta_i}$. Finally, we let $\mathbf{1}$ denote the all-ones vector with length determined by context.

An sequence $\{a_k, 0 \leq k \leq n\}$ of nonnegative real numbers is called *log-concave* if $a_k^2 \geq a_{k-1}a_{k+1}$ for all $1 \leq k \leq n-1$. Moreover, it is *ultra-log-concave* if $\left\{a_k/\binom{n}{k}, 0 \leq k \leq n\right\}$ is log-concave and has *no internal zeros*, i.e., there are no indices $i < j < k$ for which $a_i a_k \neq 0$ and $a_j = 0$. See [Brä15] and references therein, for the context behind these properties.

Throughout, we will use Stirling's approximation for factorial:

$$\frac{n!}{n^n} \,\approx\, e^{-n}\sqrt{2\pi n},$$

which holds asymptotically as $n \to \infty$. We also have the following bounds which hold for all $n \in \mathbb{N}$:

$$\left[e^{-n}\sqrt{2\pi n}\right] \,\leq\, \frac{n!}{n^n} \,\leq\, \frac{e}{\sqrt{2\pi}}\left[e^{-n}\sqrt{2\pi n}\right].$$

4.2. **Real stable and Lorentzian polynomials.** A polynomial $p \in \mathbb{C}[x_1, \ldots, x_n]$ is said to be *stable* if it is nonvanishing whenever $\Im(x_j) > 0$ for all j. If further p has real coefficients, then p is said to be *real stable.* Recall that the *Hessian* of a polynomial $p \in \mathbb{C}[x_1, \ldots, x_n]$ at $\boldsymbol{x} \in \mathbb{C}^n$, is defined as

$$\mathcal{H}_p(\boldsymbol{x}) \,=\, \left(\partial_i \partial_j p(x)\right)_{i,j=1}^n.$$

A real symmetric matrix has *Lorentzian signature* if it is nonsingular and has exactly one positive eigenvalue, i.e., its signature is $(+, -, -, \ldots, -)$.

Definition 4.1 (Brändén–Huh [BH20])**.** A homogeneous polynomial $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ of degree $d \geq 2$ is *strictly Lorentzian* if

(1) all coefficients of p are positive, and
(2) for each sequence $1 \leq i_1, i_2, \ldots, i_{d-2} \leq n$ and $\boldsymbol{x} \in \mathbb{R}^n_{>0}$, the Hessian of $\partial_{i_1} \cdots \partial_{i_{d-2}} p$ has Lorentzian signature at $\boldsymbol{x}$.

If p is the limit (in the Euclidean space of real polynomials of degree at most d in n variables) of strictly Lorentzian polynomials, we say that p is *Lorentzian.*

Proposition 4.2 ([BH20, Ex. 5.2])**.** *A bivariate homogeneous polynomial is Lorentzian if and only if its coefficients form an ultra-log-concave sequence.*

Remark 4.3. The class of Lorentzian polynomials contains homogenous stable polynomials with non-negative coefficients [BH20], which gives an easy sufficient condition for a polynomial to be Lorentzian.

Note that by definition, $\partial_i p$ is Lorentzian whenever p is, for any i. This gives the following lemma which will be useful for induction.

Lemma 4.4. *Let $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ be a Lorentzian (resp. real stable) polynomial of degree d, and let us write*

$$p(x_1, \ldots, x_n) \, = \, \sum_{k=0}^{d} x_n^{d-k} \, p_k(x_1, \ldots, x_{n-1}).$$

Then p_k is a Lorentzian (resp. real stable) polynomial of degree k, for all $k \in [d]$.

Proof. Apply the partial derivative ∂_n exactly $(d-k)$ times to the polynomial p, to obtain q, which is Lorentzian by definition. Plugging in $x_n = 0$ to q then yields p_k up to constant. To see that plugging in 0 preserves the class of Lorentzian polynomials, note that this operation preserves the class of stable polynomials with non-negative coefficients, see e.g. [Wag11, Lemma 2.4], and it also preserves homogeneity. Theorem 6.4 of [BH20] then implies that this operation also preserves the class of Lorentzian polynomials. □

And finally, we have the following by a similar argument.

Lemma 4.5. *If $p \in \mathbb{R}_+[x_1, x_2, \ldots, x_n]$ is a Lorentzian (resp. real stable) polynomial and $\lambda, \mu > 0$, then $p(\lambda x_1, \mu x_1, x_3, \ldots, x_n)$ is also Lorentzian (resp. real stable).*

Proof. Since $p \mapsto p(\lambda x_1, \mu x_1, x_3, \ldots, x_n)$ preserves the class of real stable polynomials with non-negative coefficients and also preserves homogeneity, it also preserves the class of Lorentzian polynomials by Theorem 6.4 of [BH20]. □

4.3. Denormalized Lorentzian polynomials.

Given a polynomial $p(\boldsymbol{x}) = \sum_{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}$, we define its *normalization* as

$$N[p] \, := \, \sum_{\boldsymbol{\mu}} p_{\boldsymbol{\mu}} \frac{\boldsymbol{x}^{\boldsymbol{\mu}}}{\boldsymbol{\mu}!}.$$

We say a homogeneous polynomial $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ is *denormalized Lorentzian* if $N[p]$ is Lorentzian.

Proposition 4.6 ([BH20, Cor. 6.8])**.** *Let $p_1, \ldots, p_m \in \mathbb{R}_+[x_1, \ldots, x_n]$ be denormalized Lorentzian polynomials. Then so is $p_1 \cdots p_m$.*

Recall the function P_K from Section 2, where K is an $m \times n$ matrix with entries in $\mathbb{N} \cup \{+\infty\}$:

$$P_K(\boldsymbol{x}, \boldsymbol{y}) \, := \, \prod_{i=1}^{m} \prod_{j=1}^{n} \sum_{\ell=0}^{k_{ij}} x_i^{\ell} y_j^{\ell}.$$

When K has finite entries, P_K is a polynomial, and we can further define

$$\widetilde{P}_K(\boldsymbol{x}, \boldsymbol{y}) \, := \, y_1^{\sum_i k_{i1}} \cdots y_n^{\sum_i k_{in}} \, P_K(\boldsymbol{x}, \boldsymbol{y}^{-1}) \, = \, \prod_{i=1}^{m} \prod_{j=1}^{n} \sum_{\ell=0}^{k_{ij}} x_i^{\ell} y_j^{k_{ij}-\ell}.$$

That is, $\widetilde{P}_K$ is a product of polynomials of the form $q(x, y) = x^d + x^{d-1}y + \ldots + y^d$, all of which are denormalized Lorentzian by Proposition 4.2. Therefore, $\widetilde{P}_K$ is denormalized Lorentzian by Proposition 4.6. In the next section, we will obtain bounds on the coefficients of denormalized Lorentzian polynomials, which will translate into bounds on the number of contingency tables with given marginals.

Before moving on, we give versions of Lemma 4.4 and Lemma 4.5 for denormalized Lorentzian polynomials.

Lemma 4.7. *Let $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ be a denormalized Lorentzian polynomial of degree d, and let us write*

$$p(x_1, \ldots, x_n) \,=\, \sum_{k=0}^{d} x_n^{d-k}\, p_k(x_1, \ldots, x_{n-1}).$$

Then p_k is a denormalized Lorentzian polynomial of degree k, for all $k \in [d]$.

Proof. The map $p \mapsto p_k$ commutes with N up to scalar for all k. The result then follows from Lemma 4.4. □

Lemma 4.8. *Let $p \in \mathbb{R}_+[x_1, x_2, \ldots, x_n]$ be a denormalized Lorentzian polynomial, and let $\lambda, \mu > 0$. Then $p(\lambda x_1, \mu x_1, x_3, \ldots, x_n)$ is also denormalized Lorentzian.*

Proof. Since the class of Lorentzian polynomials is closed under scaling of variables with positive numbers, so is the class of denormalized Lorentzian polynomials (since N commutes with scaling). Thus we can assume that $\mu = \lambda = 1$. Let T be the linear operator defined by

$$T[p](x_1, \ldots, x_n) \,=\, p(x_1, x_1, x_3, \ldots, x_n).$$

We need to prove that the operator $S = N \circ T \circ N^{-1}$ preserves the class of Lorentzian polynomials. The symbol of S (see [BH20, §6]), is

$$S[(x_1 + y_1)^d \cdots (x_n + y_n)^d] \,=\, (d!)^{-2}\, (x_3 + y_3)^d \cdots (x_n + y_n)^d \sum_{0 \le k, \ell \le d} \frac{x_1^{k+\ell}}{(k+\ell)!}\, \frac{y_1^{d-k}}{(d-k)!}\, \frac{y_2^{d-\ell}}{(d-\ell)!}\,.$$

Up to scalar this is the generating polynomial of an M-convex set, and so it is a Lorentzian polynomial (see [BH20, Thm 7.1]). Therefore, by [BH20, Thm 6.2], the operator S preserves the class of Lorentzian polynomials. □

5. Capacity bounds on coefficients

5.1. **Preliminaries.** Applications of polynomial capacity bounds on stable and Lorentzian polynomials to combinatorics was pioneered by Gurvits in the mid 2000s. This began with bounds for the permanent and mixed discriminant in [Gur08], and also includes applications to the mixed volume and to discrete and computational geometry more generally in [Gur09b].

In [Gur15], Gurvits used optimal improvements of bounds from [Gur09a] to prove Theorem 2.9. The main idea is that one can bound the coefficients of stable and Lorentzian (i.e. strongly log-concave or completely log-concave) polynomials. Specifically, he applies these bounds to the polynomial

$$\widetilde{P}_K(\boldsymbol{x}, \boldsymbol{y}) \,:=\, y_1^{\sum_i k_{i1}} \cdots y_n^{\sum_i k_{in}}\, P_K(\boldsymbol{x}, \boldsymbol{y}^{-1}) \,=\, \prod_{i=1}^{m} \prod_{j=1}^{n} \sum_{\ell=0}^{k_{ij}} x_i^{\ell}\, y_j^{k_{ij}-\ell},$$

where K is a matrix with 0-1 entries. The coefficients of $\widetilde{P}_K$ are precisely the number of binary contingency tables with given marginals and entrywise bound matrix K. (Note that similar bounds can be obtained from the inner product capacity bounds of [GL18+] and [AG21].)

The problem with this approach is that it does not extend to general contingency tables, since no simple operation applied to P_K yields a stable/Lorentzian polynomial in that case. To circumvent this, we instead turn to a new approach to deriving capacity bounds on coefficients of denormalized Lorentzian polynomials. We can then apply these bounds to $\widetilde{P}_K$.

Before moving on, we recall the definition of capacity and give a few basic properties that we will use throughout.

Definition 5.1. For a polynomial $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ and any non-negative vector $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, define

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) \,:=\, \inf_{\boldsymbol{x} > 0} \frac{p(\boldsymbol{x})}{\boldsymbol{x}^{\boldsymbol{\alpha}}} \,=\, \inf_{x_1, \ldots, x_n > 0} \frac{p(x_1, \ldots, x_n)}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}\,.$$

We also use this definition when p is an analytic function given by a power series with non-negative coefficients. To handle rational functions which are not analytic, see below.

Lemma 5.2 ([GL18+, Lemma 2.16]). *For any $\boldsymbol{c}, \boldsymbol{\alpha} \in \mathbb{R}_+^n$ and $m := \sum_{i=1}^n \alpha_k$, we have*

$$\mathrm{Cap}_{\boldsymbol{\alpha}}((c_1x_1 + \cdots c_nx_n)^m) \, = \, \prod_{i=1}^n \left(\frac{mc_i}{\alpha_i}\right)^{\alpha_i}.$$

Lemma 5.3 ([GL18+, Prop. 2.14]). *Let $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ be a symmetric polynomial, and let $\boldsymbol{\alpha} = \gamma \cdot \mathbf{1} \in \mathbb{R}_+^n$ be a multiple of the all-ones vector. Then:*

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) \, = \, \mathrm{Cap}_{n\gamma}\big(p(t, t, \ldots, t)\big).$$

Lemma 5.4 ([GL18+, Cor. 5.8]). *Let $p_k \in \mathbb{R}_+[x_1, \ldots, x_n]$, for $k \in \mathbb{N}$, be such that $p_k \to p$ uniformly on compact sets for some analytic function p. Then for any valid $\boldsymbol{\alpha} \in \mathbb{R}_+^n$, we have*

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) \, = \, \lim_{k\to\infty} \mathrm{Cap}_{\boldsymbol{\alpha}}(p_k).$$

We make one last comment here about the case when some of the entries of K are $+\infty$. In this case, the function P_K that we care about has some rational factors of the form $(1 - x_iy_j)^{-1}$, and hence is *not* analytic. So, we need to change notation slightly to account for this:

$$\mathrm{Cap}_{(\alpha_i,\beta_j)}\big((1 - x_iy_j)^{-1}\big) \, := \, \inf_{x_i,y_j\in(0,1)} \frac{(1 - x_iy_j)^{-1}}{x_i^{\alpha_i} y_j^{\beta_j}}.$$

Note that if we think of $(1-x_iy_j)^{-1}$ as the power series $1+x_iy_j+(x_iy_j)^2+\ldots$, then this makes intuitive sense because the infimum could only possibly be attained for $x_iy_j \in (0, 1)$. We show that this is a good definition by giving an analogue to Lemma 5.4 for this case.

Lemma 5.5. *For every $r \in \mathbb{R}_+$, we have*

$$\mathrm{Cap}_r\big((1 - t)^{-1}\big) \, = \, \lim_{k\to\infty} \mathrm{Cap}_r(1 + t + t^2 + \cdots + t^k).$$

Proof. The result follows from a standard argument about exchanging lim and inf, since both $t^{-r}(1-t)^{-1}$ and $t^{-r}(1 + t + \cdots + t^k)$ become large near the boundary and convergence is uniform elsewhere. ☐

5.2. **Weighted log-concave coefficients.** We first prove a capacity bound for bivariate homogeneous polynomials with weighted log-concave coefficients. Before saying anything more than this, we define what we mean.

Definition 5.6. Let $w(x, y) = \sum_{k=0}^n w_kx^ky^{n-k}$ and $p(x, y) = \sum_{k=0}^n p_kx^ky^{n-k}$ be bivariate homogeneous polynomials with positive coefficients. Then we say that polynomial p is *w-log-concave* if $\big\{p_k/w_k\,, 0 \le k \le n\big\}$ is a log-concave sequence.

In particular, recall that a bivariate homogeneous polynomial p is Lorentzian if and only if its coefficients form an ultra-log-concave sequence. That is, p is Lorentzian if and only if it is $(x+y)^n$-log-concave. Such weightings of log-concave coefficients have been studied in a similar context by Gurvits in [Gur09a] under the name *propagatable sequences*.

We now prove the main lemma of this section, which is a capacity bound on the coefficients of polynomials with weighted log-concave coefficients.

Lemma 5.7. *Let $w(x, y) = \sum_{k=0}^n w_kx^ky^{n-k}$ and $p(x, y) = \sum_{k=0}^n p_kx^ky^{n-k}$ be bivariate homogeneous polynomials, such that p is w-log-concave. Then for all $k \in [n]$, we have*

$$\frac{p_k}{\mathrm{Cap}_{(k,n-k)}(p)} \, \ge \, \frac{w_k}{\mathrm{Cap}_{(k,n-k)}(w)}.$$

Furthermore, this bound is sharp for every fixed k and w.

Proof. We need to compute

$$C_{k,n-k} := \sup_{\text{l.c.}\,\boldsymbol{a}>0} \inf_{x,y>0} \frac{1}{a_k x^k y^{n-k}} \sum_{j=0}^{n} w_j a_j x^j y^{n-j},$$

where the sup is over all positive log-concave sequences $\boldsymbol{a} = (a_0, \ldots, a_n)$. Since p is w-log-concave, the result is then equivalent to

$$C_{k,n-k} = \mathrm{Cap}_{(k,n-k)}(w).$$

To prove this, first note that

$$C_{k,n-k} = \sup_{\text{l.c.}\,\boldsymbol{a}>0} \inf_{x,y>0} \frac{1}{a_k x^k y^{n-k}} \sum_{j=0}^{n} w_j a_j x^j y^{n-j} = \sup_{\substack{\text{l.c.}\,\boldsymbol{a}>0\\ a_k=1}} \inf_{x>0} \left(\sum_{j=0}^{n} w_j a_j x^{j-k}\right).$$

For positive log-concave sequence $\boldsymbol{a} = (a_0, \ldots, a_n)$ with $a_k = 1$, we have the inequalities

$$a_{k-j} \le a_{k-1}^j \quad \text{and} \quad a_{k+j} \le a_{k+1}^j$$

for every valid $j \ge 0$. This implies

$$C_{k,n-k} = \sup_{a_{k-1}\,a_{k+1}\le 1} \inf_{x>0} \left[\left(\sum_{j=0}^{k-1} w_j a_{k-1}^{k-j} x^{j-k}\right) + w_k + \left(\sum_{j=k+1}^{n} w_j a_{k+1}^{j-k} x^{j-k}\right)\right].$$

We can further restrict to $a_{k-1}a_{k+1} = 1 \iff a_{k-1} = a_{k+1}^{-1}$ in the sup, which implies

$$C_{k,n-k} = \sup_{a_{k+1}>0} \inf_{x>0} \left(\sum_{j=0}^{n} w_j a_{k+1}^{j-k} x^{j-k}\right) = \inf_{x>0} \left(\sum_{j=0}^{n} w_j x^{j-k}\right).$$

Therefore $C_{k,n-k} = \mathrm{Cap}_{(k,n-k)}(w)$, which implies the result. Sharpness of the bound is then achieved by setting $p = w$. ☐

Using this lemma, we derive corollaries for specific polynomials pertinent to the polynomial $\widetilde{P}_K$. Note that in this first result we make a simplification to obtain a nice expression for the bound, and therefore the bound is not sharp.

Corollary 5.8. *Let $p(x,y) = \sum_{k=0}^{n} p_k x^k y^{n-k}$ be such that $p_0, \ldots, p_n$ is a positive log-concave sequence. We have:*

$$\frac{p_k}{\mathrm{Cap}_{(k,n-k)}(p)} \ge \max\left\{\frac{k^k}{(k+1)^{k+1}},\, \frac{(n-k)^{n-k}}{(n-k+1)^{n-k+1}}\right\},$$

for every $k \in [n]$. Further, this bound is sharp up to a factor of $\frac{e}{2}$ for every fixed $k \in [n]$.

Proof. We compute

$$\mathrm{Cap}_{(k,n-k)}\left(\sum_{j=0}^{n} x^j y^{n-j}\right) = \inf_{x>0} \sum_{j=0}^{n} x^{j-k} \le \inf_{x\in(0,1)} \left[x^{-k}(1-x)^{-1}\right].$$

Basic calculus gives

$$\sup_{x\in(0,1)} \left[x^k - x^{k+1}\right] = \frac{k^k}{(k+1)^{k+1}}.$$

Combined with the previous lemma this implies

$$p_k \ge \frac{\mathrm{Cap}_{(k,n-k)}(p)}{\mathrm{Cap}_{(k,n-k)}\left(\sum_{j=0}^{n} x^j y^{n-j}\right)} \ge \frac{k^k}{(k+1)^{k+1}}\, \mathrm{Cap}_{(k,n-k)}(p).$$

We also have

$$\mathrm{Cap}_{(k,n-k)}\left(\sum_{j=0}^{n} x^j y^{n-j}\right) = \inf_{y>0} \sum_{j=0}^{n} y^{(n-j)-(n-k)} \le \inf_{y\in(0,1)} \left[y^{n-k}(1-y)\right]^{-1}.$$

The same argument then implied the bound in the corollary and finishes the proof of the first part.

For the second part, first note that for $m := \min(k, n-k)$ we have

$$\mathrm{Cap}_{(k,n-k)}\left(\sum_{j=0}^{n} x^j y^{n-j}\right) \;\geq\; \mathrm{Cap}_{(m,m)}\left(\sum_{j=0}^{2m} x^j y^{2m-j}\right) \;=\; 2m+1,$$

by symmetry and Lemma 5.3. Therefore,

$$\frac{m^m}{(m+1)^{m+1}} \;\leq\; \left[\mathrm{Cap}_{(k,n-k)}\left(\sum_{j=0}^{n} x^j y^{n-j}\right)\right]^{-1} \;\leq\; \frac{1}{2m+1}\,.$$

By a calculus argument, we further have

$$\frac{2}{e}\cdot\frac{1}{2m+1} \;\leq\; \frac{m^m}{(m+1)^{m+1}}\,,$$

and this completes the proof. □

Corollary 5.9. *Let $p(x,y) = \sum_{k=0}^{n} p_k x^k y^{n-k}$ be a bivariate Lorentzian polynomial. For each $k \in [n]$, we have*

$$\frac{p_k}{\mathrm{Cap}_{(k,n-k)}(p)} \;\geq\; \binom{n}{k}\frac{k^k(n-k)^{n-k}}{n^n}\,.$$

Further, this bound is sharp for every fixed $k \in [n]$.

Proof. Recall that a bivariate homogeneous polynomial is Lorentzian if and only if it is $(x+y)^n$-log-concave. By Lemma 5.2, we have

$$\mathrm{Cap}_{(k,n-k)}\left(\sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\right) \;=\; \mathrm{Cap}_{(k,n-k)}\big((x+y)^n\big) \;=\; \frac{n^n}{k^k(n-k)^{n-k}}\,.$$

The k-th coefficient of $(x+y)^n$ is $\binom{n}{k}$, and this completes the proof. □

5.3. **Real stable and denormalized Lorentzian polynomials.** In this section, we emulate Gurvits's proof of Theorem 5.1 of [Gur15] to obtain bounds for real stable and denormalized Lorentzian polynomials. First, we prove our main bound on coefficients of denormalized Lorentzian polynomials.

Theorem 5.10. *Let $p \in \mathbb{R}_+[x_1,\ldots,x_n]$ be a denormalized Lorentzian polynomial of degree d, given by*

$$p(x_1,\ldots,x_n) \;=\; \sum_{\mu_1+\ldots+\mu_n=d} p_{\boldsymbol{\mu}}\,\boldsymbol{x}^{\boldsymbol{\mu}}.$$

Let d_i be the degree of x_i in

$$\partial_{i+1}^{\alpha_{i+1}}\cdots\partial_n^{\alpha_n} p\big|_{x_{i+1}=\ldots=x_n=0}\,, \quad \textit{for all} \quad 1 \leq i \leq n-1,$$

and let d_n be the degree of x_n in p. Then, for all $\boldsymbol{\alpha} \in \mathbb{N}^n$, such that $\alpha_1+\cdots+\alpha_n = d$, we have:

$$\frac{p_{\boldsymbol{\alpha}}}{\mathrm{Cap}_{\boldsymbol{\alpha}}(p)} \;\geq\; \left[\prod_{i=2}^{n}\mathrm{Cap}_{(d_i-\alpha_i,\alpha_i)}\left(\sum_{k=0}^{d_i} x^k y^{d_i-k}\right)\right]^{-1} \;\geq\; \prod_{i=2}^{n}\max\left\{\frac{\alpha_i^{\alpha_i}}{(\alpha_i+1)^{\alpha_i+1}},\, \frac{(d_i-\alpha_i)^{d_i-\alpha_i}}{(d_i-\alpha_i+1)^{d_i-\alpha_i+1}}\right\}.$$

Proof. The proof is by induction over $n \geq 2$. The case $n=2$ is Lemma 5.7 and Corollary 5.8. Let $n > 2$, and write

$$p(x_1,\ldots,x_n) \;=\; \sum_{i=0}^{d} x_n^{d-i}\,p_i(x_1,\ldots,x_{n-1}).$$

Then for positive $y_1,\ldots,y_{m-1}$, Lemma 4.8 implies that

$$p(y_1 t,\ldots,y_{n-1}t,s) \;=\; \sum_{i=0}^{d} s^{d-i}t^i p_i(y_1,\ldots,y_{n-1})$$

is denormalized Lorentzian. Now for $s, t \geq 0$, we have

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) \leq \frac{p(y_1 t, \ldots, y_{n-1} t, s)}{(ty_1)^{\alpha_1} \cdots (ty_{n-1})^{\alpha_{n-1}} s^{\alpha_n}} = \frac{\sum_{i=0}^{d_n} s^{d_n - i} t^i p_i(y_1, \ldots, y_{n-1})}{t^{d_n - \alpha_n} s^{\alpha_n} y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}}},$$

which implies

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) \leq \frac{\mathrm{Cap}_{(d_n - \alpha_n, \alpha_n)} \left(\sum_{i=0}^{d_n} s^{d_n - i} t^i p_i(y_1, \ldots, y_{n-1}) \right)}{y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}}}.$$

Clearly the polynomial $q(t,s) := t^{d_n - d} \cdot p(y_1 t, \ldots, y_{n-1} t, s) = \sum_{i=0}^{d_n} s^{d_n - i} t^i p_i(y_1, \ldots, y_{n-1})$ is also denormalized Lorentzian, with coefficients given by $p_i(y_1, \ldots, y_{n-1})$ for any chosen positive $y_1, \ldots, y_{m-1}$. In particular the coefficients of q form a log-concave sequence, and thus by Lemma 5.7 we obtain

$$\frac{\mathrm{Cap}_{(d_n - \alpha_n, \alpha_n)} \left(\sum_{i=0}^{d_n} s^{d_n - i} t^i p_i(y_1, \ldots, y_{n-1}) \right)}{y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}}} \leq \frac{\mathrm{Cap}_{(d_n - \alpha_n, \alpha_n)} \left(\sum_{k=0}^{d_n} x^k y^{d_n - k} \right) \cdot p_{\alpha_n}(y_1, \ldots, y_{n-1})}{y_1^{\alpha_1} \cdots y_{n-1}^{\alpha_{n-1}}},$$

which implies

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) \leq \mathrm{Cap}_{(d_n - \alpha_n, \alpha_n)} \left(\sum_{k=0}^{d_n} x^k y^{d_n - k} \right) \cdot \mathrm{Cap}_{(\alpha_1, \ldots, \alpha_{n-1})}(p_{\alpha_n}).$$

Therefore, by Corollary 5.8, we conclude

$$\begin{aligned}
\mathrm{Cap}_{(\alpha_1, \ldots, \alpha_{n-1})}(p_{\alpha_n}) &\geq \frac{\mathrm{Cap}_{\boldsymbol{\alpha}}(p)}{\mathrm{Cap}_{(d_n - \alpha_n, \alpha_n)} \left(\sum_{k=0}^{d_n} x^k y^{d_n - k} \right)} \\
&\geq \max \left\{ \frac{\alpha_i^{\alpha_i}}{(\alpha_i + 1)^{\alpha_i + 1}}, \frac{(d_i - \alpha_i)^{d_i - \alpha_i}}{(d_i - \alpha_i + 1)^{d_i - \alpha_i + 1}} \right\} \mathrm{Cap}_{\boldsymbol{\alpha}}(p).
\end{aligned}$$

The result follows by induction. □

Essentially the same proof also works for real stable polynomials, giving the same bound achieved by Gurvits. The proof we give here is similar to Gurvits's, but we have put it in our language for the sake of clarity and easy comparison to the proof of Theorem 5.10.

Theorem 5.11 ([Gur15, Thm 5.1]). *Let $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ be a real stable polynomial of degree d, given by*

$$p(x_1, \ldots, x_n) = \sum_{\mu_1 + \cdots + \mu_n = d} p_{\boldsymbol{\mu}} \boldsymbol{x}^{\boldsymbol{\mu}}.$$

Let d_i be the degree of x_i in

$$\partial_{i+1}^{\alpha_{i+1}} \cdots \partial_n^{\alpha_n} p \big|_{x_{i+1} = \ldots = x_n = 0}, \quad i = 1, \ldots, n-1,$$

and d_n, the degree of x_n in p. For any $\boldsymbol{\alpha} \in \mathbb{N}_+^n$ such that $\alpha_1 + \ldots + \alpha_n = d$, we have

$$\frac{p_{\boldsymbol{\alpha}}}{\mathrm{Cap}_{\boldsymbol{\alpha}}(p)} \geq \prod_{i=2}^{n} \binom{d_i}{\alpha_i} \frac{\alpha_i^{\alpha_i} (d_i - \alpha_i)^{d_i - \alpha_i}}{d_i^{d_i}}.$$

Proof. The proof is by induction over $n \geq 2$. The case of $n = 2$ is Corollary 5.9. Now, every step of the induction of the proof of Theorem 5.10 then holds for real stable polynomials, with $\sum_{i=0}^{d_n} x^i y^{d_n - i}$ replaced by $(x+y)^{d_i}$. The main difference is that in the second to last step we apply Corollary 5.9 to get

$$\mathrm{Cap}_{\boldsymbol{\alpha}}(p) \leq \binom{d_n}{\alpha_n}^{-1} \frac{d_n^{d_n}}{\alpha_n^{\alpha_n} (d_n - \alpha_n)^{d_n - \alpha_n}} \cdot \frac{p_{\alpha_n}(y_1, \ldots, y_{n-1})}{y_1^{\alpha_1} \cdots y_n^{\alpha_n}},$$

for all $y_1, \ldots, y_{n-1} > 0$. This implies

$$\mathrm{Cap}_{(\alpha_1, \ldots, \alpha_{n-1})}(p_{\alpha_n}) \geq \binom{d_n}{\alpha_n} \frac{\alpha_n^{\alpha_n} (d_n - \alpha_n)^{d_n - \alpha_n}}{d_n^{d_n}} \cdot \mathrm{Cap}_{\boldsymbol{\alpha}}(p),$$

which proves the step of induction. □

6. Proofs of the results

In this section we prove the results in Sections 2 and 3. We obtain bounds on $\mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta})$ for various K, and on probabilities that a random contingency table will have marginals $(\boldsymbol{\alpha},\boldsymbol{\beta})$ when the entries are chosen from binomial and Poisson distributions. The proofs of these facts all have the same form:
(1) determine a polynomial whose coefficients hold the information that we want to bound,
(2) transform that polynomial until it is real stable or denormalized Lorentzian, and then
(3) apply the capacity bounds of the previous section.

6.1. **Proof of Theorem 2.1 and Corollary 2.2.** Recall that $\mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta})$ is the coefficient of $\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{y}^{\boldsymbol{\beta}}$ in the polynomial

$$P_K(\boldsymbol{x},\boldsymbol{y}) \;:=\; \prod_{i=1}^m \prod_{j=1}^n \sum_{\ell=0}^{k_{ij}} x_i^\ell y_j^\ell$$

for every $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and every $K=(k_{ij})$. When the entries of K are finite, we invert the $\boldsymbol{y}$ variables to get

$$\widetilde{P}_K(\boldsymbol{x},\boldsymbol{y}) \;:=\; y_1^{\sum_i k_{i1}}\cdots y_n^{\sum_i k_{in}}\, P_K(\boldsymbol{x},\boldsymbol{y}^{-1}) \;=\; \prod_{i=1}^m \prod_{j=1}^n \sum_{\ell=0}^{k_{ij}} x_i^\ell y_j^{k_{ij}-\ell}.$$

The polynomial $\widetilde{P}_K$ is a product of denormalized Lorentzian polynomials. By Proposition 4.6, this implies $\widetilde{P}_K$ is also denormalized Lorentzian. Therefore, we can apply Theorem 5.10 to $\widetilde{P}_K$. Since $\widetilde{P}_K$ is of degree $\lambda_i := \sum_j k_{ij}$ in x_i for all i, and of degree $\gamma_j := \sum_i k_{ij}$ in y_j for all j, we obtain the following for any valid $\boldsymbol{\alpha},\boldsymbol{\beta}$:

$$\frac{[\widetilde{P}_K]_{\boldsymbol{\alpha}\,(\boldsymbol{\gamma}-\boldsymbol{\beta})}}{\mathrm{Cap}_{\boldsymbol{\alpha}\,(\boldsymbol{\gamma}-\boldsymbol{\beta})}(\widetilde{P}_K)} \;\geq\; \prod_{i=2}^m \frac{\alpha_i^{\alpha_i}}{(\alpha_i+1)^{\alpha_i+1}} \;\prod_{j=1}^n \frac{\beta_j^{\beta_j}}{(\beta_j+1)^{\beta_j+1}}\,.$$

Here $[\widetilde{P}_K]_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}$ denotes the coefficient of $\widetilde{P}_K$ corresponding to the monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{y}^{\boldsymbol{\beta}}$. Finally, it is straightforward to see that

$$\frac{\mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta})}{\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_K)} \;=\; \frac{[P_K]_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}}{\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_K)} \;=\; \frac{[\widetilde{P}_K]_{\boldsymbol{\alpha}\,(\boldsymbol{\gamma}-\boldsymbol{\beta})}}{\mathrm{Cap}_{\boldsymbol{\alpha}\,(\boldsymbol{\gamma}-\boldsymbol{\beta})}(\widetilde{P}_K)}\,.$$

We now simplify this bound. Note first that for any $k\in\mathbb{N}$, we have:

$$\frac{k^k}{(k+1)^{k+1}} \;=\; \frac{1}{k+1}\left(\frac{k}{k+1}\right)^k \;\geq\; \frac{1}{e(k+1)}\,.$$

Combining this with the above bound gives

$$\begin{aligned}\frac{\mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta})}{\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_K)} \;&\geq\; \prod_{i=2}^m \frac{\alpha_i^{\alpha_i}}{(\alpha_i+1)^{\alpha_i+1}} \;\prod_{j=1}^n \frac{\beta_j^{\beta_j}}{(\beta_j+1)^{\beta_j+1}} \\ &\geq\; \frac{1}{e^{m+n-1}} \prod_{i=2}^m \frac{1}{\alpha_i+1} \;\prod_{j=1}^n \frac{1}{\beta_j+1}\,.\end{aligned}$$

Finally if K has some entries which are $+\infty$, then we can choose large finite numbers for the those entries, apply the previous argument, and limit to $+\infty$ (see Lemma 5.4). $\square$

Remark 6.1. Note that we did not use the full strength of Theorem 5.10 here, which allows us to make the following replacements:

$$\frac{\alpha_i^{\alpha_i}}{(\alpha_i+1)^{\alpha_i+1}} \;\longrightarrow\; \max\left\{\frac{\alpha_i^{\alpha_i}}{(\alpha_i+1)^{\alpha_i+1}},\, \frac{(\lambda_i-\alpha_i)^{\lambda_i-\alpha_i}}{(\lambda_i-\alpha_i+1)^{\lambda_i-\alpha_i+1}}\right\}$$

and

$$\frac{\beta_j^{\beta_j}}{(\beta_j+1)^{\beta_j+1}} \;\longrightarrow\; \max\left\{\frac{\beta_j^{\beta_j}}{(\beta_j+1)^{\beta_j+1}},\, \frac{(\gamma_j-\beta_j)^{\gamma_j-\beta_j}}{(\gamma_j-\beta_j+1)^{\gamma_j-\beta_j+1}}\right\}.$$

Via the above simplification, we then obtain the stronger bound

$$\frac{\mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta})}{\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_K)} \;\geq\; \frac{1}{e^{m+n-1}} \prod_{i=2}^m \frac{1}{\min\{\alpha_i,\lambda_i-\alpha_i\}+1} \;\prod_{j=1}^n \frac{1}{\min\{\beta_j,\gamma_j-\beta_j\}+1}\,,$$

where $\lambda_i := \sum_j k_{ij}$ for all i, and $\gamma_j := \sum_i k_{ij}$ for all j.

6.2. **Proof of Theorem 2.3.** In this section, we obtain a simply exponential approximation factor for $\mathrm{CT}(\boldsymbol{\alpha},\boldsymbol{\beta})$ in the case that $\alpha_i, \beta_j \leq c$ for $i, j \geq 2$. Letting $N := \sum_{i=1}^m \alpha_i = \sum_{j=1}^n \beta_i$ (which is dominated by the value of α_1 and β_1), we will compute $\mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta})$, where we may assume K to be the matrix with each entry defined by $k_{ij} := \min(\alpha_i, \beta_j)$. By Remark 6.1, we obtain the bound

$$\begin{aligned}
\frac{\mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta})}{\mathrm{Cap}_{\boldsymbol{\alpha}\boldsymbol{\beta}}(P_K)} &\geq \frac{1}{e^{m+n-1}} \prod_{i=2}^m \frac{1}{\min\{\alpha_i, \lambda_i - \alpha_i\}+1} \prod_{j=1}^n \frac{1}{\min\{\beta_j, \gamma_j - \beta_j\}+1} \\
&\geq \frac{1}{e^{m+n-1}} \cdot \frac{1}{\gamma_1 - \beta_1 + 1} \prod_{i=2}^m \frac{1}{c+1} \prod_{j=2}^n \frac{1}{c+1} \\
&\geq \frac{1}{e^{m+n-1}} \cdot \frac{1}{(m-1)c+1} \cdot \frac{1}{(c+1)^{m+n-2}} \\
&\geq \frac{1}{(m+n-1) \cdot (e(c+1))^{m+n-1}},
\end{aligned}$$

where $\lambda_i := \sum_j k_{ij}$ for all i, and $\gamma_j := \sum_i k_{ij}$ for all j. Since $\mathrm{CT}_K(\boldsymbol{\alpha},\boldsymbol{\beta}) = \mathrm{CT}(\boldsymbol{\alpha},\boldsymbol{\beta})$ in this case, we obtain

$$\mathrm{Cap}_{\boldsymbol{\alpha}\boldsymbol{\beta}}(P_K) \geq \mathrm{CT}(\boldsymbol{\alpha},\boldsymbol{\beta}) \geq \frac{1}{(m+n-1) \cdot (e(c+1))^{m+n-1}} \cdot \mathrm{Cap}_{\boldsymbol{\alpha}\boldsymbol{\beta}}(P_K).$$

This finishes the proof. $\square$

6.3. **Proof of Theorem 3.1.** We now prove the probability bound in the case where the (i,j)-th entry of the table is sampled according to a binomial distribution on $\{0, 1, \ldots, k_{ij}\}$ with parameter $s \in [0,1]$. The probability of selecting a contingency table with marginals $(\boldsymbol{\alpha},\boldsymbol{\beta})$ in this case is given by the coefficient of $\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{y}^{\boldsymbol{\beta}}$ in the polynomial

$$Q_{K,s}(\boldsymbol{x},\boldsymbol{y}) := \prod_{i=1}^m \prod_{j=1}^n \big(s x_i y_j + (1-s)\big)^{k_{i,j}}$$

for all $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $K = (k_{ij})$ with finite entries. Note that $Q_{K,s}(\mathbf{1},\mathbf{1}) = 1$. We can invert the $\boldsymbol{y}$ variables to get

$$\widetilde{Q}_{K,s}(\boldsymbol{x},\boldsymbol{y}) := y_1^{\sum_i k_{i1}} \cdots y_n^{\sum_i k_{in}} Q_{K,s}(\boldsymbol{x},\boldsymbol{y}^{-1}) = \prod_{i=1}^m \prod_{j=1}^n \big(s x_i + (1-s) y_j\big)^{k_{ij}}.$$

This polynomial is real stable, and we can apply Theorem 5.11. Since $\widetilde{Q}_{K,s}$ is of degree $\lambda_i := \sum_j k_{ij}$ in x_i for all i, and of degree $\gamma_j := \sum_i k_{ij}$ in y_j for all j, we obtain the following for all valid $\boldsymbol{\alpha},\boldsymbol{\beta}$:

$$\frac{[\widetilde{Q}_{K,s}]_{\boldsymbol{\alpha},(\boldsymbol{\gamma}-\boldsymbol{\beta})}}{\mathrm{Cap}_{\boldsymbol{\alpha},(\gamma-\boldsymbol{\beta})}(\widetilde{Q}_{K,s})} \geq \prod_{i=2}^m \binom{\lambda_i}{\alpha_i} \frac{\alpha_i^{\alpha_i}(\lambda_i - \alpha_i)^{\lambda_i - \alpha_i}}{\lambda_i^{\lambda_i}} \prod_{j=1}^n \binom{\gamma_j}{\beta_j} \frac{\beta_j^{\beta_j}(\gamma_j - \beta_j)^{\gamma_j - \beta_j}}{\gamma_j^{\gamma_j}}.$$

Here, $[\widetilde{Q}_{K,s}]_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ denotes the coefficient of $\widetilde{Q}_{K,s}$ corresponding to the monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{y}^{\boldsymbol{\beta}}$. It is then straightforward to see that

$$\frac{[Q_{K,s}]_{\boldsymbol{\alpha},\boldsymbol{\beta}}}{\mathrm{Cap}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(Q_{K,s})} = \frac{[\widetilde{Q}_{K,s}]_{\boldsymbol{\alpha},(\boldsymbol{\gamma}-\boldsymbol{\beta})}}{\mathrm{Cap}_{\boldsymbol{\alpha},(\boldsymbol{\gamma}-\boldsymbol{\beta})}(\widetilde{Q}_{K,s})}.$$

This gives the desired bound. $\square$

6.4. **Proof of Theorem 2.9.** The bound in this case follows from the binomial-distributed case (§6.3 above), up to scalar when K is a 0-1 matrix and $s = \frac{1}{2}$. The details are straightforward. $\square$

6.5. **Proof of Theorem 3.2.** The equality follows from the same sort of arguments used to prove Lemma 5.3 (2) of [Bar12]. However, another proof can be given using the following nice capacity-theoretic result, which we present for completeness.

Proposition 6.2. *Given $p_1, \ldots, p_m \in \mathbb{R}_+[x_1, \ldots, x_n]$ and $\boldsymbol{\alpha} \in \mathbb{R}^n_+$, we have:*

$$\operatorname{Cap}_{\boldsymbol{\alpha}}\left(\prod_{k=1}^m p_k\right) = \sup_{\substack{\boldsymbol{\beta}^1, \ldots, \boldsymbol{\beta}^m \in \mathbb{R}^n_+ \\ \boldsymbol{\beta}^1 + \ldots + \boldsymbol{\beta}^m = \boldsymbol{\alpha}}} \prod_{k=1}^m \operatorname{Cap}_{\boldsymbol{\beta}^k}(p_k).$$

Proof outline. Define $p := \prod_{k=1}^m p_k$. First, the case where $\nabla \log(p)|_{\boldsymbol{x}=\boldsymbol{1}} = \boldsymbol{\alpha}$ follows from the fact that $\operatorname{Cap}_{\boldsymbol{\gamma}}(p)$ is maximized over $\boldsymbol{\gamma} \in \mathbb{R}^n_+$ at $\boldsymbol{\gamma} = \nabla \log(p)|_{\boldsymbol{x}=\boldsymbol{1}}$, and $\operatorname{Cap}_{\boldsymbol{\alpha}}(p) = p(\boldsymbol{1})$ in this case (see Fact 2.10 of [GL18+]). Then, to handle $\boldsymbol{\alpha}$ in the relative interior of the Newton polytope of p, one can choose $\boldsymbol{r} > 0$ such that $\nabla \log\big(p(\boldsymbol{r} \cdot \boldsymbol{x})\big)\big|_{\boldsymbol{x}=\boldsymbol{1}} = \boldsymbol{\alpha}$. The result then follows from the first case. Since the case of $\boldsymbol{\alpha}$ outside the Newton polytope of p is trivial (because the capacity is 0), we only need to handle the case when $\boldsymbol{\alpha}$ is on the relative boundary of the Newton polytope of p. This can be done by a limiting argument; the details are straightforward. $\square$

Once we have this result, Theorem 3.2 follows from a straightforward application to the polynomial $Q_{K,s}$, using Lemma 5.2 to obtain explicit expressions for the capacity of the terms of the product.

6.6. **Proof of Theorem 3.3.** We now prove the probability bound in the case where the entries of the table are sampled according to the Poisson distribution on $\{0, 1, 2, \ldots\}$ with parameter $s > 0$. The probability of selecting a contingency table with marginals $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ in this case is given by the coefficient of $\boldsymbol{x}^{\boldsymbol{\alpha}} \boldsymbol{y}^{\boldsymbol{\beta}}$ in the power series of

$$Q_{\infty,s}(\boldsymbol{x}, \boldsymbol{y}) := \prod_{i=1}^m \prod_{j=1}^n e^{s x_i y_j - s} \quad \text{for all } \boldsymbol{\alpha} \text{ and } \boldsymbol{\beta}.$$

Note that $Q_{\infty,s}(\boldsymbol{1}, \boldsymbol{1}) = 1$. Because this is not a polynomial, we can't invert the $\boldsymbol{y}$ variables as we have done above. Instead, we view this case as a limit of the binomial case. In particular, note that

$$e^{s x_i y_j - s} = \lim_{d \to \infty} \frac{1}{e^s} \left(\frac{s x_i y_j}{d} + 1\right)^d$$

uniformly on compact sets. Therefore, we can consider the polynomials

$$R_{d,s}(\boldsymbol{x}, \boldsymbol{y}) := \prod_{i=1}^m \prod_{j=1}^n \frac{1}{e^s} \left(\frac{s x_i y_j}{d} + 1\right)^d \quad \text{and} \quad \widetilde{R}_{d,s}(\boldsymbol{x}, \boldsymbol{y}) := \prod_{i=1}^m \prod_{j=1}^n \frac{1}{e^s} \left(\frac{s x_i}{d} + y_j\right)^d.$$

Since $\widetilde{R}_{d,s}$ is real stable, we can apply Theorem 5.11 to get

$$\frac{[\widetilde{R}_{d,s}]_{\boldsymbol{\alpha}\,(md-\boldsymbol{\beta})}}{\operatorname{Cap}_{\boldsymbol{\alpha}\,(md-\boldsymbol{\beta})}(\widetilde{R}_{d,s})} \geq \prod_{i=2}^m \binom{nd}{\alpha_i} \frac{\alpha_i^{\alpha_i} (nd - \alpha_i)^{nd - \alpha_i}}{(nd)^{nd}} \prod_{j=1}^n \binom{md}{\beta_j} \frac{\beta_j^{\beta_j} (md - \beta_j)^{md - \beta_j}}{(md)^{md}}.$$

Further, by Stirling's approximation we have

$$\lim_{d \to \infty} \binom{nd}{\alpha_i} \frac{\alpha_i^{\alpha_i} (nd - \alpha_i)^{nd - \alpha_i}}{(nd)^{nd}} = \frac{\alpha_i^{\alpha_i}}{\alpha_i!} e^{-\alpha_i},$$

and the same holds for β_j. Combining this with the above bound gives

$$\lim_{d \to \infty} \frac{[\widetilde{R}_{d,s}]_{\boldsymbol{\alpha}\,(md-\boldsymbol{\beta})}}{\operatorname{Cap}_{\boldsymbol{\alpha}\,(md-\boldsymbol{\beta})}(\widetilde{R}_{d,s})} \geq e^{-2N + \alpha_1} \prod_{i=2}^m \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \prod_{j=1}^n \frac{\beta_j^{\beta_j}}{\beta_j!},$$

where $N := \sum_{i=1}^m \alpha_i = \sum_{j=1}^n \beta_j$. Finally since $R_{d,s} \to Q_{\infty,s}$ coefficient-wise as $d \to \infty$, Lemma 5.4 implies

$$\frac{[Q_{\infty,s}]_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}}{\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(Q_{\infty,s})} = \lim_{d\to\infty} \frac{[R_{d,s}]_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}}{\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(R_{d,s})} = \lim_{d\to\infty} \frac{[\widetilde{R}_{d,s}]_{\boldsymbol{\alpha}\,(md-\boldsymbol{\beta})}}{\mathrm{Cap}_{\boldsymbol{\alpha}\,(md-\boldsymbol{\beta})}(\widetilde{R}_{d,s})}$$
$$\geq e^{-2N+\alpha_1} \prod_{i=2}^m \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \prod_{j=1}^n \frac{\beta_j^{\beta_j}}{\beta_j!} \geq e^{-2N} \prod_{i=1}^m \frac{\alpha_i^{\alpha_i}}{\alpha_i!} \prod_{j=1}^n \frac{\beta_j^{\beta_j}}{\beta_j!}.$$

We now compute the capacity of $Q_{\infty,s}$, which has an explicit formula due to the nice form of the function. Note first that we have

$$Q_{\infty,s}(\boldsymbol{x},\boldsymbol{y}) = \prod_{i=1}^m e^{-sn + sx_i \sum_{j=1}^n y_j}.$$

So the capacity expression can be broken up as follows:

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(Q_{\infty,s}) = \inf_{\boldsymbol{y}>0} \frac{1}{\boldsymbol{y}^{\boldsymbol{\beta}}} \prod_{i=1}^m \inf_{x_i>0} \frac{e^{-sn + sx_i \sum_{j=1}^n y_j}}{x_i^{\alpha_i}}.$$

For every $i \in [m]$, note that

$$\frac{e^{-sn+sx_i \sum_{j=1}^n y_j}}{x_i^{\alpha_i}} = e^{-\alpha_i \log x_i - sn + sx_i \sum_{j=1}^n y_j}.$$

To minimize this expression, we only need to minimize the exponent. Applying calculus, we have

$$0 = \partial_{x_i}\left[-\alpha_i \log x_i - sn + sx_i \sum_{j=1}^n y_j\right] = -\frac{\alpha_i}{x_i} + s\sum_{j=1}^n y_j \implies x_i = \frac{\alpha_i}{s\sum_{j=1}^n y_j}.$$

This gives

$$\inf_{x_i>0} \frac{e^{-sn+sx_i \sum_{j=1}^n y_j}}{x_i^{\alpha_i}} = \frac{(se\sum_{j=1}^n y_j)^{\alpha_i}}{\alpha_i^{\alpha_i} e^{sn}} = \frac{(se)^{\alpha_i}}{\alpha_i^{\alpha_i} e^{sn}} \cdot \left(\sum_{j=1}^n y_j\right)^{\alpha_i},$$

which in turn implies

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(Q_{\infty,s}) = \frac{(se)^N}{\alpha^\alpha e^{smn}} \cdot \inf_{\boldsymbol{y}>0} \frac{\left(\sum_{j=1}^n y_j\right)^N}{\boldsymbol{y}^{\boldsymbol{\beta}}}.$$

By Lemma 5.2, we then have

$$\inf_{\boldsymbol{y}>0} \frac{\left(\sum_{j=1}^n y_j\right)^N}{\boldsymbol{y}^{\boldsymbol{\beta}}} = \frac{N^N}{\boldsymbol{\beta}^{\boldsymbol{\beta}}},$$

which finally implies

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(Q_{\infty,s}) = \frac{(seN)^N}{\boldsymbol{\alpha}^{\boldsymbol{\alpha}}\,\boldsymbol{\beta}^{\boldsymbol{\beta}}\, e^{smn}}.$$

Combining everything then gives the desired bounds. □

7. Comparing Bounds

In this section, we compare Barvinok's bounds to the bounds we are able to achieve in this paper for counting contingency tables. To simplify the computations, we use Stirling's approximation indiscriminately for every factorial that appears. The approximation is in general only off by at most a factor of $\frac{e}{\sqrt{2\pi}}$, and it holds asymptotically as $\min\{\alpha_i, \beta_j\} \to \infty$ and $\min\{m, n\} \to \infty$. This is the meaning in which we use "$\approx$" and "$\gtrsim$".

7.1. **New bound vs. Barvinok's first bound.** In [Bar09], Barvinok achieves the following constant in the general case

$$C_{\mathrm{Barv}}(K,\boldsymbol{\alpha},\boldsymbol{\beta}) \;=\; \frac{\Gamma(\frac{m+n}{2})}{2\,e^5\,\pi^{\frac{m+n-2}{2}}\,mn\,(N+mn)}\left(\frac{2}{(mn)^2\,(N+1)\,(N+mn)}\right)^{m+n-1}$$
$$\times\;\frac{N!\,(N+mn)!\,(mn)^{mn}}{N^N\,(N+mn)^{N+mn}\,(mn)!}\prod_{i=1}^m\frac{\alpha_i^{\alpha_i}}{\alpha_i!}\prod_{j=1}^n\frac{\beta_j^{\beta_j}}{\beta_j!}\,,$$

where $N=\sum_i\alpha_i=\sum_j\beta_j$. We now compare to our constant:

$$C_{\mathrm{new}}(K,\boldsymbol{\alpha},\boldsymbol{\beta}) \;=\; \prod_{i=2}^m\frac{\alpha_i^{\alpha_i}}{(\alpha_i+1)^{\alpha_i+1}}\prod_{j=1}^n\frac{\beta_j^{\beta_j}}{(\beta_j+1)^{\beta_j+1}} \;\approx\; \frac{1}{e^{m+n-1}}\prod_{i=2}^m\frac{1}{\alpha_i}\prod_{j=1}^n\frac{1}{\beta_j}\,.$$

First note that

$$\frac{N!\,(N+mn)!\,(mn)^{mn}}{N^N\,(N+mn)^{N+mn}\,(mn)!}\prod_{i=1}^m\frac{\alpha_i^{\alpha_i}}{\alpha_i!}\prod_{j=1}^n\frac{\beta_j^{\beta_j}}{\beta_j!} \;\approx\; \sqrt{\frac{N(N+mn)}{(2\pi)^{m+n-1}mn}\prod_{i=1}^m\frac{1}{\alpha_i}\prod_{j=1}^n\frac{1}{\beta_j}}\,.$$

We then further have

$$\frac{\Gamma(\frac{m+n}{2})}{2\,e^5\,\pi^{\frac{m+n-2}{2}}\,mn\,(N+mn)}\left(\frac{2}{(mn)^2\,(N+1)\,(N+mn)}\right)^{m+n-1}$$
$$\approx\;\frac{\pi}{2\,e^{\frac{11}{2}}\,mn\,(N+mn)}\left(\frac{\sqrt{2(m+n)}}{\sqrt{e\pi}(mn)^2\,(N+1)\,(N+mn)}\right)^{m+n-1}.$$

Combining these approximate equalities, gives

$$C_{\mathrm{Barv}}(K,\boldsymbol{\alpha},\boldsymbol{\beta}) \;\approx\; \left(\frac{\sqrt{m+n}}{\pi\sqrt{e}\,(mn)^2\,(N+1)\,(N+mn)}\right)^{n+m-1}\sqrt{\frac{\pi^2\,N}{4\,e^{11}\,(mn)^3\,(N+mn)}\prod_{i=1}^m\frac{1}{\alpha_i}\prod_{j=1}^n\frac{1}{\beta_j}}\,.$$

We also have the more amenable bound

$$C_{\mathrm{Barv}}(K,\boldsymbol{\alpha},\boldsymbol{\beta}) \lesssim \left(\frac{\sqrt{m+n}}{\pi\sqrt{e}\,(mn)^2\,(N+1)\,(N+mn)}\right)^{n+m-1}\sqrt{\prod_{i=2}^m\frac{1}{\alpha_i}\prod_{j=1}^n\frac{1}{\beta_j}}\,.$$

To compare to our constant $C_{\mathrm{new}}(K,\alpha,\beta)$ we use the easy bound

$$\prod_{i=2}^m\alpha_i\prod_{j=1}^n\beta_j \;\le\; N^{m+n-1}\,,$$

which leads to

$$\frac{C_{\mathrm{new}}(K,\boldsymbol{\alpha},\boldsymbol{\beta})}{C_{\mathrm{Barv}}(K,\boldsymbol{\alpha},\boldsymbol{\beta})} \gtrsim \left(\frac{\pi\,(mn)^2\,(N+1)\,(N+mn)}{\sqrt{e\,(m+n)}}\right)^{m+n-1}\sqrt{\prod_{i=2}^m\frac{1}{\alpha_i}\prod_{j=1}^n\frac{1}{\beta_j}}$$
$$\gtrsim\;\left(N^{m+n-1}\right)^2\sqrt{\prod_{i=2}^m\frac{1}{\alpha_i}\prod_{j=1}^n\frac{1}{\beta_j}} \;\gtrsim\; \left(N^{m+n-1}\right)^{\frac{3}{2}}\,.$$

That is, our lower bound (and approximation ratio) improves upon Barvinok's by at least the above factor.

7.2. **New bound vs. Barvinok's second bound.** There are now two features of this bound that we want to compare to ours: the approximation ratio and the actual lower bound. For every valid K, we first note that

$$H_N(\boldsymbol{x},\boldsymbol{y}) \,\le\, P_K(\boldsymbol{x},\boldsymbol{y}) \quad \text{for all} \quad \boldsymbol{x},\boldsymbol{y} \,>\, 0, \tag{7.1}$$

since H_N and P_K have the same coefficients on the support of H_N. (See Section 2.6 for the definition of H_N.) So in fact, if our approximation ratio is better than Barvinok's ration $C_{\mathrm{H}}(\boldsymbol{\alpha},\boldsymbol{\beta})$, then so is our lower bound.

To compare approximation ratios, we assume the β_j's maximize $C_{\mathrm{H}}(\boldsymbol{\alpha},\boldsymbol{\beta})$ and partially apply Stirling's approximation to get

$$C_{\mathrm{H}}(\boldsymbol{\alpha},\boldsymbol{\beta}) \,\approx\, \binom{N+m-1}{m-1}^{-1}\binom{N+n-1}{n-1}^{-1}\left(\frac{1}{\sqrt{2\pi}}\right)^{n-1}\sqrt{\frac{N}{\prod_{j=1}^n \beta_j}}\,.$$

This then gives

$$\frac{C_{\mathrm{new}}(\boldsymbol{\alpha},\boldsymbol{\beta})}{C_{\mathrm{H}}(\boldsymbol{\alpha},\boldsymbol{\beta})} \,\approx\, \binom{N+m-1}{m-1}\binom{N+n-1}{n-1}\frac{(\sqrt{2\pi})^{n-1}}{e^{m+n-1}\sqrt{N}}\prod_{i=2}^m \frac{1}{\alpha_1+1}\prod_{j=1}^n \frac{\sqrt{\beta_j}}{\beta_j+1}\,.$$

We now make a few simple observations. First, if $k \ll n$, by Stirling's approximation we have

$$\binom{n+k}{k} \,\approx\, \sqrt{\frac{n+k}{2\pi nk}}\cdot\frac{(n+k)^{n+k}}{k^k n^n} \,\approx\, \sqrt{\frac{n+k}{2\pi nk}}\left(\frac{e(n+k)}{k}\right)^k.$$

Then by the AM–GM inequality, we obtain:

$$\prod_{j=1}^n \frac{1}{\beta_j+1} \,\ge\, \left(\frac{n}{N+n}\right)^n \,\approx\, e^n\binom{N+n}{n}^{-1}\sqrt{\frac{N+n}{2\pi Nn}} \,=\, e^n\binom{N+n-1}{n-1}^{-1}\sqrt{\frac{n}{2\pi N(N+n)}}\,.$$

Similarly,

$$\prod_{i=2}^m \frac{1}{\alpha_i+1} \,\ge\, \left(\frac{m-1}{N-\alpha_1+m-1}\right)^{m-1} \,\gtrsim\, e^{m-1}\binom{N+m-1}{m-1}^{-1}\sqrt{\frac{N+m-1}{2\pi N(m-1)}}\,.$$

Further, it is easy to see that

$$\frac{\sqrt{\beta_j}}{\beta_j+1} \ge \frac{1}{\sqrt{2}\cdot\sqrt{\beta_j+1}} \quad \text{for} \quad \beta_j \,\ge\, 1,$$

which implies

$$\begin{aligned}\frac{C_{\mathrm{new}}(\boldsymbol{\alpha},\boldsymbol{\beta})}{C_{\mathrm{H}}(\boldsymbol{\alpha},\boldsymbol{\beta})} \,&\gtrsim\, \binom{N+n-1}{n-1}^{\frac12}\left(\frac{\pi^{2n-5}n(N+m-1)^2}{e^{2n}2^5N^5(N+n)(m-1)^2}\right)^{\frac14}\\ &\gtrsim\, \left(\frac{N+n-1}{n-1}\right)^{\frac{n-1}{2}}\left(\frac{\pi^{2n-6}n(N+m-1)^2(N+n-1)}{e^2 2^6 N^6(N+n)(m-1)^2(n-1)}\right)^{\frac14}\\ &\gtrsim\, \frac{1}{\pi^3 N\sqrt{m}}\left(\frac{\pi(N+n-1)}{n-1}\right)^{\frac{n-1}{2}}.\end{aligned}$$

Therefore, our approximation ratio improves upon Barvinok's second approximation ratio.

8. Volumes of flow and transportation polytopes

8.1. **The setup.** Let $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_m) \in \mathbb{N}^m$ and $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_n) \in \mathbb{N}^n$ be integer vectors. A *transportation polytope* $\mathcal{T}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ is the set of $m \times n$ real matrices $Z = (z_{ij})$, such that $z_{ij} \ge 0$, and

$$\sum_{i=1}^m z_{ij} \,=\, \beta_j \ \text{for all} \ \ 1 \le j \le n\,, \quad \text{and} \quad \sum_{j=1}^n z_{ij} \,=\, \alpha_i \ \text{for all} \ \ 1 \le i \le m\,. \tag{8.1}$$

Transportation polytopes are classical objects of study in geometric combinatorics and combinatorial optimization [DK14, EKK84], and their volume is one of the motivations to study contingency tables, see e.g. [Bar09, CM09].

The celebrated *Birkhoff polytope* $\mathcal{B}_n$ is a special case of the transportation polytope $\mathcal{T}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$, when $m = n$, and $\boldsymbol{\alpha} = \boldsymbol{\beta} = (1, \ldots, 1) \in \mathbb{R}^n$. It is especially well studied in its own right, see e.g. [EKK84, Pak00]. First few values of $\operatorname{vol}(\mathcal{B}_n)$ are given in [CR99, BP03a, BP03b], see also [OEIS, A037302]. This is also one of the few cases when the exact asymptotics for the volumes is known, see Example 8.4 below.

For a matrix $K = (k_{ij})$, a *flow polytope* $\mathcal{F}_{K,\boldsymbol{\alpha},\boldsymbol{\beta}}$ is defined by (8.1) and $0 \le z_{ij} \le k_{ij}$. For $K = \infty$ we obtain the transportation polytopes. Note also that the K-contingency tables are integer points in $\mathcal{F}_{K,\boldsymbol{\alpha},\boldsymbol{\beta}}$.

The volume of flow polytopes has been actively studied in connection to both discrete geometry and enumerative combinatorics. We refer to [BV08, BDV04, CDR10], and to more recent papers [B+19, MM19] for further references.

8.2. **The results.** The connection between the number of K-contingency tables and the volume of the corresponding flow polytope is given by the following:

$$\operatorname{vol}(\mathcal{F}_{K,\boldsymbol{\alpha},\boldsymbol{\beta}}) \; = \; f(S,m,n) \cdot \lim_{M\to\infty} \frac{\mathrm{CT}_{MK}(M\boldsymbol{\alpha}, M\boldsymbol{\beta})}{M^{(m-1)(n-1)}}\,,$$

where $S = \operatorname{Supp}(K)$ and the $f(S,m,n)^2$ is the covolume of the lattice $\mathbb{Z}\langle S\rangle \cap \mathbb{R}\langle \mathcal{F}_{K,\boldsymbol{\alpha},\boldsymbol{\beta}}\rangle$.

For the transportation polytopes, we have:

$$\operatorname{vol}(\mathcal{T}_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \; = \; \sqrt{m^{n-1}n^{m-1}} \cdot \lim_{M\to\infty} \frac{\mathrm{CT}(M\boldsymbol{\alpha}, M\boldsymbol{\beta})}{M^{(m-1)(n-1)}}\,,$$

where the covolume $m^{n-1}n^{m-1}$ computed e.g. in [DE85, Lemma 3].[1]

Theorem 8.1 (General lower bound)**.** *Let $\boldsymbol{\alpha} \in \mathbb{N}^m$ and $\boldsymbol{\beta} \in \mathbb{N}^n$ be such that $\sum_i \alpha_i = \sum_j \beta_j$. Let $K = (k_{ij})$ be an $m \times n$ matrix with all entries $k_{ij} \in \mathbb{N} \cup \{+\infty\}$. Then:*

$$\operatorname{vol}(\mathcal{F}_{K,\boldsymbol{\alpha},\boldsymbol{\beta}}) \; \ge \; \frac{f(S,m,n)}{e^{m+n-1}} \prod_{i=2}^{m} \frac{1}{\alpha_i} \prod_{j=1}^{n} \frac{1}{\beta_j} \; \mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}\left(\prod_{i=1}^{m} \prod_{j=1}^{n} \frac{(x_i y_j)^{k_{ij}} - 1}{\log(x_i y_j)} \right),$$

where $S = \operatorname{Supp}(K)$ and $f(S,m,n)$ are as above.

For the transportation polytopes, we get:

Theorem 8.2. *Let $\boldsymbol{\alpha} \in \mathbb{N}^m$ and $\boldsymbol{\beta} \in \mathbb{N}^n$ be such that $\sum_i \alpha_i = \sum_j \beta_j$. Then we have:*

$$\operatorname{vol}(\mathcal{T}_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \; \ge \; \frac{\sqrt{m^{n-1}n^{m-1}}}{e^{m+n-1}} \prod_{i=2}^{m} \frac{1}{\alpha_i} \prod_{j=1}^{n} \frac{1}{\beta_j} \; \mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}\left(\prod_{i=1}^{m} \prod_{j=1}^{n} \frac{-1}{\log(x_i y_j)} \right),$$

where the $\inf$ *in the capacity is over $0 < \boldsymbol{x}, \boldsymbol{y} < 1$.*

Note that $\frac{(x_i y_j)^{k_{ij}}-1}{\log(x_i y_j)}$ becomes convex after you plug in e^x and e^y and then take log on the outside. Thus, one can easily compute this capacity value using convex optimization as in the case of counting contingency tables, see §11.2.

Before we present a proof, let us single out the case of uniform marginals which are especially interesting and important in applications.

Corollary 8.3 (Uniform marginals)**.** *For $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_0) \in \mathbb{N}^m$ and $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_0) \in \mathbb{N}^n$, we have*

$$\operatorname{vol}(\mathcal{T}_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \; \ge \; \frac{(eN)^{(m-1)(n-1)}}{m^{(m-\frac{1}{2})(n-1)+1}\, n^{(n-\frac{1}{2})(m-1)}}\,.$$

Note that the results in [CM09] give the exact asymptotics only for uniform marginals with $\max\left\{\frac{m}{n}, \frac{n}{m}\right\} = O(\log n)$, while the lower bound above applies unconditionally.

[1] The covolume $m^{n-1}n^{m-1}$ is equal to the number of spanning graphs in a complete bipartite graph K_{mn}, an observation which extends to all multigraphical matrices K, cf. §11.1.

Example 8.4. For the Birkhoff polytope, the corollary gives:

$$\operatorname{vol}(\mathcal{B}_n) \,\geq\, \frac{(en)^{(n-1)^2}}{n^{2(n-\frac{1}{2})(n-1)+1}} \,=\, e^{(n-1)^2}\, n^{-(n^2-n+1)}.$$

This lower bound can be compared with the exact asymptotics given in [CM09]:

$$\operatorname{vol}(\mathcal{B}_n) \,\approx\, C\cdot(2\pi)^{-n}\, e^{n^2+O(1)}\, n^{-(n-1)^2},$$

for some known $C > 0$. Note that our lower bound coincides with the actual asymptotic bound in the first two terms:

$$\log\operatorname{vol}(\mathcal{B}_n) \,=\, -n^2\log n \,+\, n^2 \,+\, O(n\log n).$$

8.3. **Proof of Theorem 8.1.** By Corollary 2.2, we have:

$$\mathrm{CT}_{MK}(M\boldsymbol{\alpha}, M\boldsymbol{\beta}) \,\geq\, e^{1-m-n}\prod_{i=2}^{m}\frac{1}{M\alpha_i+1}\prod_{j=1}^{n}\frac{1}{M\beta_j+1}\,\mathrm{Cap}_{M\boldsymbol{\alpha}\,M\boldsymbol{\beta}}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1-(x_i y_j)^{Mk_{ij}+1}}{1-x_i y_j}\right).$$

The first thing to note is that the constant in front of the capacity is asymptotically

$$e^{1-m-n}\prod_{i=2}^{m}\frac{1}{M\alpha_i+1}\prod_{j=1}^{n}\frac{1}{M\beta_j+1} \,\approx\, M^{1-m-n}\, e^{1-m-n}\prod_{i=2}^{m}\frac{1}{\alpha_i}\prod_{j=1}^{n}\frac{1}{\beta_j}$$

as $M \to \infty$. Since the constant in the denominator of the volume limit expression is order $M^{(m-1)(n-1)}$, this means we need the order of the capacity term to be M^{mn}. This is in fact the case, and also we can get another capacity expression for the capacity term divided by M^{mn}. Specifically, note that

$$\frac{1}{M^{mn}}\inf_{\boldsymbol{x},\boldsymbol{y}>0}\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{(1-(x_i y_j)^{Mk_{ij}+1})}{x_i^{M\alpha_i}y_j^{M\beta_j}(1-x_iy_j)} \,=\, \inf_{\boldsymbol{x},\boldsymbol{y}>0}\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{x_i^{\alpha_i}y_j^{\beta_j}}\cdot\frac{(1-(x_i y_j)^{k_{ij}+M^{-1}})}{M(1-(x_iy_j)^{M^{-1}})}.$$

Next, we pass the limit on M into the infimum in the capacity (swapping lim and inf is valid here by a standard argument, since we only need to prove a lower bound). We then have:

$$\begin{aligned}
\lim_{M\to\infty}\frac{(1-(x_i y_j)^{k_{ij}+M^{-1}})}{M(1-(x_iy_j)^{M^{-1}})} &= \lim_{M\to 0^+}\frac{M(1-e^{(k_{ij}+M)\log(x_i y_j)})}{1-e^{M\log(x_i y_j)}}\\
&= \lim_{M\to 0^+}\frac{1-(1+M\log(x_i y_j))\,\cdot\, e^{(k_{ij}+M)\log(x_i y_j)}}{-\log(x_i y_j)\,\cdot\, e^{M\log(x_i y_j)}}\\
&= \frac{(x_i y_j)^{k_{ij}}-1}{\log(x_i y_j)}.
\end{aligned}$$

With this, we have

$$\operatorname{vol}(\mathcal{F}_{K,\boldsymbol{\alpha},\boldsymbol{\beta}}) \,\geq\, \frac{f(S,m,n)}{e^{m+n-1}\prod_{i=2}^{m}\alpha_i\prod_{j=1}^{n}\beta_j}\,\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{(x_i y_j)^{k_{ij}}-1}{\log(x_i y_j)}\right),$$

as desired. □

8.4. **Proof of Theorem 8.2.** The same proof works here as was used above for Theorem 8.1. The main difference is that we consider

$$\mathrm{Cap}_{M\boldsymbol{\alpha}\,M\boldsymbol{\beta}}\left(\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-x_i y_j}\right),$$

and so the infimum is over $0 < \boldsymbol{x}, \boldsymbol{y} < 1$. Another way to see this is as a limit of the lower bound of Theorem 8.1 as $K \to \infty$.

8.5. **Proof of Corollary 8.3.** We can explicitly compute the capacity in this case. Consider

$$\log\left(\frac{\prod_{i=1}^m \prod_{j=1}^n \frac{-1}{\log(e^{x_i+y_j})}}{e^{\langle \boldsymbol{x},\boldsymbol{\alpha}\rangle + \langle \boldsymbol{y},\boldsymbol{\beta}\rangle}}\right) = -\langle \boldsymbol{x},\boldsymbol{\alpha}\rangle - \langle \boldsymbol{y},\boldsymbol{\beta}\rangle - \sum_{i=1}^m \sum_{j=1}^n \log(-x_i - y_j).$$

The gradient of this expression at $x = y = -\frac{mn}{2N}$ where $N = m \cdot \alpha_0 = n \cdot \beta_0$ is then computed as

$$\left(-\alpha_0 - \sum_{j=1}^n \frac{1}{x_1 + y_j}, \ldots, -\beta_0 - \sum_{i=1}^m \frac{1}{x_i + y_1}, \ldots\right)\Bigg|_{x=y=-\frac{mn}{2N}}$$

So the above expression is minimized at $x = y = -\frac{mn}{2N}$, which means the above capacity value is

$$\frac{\prod_{i=1}^m \prod_{j=1}^n \frac{-1}{x_i+y_j}}{e^{\langle \boldsymbol{x},\boldsymbol{\alpha}\rangle+\langle \boldsymbol{y},\boldsymbol{\beta}\rangle}}\Bigg|_{x=y=-\frac{mn}{2N}} = \frac{(\frac{N}{mn})^{mn}}{e^{-mn}} = \left(\frac{eN}{mn}\right)^{mn}.$$

Combining this with the above lower bound gives an explicit lower bound on the volume in the uniform case:

$$\mathrm{vol}(\mathcal{T}_{\boldsymbol{\alpha},\boldsymbol{\beta}}) \geq \frac{\sqrt{m^{n-1}n^{m-1}}}{e^{m+n-1}\left(\frac{N}{m}\right)^{m-1}\left(\frac{N}{n}\right)^n}\left(\frac{eN}{mn}\right)^{mn} = \frac{(eN)^{(m-1)(n-1)}}{m^{(m-\frac{1}{2})(n-1)+1}\, n^{(n-\frac{1}{2})(m-1)}},$$

as desired. □

9. Uniform marginals

In the case of uniform marginals, i.e., $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are both multiples of the all-ones vector, we can explicitly compute the capacity value in the upper and lower bounds of Theorem 2.1. This then gives the explicit upper and lower bounds for the number of contingency tables.

9.1. **Explicit upper and lower bounds.** In this and the next section, we adopt the following conveenient shorthand for bounds on $\mathrm{CT}(\boldsymbol{\alpha},\boldsymbol{\beta})$:

UB1 is the Barvinok first upper bound (see Theorem 2.4),
LB1 is the Barvinok first lower bound (ibid.),
UB2 is the Barvinok second upper bound (see Theorem 2.11),
LB2 is the Barvinok second lower bound (ibid.),
UB3 is the Shapiro upper bound (see Remark 2.6), and
New LB is our lower bound in the Main Theorem 2.1.

Theorem 9.1. *Let $\boldsymbol{\alpha} = (s,\ldots,s) \in \mathbb{N}^m$, $\boldsymbol{\beta} = (t,\ldots,t) \in \mathbb{N}^n$, where $m,n,s,t \in \mathbb{N}$, such that $ms = nt = N$, and $m \leq n$. Then we have the following bounds on* $\mathrm{CT}(\boldsymbol{\alpha},\boldsymbol{\beta})$:[2]

$$\begin{aligned}
\mathrm{UB1} &= \frac{(N+mn)^{N+mn}}{N^N\,(mn)^{mn}}\\
\mathrm{LB1} &= \frac{2^{m+n-2}\,\Gamma(\frac{m+n}{2})\,N!\,(N+mn)!\;s^{sm}\,t^{tn}}{e^5\,\pi^{\frac{m+n-2}{2}}\,(N+mn)^{m+n}\,(N+1)^{m+n-1}\,N^{2N}\,(mn)^{2m+2n-1}\,(mn)!\;(s!)^m\,(t!)^n}\\
\mathrm{UB2} &= \binom{N+mn-1}{N}\\
\mathrm{LB2} &= \mathrm{UB2}\binom{N+m-1}{m-1}^{-1}\binom{N+n-1}{n-1}^{-1}\frac{N!\,s^{sm}}{N^N\,(s!)^m}\\
\mathrm{UB3} &= \mathrm{UB1}\,\frac{1}{\left(1+\frac{N}{mn}\right)^{m+n-1}} = \frac{(N+mn)^{N+(m-1)(n-1)}}{N^N(mn)^{(m-1)(n-1)}}\\
\text{New LB} &= \mathrm{UB1}\,\frac{s^{s(m-1)}\,t^{tn}}{(s+1)^{(s+1)(m-1)}\,(t+1)^{(t+1)n}}
\end{aligned}$$

[2]The LB1 is given only for $m+n \geq 10$.

Note that UB1 > UB2 in this case. This is a general fact which follows from (7.1). Note also that we trivially have UB2 > $\mathrm{CT}(\boldsymbol{\alpha},\boldsymbol{\beta})$, since UB2 counts *all* tables with sum N irrespective of the row/column constraints. That makes only the lower bounds nontrivial in this case, and possibly UB3 when s and t are large. This is confirmed by the numerical results in the next section.

9.2. **Capacity calculations.** Recall the notation

$$P_\infty(\boldsymbol{x},\boldsymbol{y}) := \prod_{i=1}^m \prod_{j=1}^n (1-x_i y_j)^{-1}.$$

The explicit computation of the capacity of P_∞ in the uniform case is then given by following result.

Lemma 9.2. *Let $\boldsymbol{\alpha}=(s,\ldots,s)\in\mathbb{N}^m$, $\boldsymbol{\beta}=(t,\ldots,t)\in\mathbb{N}^n$, where $m,n,s,t\in\mathbb{N}$, such that $ms=nt=N$. Then we have:*

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_\infty) = \frac{(N+mn)^{N+mn}}{N^N(mn)^{mn}}\,.$$

Proof. In [Bar12, Bar17], using prior work and duality, Barvinok shows (in greater generality) that the capacity bound is equal to $\exp g(Z)$, where $Z=(z_{ij})$ is the unique maximum of the strictly convex function

$$g(Z) := \sum_{i=1}^m \sum_{j=1}^n (z_{ij}+1)\log(z_{ij}+1) - z_{ij}\log z_{ij}\,,$$

and the maximum is over the transportation polytope $\mathcal{T}_{\alpha,\beta}$. By the symmetry, this unique maximum is attained at $z_{ij}=N/mn$, and we have:

$$g(Z) = mn\left(\frac{N}{mn}+1\right)\log\left(\frac{N}{mn}+1\right) - mn\,\frac{N}{mn}\log\frac{N}{mn}\,.$$

Therefore,

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(P_\infty) = e^{g(Z)} = \frac{(N+mn)^{N+mn}}{N^N(mn)^{mn}}\,,$$

as desired. □

Additionally, we need to be able to compute the capacity of the complete symmetric polynomials, used in Theorem 2.11. Recall from Section 2.6 the notation

$$H_N(\boldsymbol{x},\boldsymbol{y}) = \sum_K \prod_{i=1}^m \prod_{j=1}^n (x_i y_j)^{k_{ij}} = h_N(\boldsymbol{x}\cdot\boldsymbol{y}),$$

where the sum is over all $K=(k_{ij})$ with total sum N of the entries: $\sum_{i,j} k_{ij}=N$, and h_N is the complete homogeneous symmetric polynomial of degree N in mn variables. The explicit computation of the capacity of H_N in the uniform case is then given as follows.

Lemma 9.3. *Let $\boldsymbol{\alpha}=(s,\ldots,s)\in\mathbb{N}^m$, $\boldsymbol{\beta}=(t,\ldots,t)\in\mathbb{N}^n$, where $m,n,s,t\in\mathbb{N}$, such that $ms=nt=N$. Then we have:*

$$\mathrm{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(H_N) = H_N(\mathbf{1},\mathbf{1}) = \binom{N+mn-1}{N}.$$

Proof. The second equality comes from the fact that the complete symmetric polynomial of degree N in mn variables, evaluated at the all-ones vector, counts the number of degree N monomials in mn variables. For the first equality, note that by symmetry we have

$$\partial_{x_1} H_N(\mathbf{1},\mathbf{1}) = \frac{1}{m}\sum_{\ell=1}^m \partial_{x_\ell} H_N(\mathbf{1},\mathbf{1}) = \frac{1}{m}\sum_{\ell=1}^m \sum_{\sum_{i,j} k_{ij}=N} \sum_{j=1}^n k_{\ell j} = \frac{N}{m}\,H_N(\mathbf{1},\mathbf{1}),$$

and similarly,

$$\partial_{y_1} H_N(\mathbf{1},\mathbf{1}) = \frac{N}{n}\,H_N(\mathbf{1},\mathbf{1}).$$

Therefore, we in fact have

$$\nabla \log(H_N)|_{\boldsymbol{x}=\mathbf{1},\boldsymbol{y}=\mathbf{1}} = \left(\frac{N}{m}, \ldots, \frac{N}{m}, \frac{N}{n}, \ldots, \frac{N}{n}\right) = (s, \ldots, s, t, \ldots, t).$$

By Fact 2.10 of [GL18+], this implies $\operatorname{Cap}_{\boldsymbol{\alpha}\,\boldsymbol{\beta}}(H_N) = H_N(\mathbf{1}, \mathbf{1})$. □

Proof of Theorem 9.1. The exact values of UB1 and UB2 are given by the lemmas above. The formulas for LB1, LB2 and New LB follow immediately from Theorem 2.4, Theorem 2.11 and Main Theorem 2.1, respectively. For LB2, we use the fact that $s \geq t$. Finally, the Shapiro correction term simplifies in the uniform case. Indeed, the product in the minimum the Remark 2.6 is equal on all spanning trees $\tau \in K_{mn}$, which all have $(m + n - 1)$ edges, and all edges have the same weight $\frac{1}{1+z_{ij}} = \frac{1}{1+N/mn}$ from the proof of Lemma 9.2. We omit the details. □

10. Numerical examples

In this section we compare the bounds numerically in specific cases where the exact number of contingency tables is known either exactly or approximately. Such comparisons are particulary easy when the marginals are uniform, see Theorem 9.1.

10.1. **Uniform marginals.** Our first table consists of comparisons in the case of uniform marginals. We give bounds for the number $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ of $m \times n$ contingency tables with row sums s and column sums t, so $N = ms = nt$. We use the notation in §9.1 and explicit formulas from Theorem 9.1.

Case	m	n	s	t	UB1	UB2	UB3	**Actual**	**New LB**	LB2	LB1
1	3	3	100	100	4.7×10^{17}	1.8×10^{15}	3.4×10^{11}	1.3×10^{7}	3.1×10^{5}	2.4×10^{3}	1.5×10^{-28}
2	3	9	99	33	2.3×10^{40}	1.5×10^{38}	3.7×10^{29}	2.8×10^{21}	7.3×10^{17}	5.6×10^{15}	1.2×10^{-62}
3	3	49	98	6	8.1×10^{121}	1.1×10^{120}	1.1×10^{98}	1.0×10^{68}	9.1×10^{55}	6.4×10^{53}	4.1×10^{-381}
4	10	10	20	20	8.5×10^{82}	1.4×10^{81}	2.2×10^{74}	1.1×10^{59}	5.7×10^{49}	4.8×10^{41}	5.2×10^{-104}
5	18	18	13	13	6.4×10^{164}	1.3×10^{163}	6.0×10^{156}	7.9×10^{127}	1.1×10^{110}	2.7×10^{95}	1.1×10^{-214}
6	30	30	3	3	9.5×10^{130}	3.8×10^{129}	3.8×10^{128}	2.2×10^{92}	2.2×10^{73}	1.6×10^{56}	2.2×10^{-522}
7	100	100	3	3	1.2×10^{589}	2.8×10^{587}	3.4×10^{586}	5.3×10^{459}	4.9×10^{394}	4.1×10^{332}	1.5×10^{-2267}
8	4	4	300	300	9.9×10^{36}	1.3×10^{34}	5.1×10^{25}	2.0×10^{19}	4.1×10^{16}	3.8×10^{12}	2.5×10^{-39}
9	9	9	10^3	10^3	1.1×10^{201}	4.4×10^{197}	1.8×10^{168}	8.0×10^{151}	4.5×10^{142}	7.3×10^{128}	1.8×10^{-32}
10	9	9	10^5	10^5	7.7×10^{362}	3.1×10^{357}	1.4×10^{298}	6.1×10^{279}	3.2×10^{270}	5.2×10^{248}	1.5×10^{44}
11	15	15	10^3	10^3	6.7×10^{508}	2.6×10^{505}	3.8×10^{457}	$\approx 1.7 \times 10^{427}$	1.7×10^{409}	2.3×10^{384}	1.3×10^{80}
12	15	15	10^5	10^5	1.3×10^{958}	5.1×10^{952}	1.1×10^{851}	$\approx 1.7 \times 10^{819}$	3.2×10^{800}	4.5×10^{761}	4.0×10^{383}
13	100	100	10^3	10^3	1.3×10^{14553}	6.0×10^{14549}	8.2×10^{14346}	$\approx 6.3 \times 10^{14072}$	5.3×10^{13869}	4.6×10^{13684}	5.0×10^{10741}
14	100	100	10^5	10^5	1.3×10^{34345}	5.2×10^{34339}	1.1×10^{33751}	$\approx 6.3 \times 10^{33470}$	4.9×10^{33263}	4.4×10^{32979}	6.2×10^{29545}

Here the actual values in cases 1–6 are taken from [CM10, Table 1], in case 7 from [OEIS, A001500] (computed by Heinz), and in case 8 is from [D09b, Table 3] (see also [Lan11, p. 27]). Actual values in cases 9–10 are computed from the exact form of the *Ehrhart polynomial* for the Birkhoff polytope $\mathcal{B}_9$ given in [BP03b].

In the last four cases 11–14, the number of tables is only given approximately and likely imprecise, but giving the right order of magnitude. In cases 11–12, we used a numerical approximation for the volume of the Birkhoff polytope $\mathcal{B}_{15}$ given in [CV16, Table 6] (see also [EF18, Table 1]). Finally, in cases 13–14, we used the exact asymptotics, given in [CM10, Thm 1].

10.2. **Non-uniform marginals.** In the table below, the last column "Time" gives the CPU time it took to compute Barvinok's UB2 and LB2. To compute UB1, LB1, and our lower bound, the time never exceeds 2 seconds. The stark difference between these two cases comes from the fact that the complete symmetric polynomials associated to UB2/LB2 (see Section 7.2) take much longer to compute than the rational function $P_\infty(\boldsymbol{x}, \boldsymbol{y}) = \prod_{ij}(1 - x_i y_j)^{-1}$ (see, however §11.2).

Case	m	n	N	UB1	UB2	UB3	**Actual**	**New LB**	LB2	LB1	Time
1	4	4	592	3.0×10^{30}	6.0×10^{27}	7.1×10^{18}	1.2×10^{15}	9.5×10^{12}	4.6×10^{8}	3.8×10^{-40}	79 sec
2	5	4	1269	1.4×10^{34}	1.2×10^{31}	8.3×10^{20}	3.4×10^{16}	2.0×10^{14}	3.0×10^{7}	1.5×10^{-52}	550 sec
3	4	4	65159458	1.3×10^{112}	?	2.1×10^{65}	4.3×10^{61}	5.8×10^{58}	?	2.3×10^{-49}	N/A
4	50	50	486	7.2×10^{562}	?	1.3×10^{551}	??	5.2×10^{421}	?	6.4×10^{-749}	N/A
5	50	50	302	1.2×10^{350}	?	7.3×10^{338}	??	1.1×10^{239}	?	2.0×10^{-922}	N/A

In the above table, case 1 is a celebrated example given by $\boldsymbol{\alpha} = (220, 215, 93, 64)$ and $\boldsymbol{\beta} = (108, 286, 71, 127)$. It was first studied in [DE85], the exact value was reported in [DG95, §6], and further discussed as a benchmark in [Bar17, BH12, D09a, DS98]. Case 2 is given by $\boldsymbol{\alpha} = (9, 49, 182, 478, 551)$ and $\boldsymbol{\beta} = (9, 309, 355, 596)$, was studied in [CDHL05, §6.4]. Case 3 is a large example computed in [D09b, Table 3], and is given by

$$\boldsymbol{\alpha} = (13070380, 18156451, 13365203, 20567424), \quad \boldsymbol{\beta} = (12268303, 20733257, 17743591, 14414307).$$

Case 4 is given by

$$\boldsymbol{\alpha} = (10, 8, 11, 11, 13, 11, 10, 9, 7, 9, 10, 16, 11, 9, 12, 14, 12, 7, 9, 10, 10, 6, 11, 8, 9, 8, 14, 12, 5, 10, 10, 8, 7, 8, 10, 10, 14, 6, 10, 7, 13, 4, 6, 8, 9, 15, 11, 12, 10, 6),$$

$$\boldsymbol{\beta} = (9, 6, 12, 11, 9, 8, 8, 11, 9, 11, 13, 7, 10, 8, 9, 7, 8, 3, 10, 11, 13, 7, 5, 11, 10, 9, 10, 13, 9, 9, 7, 7, 6, 8, 10, 12, 8, 12, 16, 12, 15, 12, 13, 13, 10, 7, 12, 13, 6, 11).$$

These marginals were introduced in [CDHL05, §6.1], where an estimate $\mathrm{BCT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (7.7 \pm .1) \times 10^{432}$ was given for the number of binary contingency tables using the sequential importance sampling (SIS). For comparison, Gurvits's Theorem 2.9 gives a large upper bound 1.3×10^{515}, but a very sharp lower bound 8.9×10^{431}, with a 0.5 sec. CPU time.

Finally, case 5 is given by

$$\boldsymbol{\alpha} = (14, 14, 19, 18, 11, 12, 12, 10, 13, 16, 8, 12, 6, 15, 6, 7, 12, 1, 12, 3, 8, 5, 9, 4, 2, 4, 1, 4, 4, 5, 2, 3, 3, 1, 1, 1, 2, 1, 1, 2, 1, 3, 3, 1, 3, 2, 1, 1, 1, 2),$$

$$\boldsymbol{\beta} = (14, 13, 14, 13, 13, 12, 14, 8, 11, 9, 10, 8, 9, 8, 4, 7, 10, 9, 6, 7, 6, 5, 6, 8, 1, 6, 6, 3, 2, 3, 5, 4, 5, 2, 2, 2, 3, 2, 4, 3, 1, 1, 1, 3, 2, 2, 3, 5, 2, 5).$$

These marginals were also introduced in [CDHL05, §6.1], with an estimate $\mathrm{BCT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (8.78 \pm .05) \times 10^{242}$. Here Gurvits's Theorem 2.9 gives an upper bound 1.7×10^{309}, and a sharp lower bound 3.0×10^{240}, again with a 0.5 sec. CPU time.

Note that we were unable to finish computation of the UB2 and LB2 in cases 3–5, which overwhelmed our computer system. This could be a problem with our implementation, of course. We would be curious to see these bounds if someone could compute them.

10.3. **Discussion.** All bounds above are best viewed on the log-scale, since they are multiplicative in nature and the approximation ratios grow exponentially otherwise. However, as $\min\{\alpha_i, \beta_j\} \to \infty$, we have logs of all bounds equal to $(1 + o(1)) \log \mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$, even if the rate of convergence implied by $o(1)$ notation vary greatly between the bounds.

Now, in all examples we computed, New LB dominates LB2 and dwarfs LB1, confirming the asymptotic bounds in Section 7. In fact, the latter is smaller than 1 in many cases. In addition, as discussed above, New LB is much faster to compute than the LB2, sometimes by orders of magnitude faster. Furthermore, in many examples, especially with non-uniform margins, the upper bounds are rather far from the actual number of contingency tables, while our New LB is much closer (on a log-scale).

We should mention an unusual situation in cases 4 and 5, when New LB for $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is smaller than Gurvits's LB for $\mathrm{BCT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. This is very counterintuitive, and suggests that for relatively small marginals sometimes taking smaller $K = (k_{ij})$ can give greater lower bounds for $\mathrm{CT}_K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ than taking $K = \infty$ gives for $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Although we concentrate our efforts on the lower bounds, let us make some observations about the upper bounds. It is clear from definition that UB1 > UB3, but the relationship of UB2 vs. UB3 is not so clear. In fact, for s and t small relative to m and n, the Shapiro correction term is very small, and UB3 becomes close to UB1.

11. Final remarks

11.1. There are many variations on the problem of counting general (unrestricted) and binary contingency tables. These include symmetric tables with zero diagonal, which correspond to graphs with fixed degrees, see [Wor18]. The technique of typical matrices was extended to this setting in [BH13]. High-dimensional tables are especially important in statistical applications [Eve92, Kat14], but even harder to analyze computationally [DO04]. We refer to [Ben14] for a recent extension of Barvinok's first lower bound to this setting. It would be interesting if the Lorentzian polynomials technique can be extended or modified in either of these two directions.

In the paper, we consider only a special case of flow polytopes and integer flows corresponding to weighted bipartite graphs, see e.g. [BV08, B+19]. In fact, flow polytopes for general directed graphs can be reduced to this case via a simple *BDV–transformation* of graphs, which gives a bijection between the flows [BDV04]. Finally, our lower bound can be further extended to *weighted contingency tables*, which can be viewed as evaluations of the natural generating function of $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$, see e.g. [Bar16, §8.5].

11.2. From the computational complexity point of view, the upper and lower bounds in our paper can be viewed as a deterministic approximation algorithm. The algorithm has an exponential approximation ratio, of course. For some special cases of the problem, such as the permanent, there are probabilistic *strong polynomial time* approximation algorithms, see [JSV04].

Now, capacities in the paper are defined to be solutions of a convex polynomial optimization problem. Thus, they can be solved in polynomial time by the classical *interior point method*, see e.g. [NN94, Ch. 6]. More specifically, capacity can be seen as the convex dual program to a certain maximum-entropy program, and strong polynomial time algorithms based on the *ellipsoid method* have also been developed to solve such problems, see e.g. [Vis21, §13.3]. For example, the capacity $\mathrm{Cap}_{\boldsymbol{\alpha}\boldsymbol{\beta}}$ in Theorem 1.1 can be ε*-approximated*, i.e. estimated up to a multiplicative factor $(1 \pm \varepsilon)$, in time $\mathrm{poly}(\phi + \log \frac{1}{\varepsilon})$, where

$$\phi := \sum_{i=1}^{m} \lceil \log \alpha_i \rceil + \sum_{j=1}^{n} \lceil \log \beta_j \rceil$$

is the *size* of the input.

The minimization of Shapiro's correction term in Remark 2.6 is an instance of the minimum spanning tree problem and can be solved by the greedy algorithm. On the other hand, computing the complete homogeneous polynomial necessary for Barvinok's second bound (Theorem 2.11) is harder, but can be done efficiently in several different ways, e.g. via [BIM16, Thm 1].

For flow polytopes, the covolume $f(S, m, n)$ in §8.2 is a matrix determinant of size at most mn, and thus easy to compute from the computational complexity point of view. Finally, the *BDV–transformation* mentioned above gives at most a quadratic blowup in the size of the graph, and is also easy to compute.

11.3. We can interpret a graphical matrix K as an $m \times n$ adjacency matrix of a bipartite graph G. Then $\mathrm{CT}_K(\alpha, \beta)$ is the number of subgraphs of G with degree sequences given by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. In this case, the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\gamma}$ in Theorem 2.9 correspond to the degree sequences of G.

11.4. There is a surprisingly strong (unproven) *independence heuristic* due to Good [Goo76], for the number of (unconstrained) contingency tables:

$$\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \approx \mathrm{I}(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \binom{N + mn - 1}{mn - 1}^{-1} \prod_{i=1}^{m} \binom{\alpha_i + n - 1}{n - 1} \prod_{j=1}^{n} \binom{\beta_j + m - 1}{m - 1}.$$

This heuristic is discussed further in [Bar09, DE85, DG95] and most recently in [LP20]. For the uniform marginals, a weak version of the heuristic is stated as a conjecture in [CM10, Conj. 1], where it is proved asymptotically in some "near-square" cases.

For example, in the uniform case 4 in §10.1, we have $\mathrm{I}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 7.4 \times 10^{58}$, which is much closer to the actual value $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 1.1 \times 10^{59}$ than any of the bounds. Similarly, in the non-uniform case 3 in §10.2, we have $\mathrm{I}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 3.7 \times 10^{61}$, which is again much closer to the actual value $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 4.3 \times 10^{61}$ than any of the bounds. Finally, for the non-uniform case 4 in §10.2, we have $\mathrm{I}(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 7.8 \times 10^{471}$, a very reasonable guess given that the New LB is clearly undercounting $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ in this case, cf. §10.3.

11.5. Let m, n and N be fixed, and let $K = \infty$. When we vary the marginals $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ over all partitions of N, the uniform case has the largest approximation ratio in the Main Theorem 2.1, while $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is also the largest of all such marginals, see [Bar10b, PP20]. This explains why our New LB can be still far away from the actual value in §10.1. This can also be seen in the approximation ratio in Theorem 2.3 and in the lower term gap in the volume of the Birkhoff polytope (Example 8.4).

On the other hand, all previously known lower bounds and other techniques tend to behave rather poorly when the matrix is far from uniform, and this includes the MCMC algorithms, see [B+10, DKM97]. So perhaps our lower bound coupled with Shapiro's upper bound are the only provably good bounds in that case.

Let us also mention that it is unlikely there is a universally good lower and upper bound for the general $\mathrm{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Sidestepping conjectural hardness of approximation results in computational complexity, there is also a probabilistic evidence of this phenomenon. In the simplest nonuniform case with marginals of two types, we already have a phase transition for the number of contingency tables, first predicted in [Bar10b], and recently proved in [DLP20].

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