The Future of Bijections

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Permutation Patterns, San Luis Obispo, June 23, 2011

The future is already here — it's just not very evenly distributed.

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Joint work with Matjaž Konvalinka

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Q1: What are combinatorial bijections?

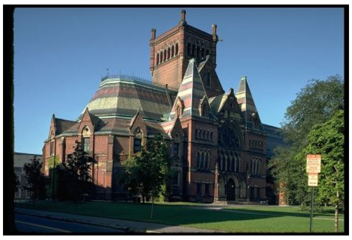
Q2: What is Combinatorics?

Q3: What is Mathematics?

Answer: see next page...

Harvard University:

Mathematics is an art!

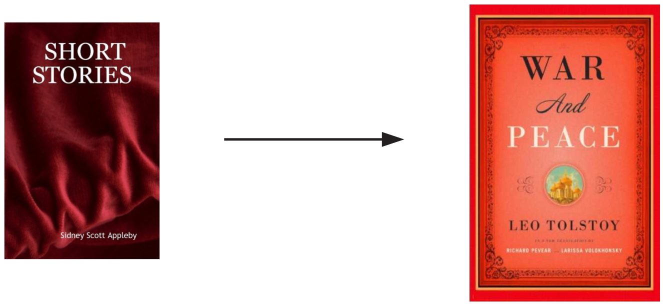


My answer:

Bijective combinatorics is an art!

Past: bijections as essays or short stories.

Future: bijections as long form storytelling, bordering on novels.



Common features of long bijection stories:

- they are long and occasionally technical;
- they begin as basic observations which are later greatly generalized;
- throughout the work or its followups, they have common characters, threads or features going to the heart of the bijection in the simplest case;
- they reveal new underlying (combinatorial, algebraic, geometric, etc.) structure of the objects of study, which was not previously transparent.




Examples of long form bijective stories:

1) Involution principle and the followup applications to the Andrews identities:

[Garsia & Milne, 1981], [Remmel, 1982], [Gordon, 1983], [O'Hara, 1988], [KP, 2009]

2) Dyson's rank and Rogers–Ramanujan identities:

[Dyson, 1944,'69,'88], [Bressoud & Zeilberger, 1988], [Boulet & P., 2006], [Boulet, 2010]

3) MacMahon's Master Theorem and GLZ's quantum generalization:

[Cartier & Foata, 1969], [P., Postnikov & Retakh, 1995], [KP, 2007]

4) Bijections for planar maps and Tutte formulas:

[Cori & Vauquelin, 1981], [Schaeffer, 1998], [Poulalhon & Schaeffer, 2006], [Fusy, Poulalhon, & Schaeffer, 2008], [Bernardi & Fusy, 2011+]

5) RSK, jeu-de-taquin, Littlewood-Robinson map, tableaux switching, Schützenberger involution, fundamental symmetry for LR-coefficients:

[... too many to list ...], [P. & Vallejo, 2010]

6) Hook walks and weighted hook walks:

[Green, Nijenhuis & Wilf, 1979,'84], [Sagan, 1980], [Zeilberger,1984]
[Ciocan-Fontanine, KP, 2010,'11], [Konvalinka, 2010,'11+]

.... $\Longrightarrow$ **future.**

What to expect today: two long bijection stories

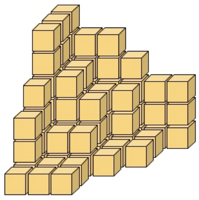
0) *Motivating problem:* Cayley theorem and Braun's conjecture on Cayley polytopes.

1) From statistics on trees and graph polynomials to dissections of Cayley polytopes.

2) Cayley compositions, partitions, and integer points in Cayley polytopes.

3) How these stories come together: a new explanation of the Cayley polytopes.

Motivating Problem



Cayley's Theorem (1857)

The number of integer sequences $(a_1, \ldots, a_n)$ *such that*

$$1 \le a_1 \le 2\,, \quad \textit{and} \quad 1 \le a_{i+1} \le 2a_i \quad \textit{for} \;\; 1 \le i < n\,,$$

is equal to the total number of partitions of integers $\le 2^n - 1$ *into parts* $1, 2, 4, \ldots, 2^{n-1}$.

These are called *Cayley compositions* $\mathcal{A}_n$ and *Cayley partitions* $\mathcal{B}_n$.

Example: $n = 2,\ |\mathcal{A}_2| = |\mathcal{B}_2| = 6$

$\mathcal{A}_2 = \big\{\, (1,1),\, (1,2),\, (2,1),\, (2,2),\, (2,3),\, (2,4) \,\big\}$,

$\mathcal{B}_2 = \big\{\, 21,\, 1^3,\, 2,\, 1^2,\, 1,\, \varnothing \,\big\}$.

Braun's Conjecture (2011)

Define *Cayley polytope* $\mathbf{C}_n \subset \mathbb{R}^n$ by inequalities:

$1 \le x_1 \le 2$, and $1 \le x_i \le 2x_{i-1}$ for $i = 2, \ldots, n$,

so that $\mathcal{A}_n$ are integer points in $\mathbf{C}_n$.

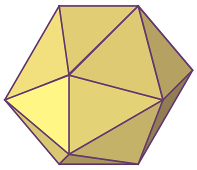
Theorem 1. [Formerly Braun's Conjecture]

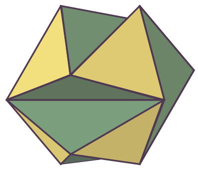
$$\operatorname{vol} \mathbf{C}_n = C_{n+1}/n!,$$

where C_n is the number of connected labeled graphs on n vertices.

Remark: Polytope $\mathbf{C}_n$ is combinatorially equivalent to a n-cube.
$\{C_n\}$ is A001187 in Sloane's *Encyclopedia of Integer Sequences*:
$1, 1, 4, 38, 728, 26704, 1866256, 251548592, 66296291072, 34496488594816, \ldots$

First Story





First prequel: Tutte's external activities

Theorem [Tutte, 1954]: $|\mathcal{C}(G)| = \sum_{\tau \in G} 2^{\mathrm{ea}(\tau)}$, where

- G is a connected graph with a fixed ordering $\prec$ of edges,
- τ are spanning trees in G,
- $\mathrm{ea}(\tau)$ is the number of *externally active* edges in τ,
- $\mathcal{C}(G)$ is the set of connected subgraphs in G.

Bijection: [Crapo, 1969]

- ∘ Let $\varphi : H \to \tau$ be the minimal spanning tree (MST) map.
- ∘ Observe that edges in $H - \tau$ are externally active edges (to τ).
- ∘ Conclude that $\left|\varphi^{-1}(\tau)\right| = 2^{\mathrm{ea}(\tau)}$.

Second prequel: tree inversions and DFS

Theorem [Mallows & Riordan, 1968]: $C_n = \sum_{\tau \in K_n} 2^{\mathrm{inv}(\tau)}$, where

- K_n is a complete graph on $\{1, \ldots, n\}$,
- τ are spanning trees in K_n,
- $\mathrm{inv}(\tau)$ is the number of *inversions* in τ,
- C_n is the number of connected subgraphs in K_n.

Bijection: [Gessel & Wang, 1979]

- Let $\varphi : H \to \tau$ be the *depth first search* (DFS) tree.
- Observe that edges in $K_n - \tau$ correspond to inversions in τ.
- Conclude that $\left|\varphi^{-1}(\tau)\right| = 2^{\mathrm{inv}(\tau)}$.

Third prequel: neighbor-first search (NFS)

NFS Algorithm

INPUT: graph G on $\{1, \ldots, n\}$.

START at n. Make node n *active*. DO:

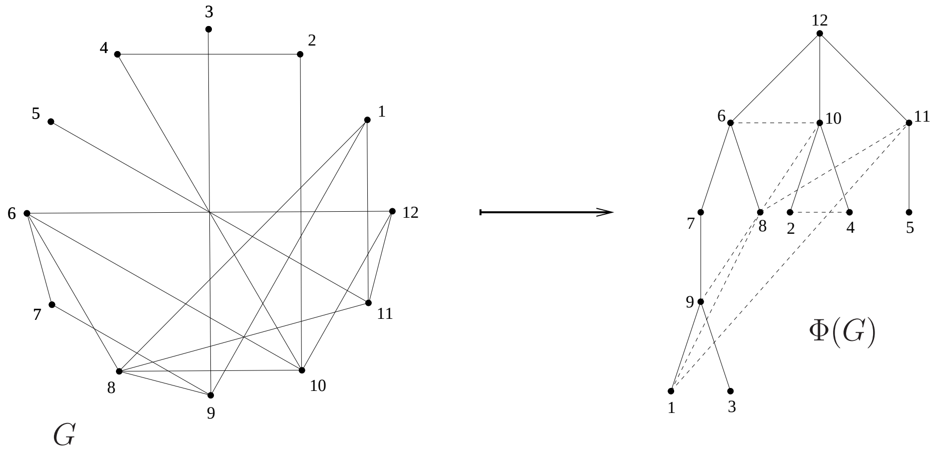
- Visit unvisited neighbors of the active node in decreasing order of their labels; make the one with the smallest label the new active vertex.
- If all the neighbors of the active vertex have been visited, backtrack to the last visited vertex that has not been an active vertex, and make it the new active vertex.

REPEAT: until all vertices have been active.

OUTPUT: the resulting search tree $\tau = \Phi(G)$.

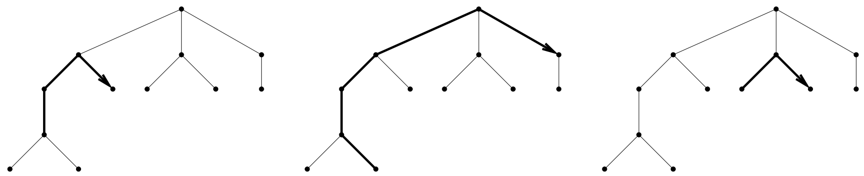
Remark: The NFS is a mixture of BFS and DFS.

Example: Graph G and its NFS tree $\Phi(G)$.



Note: Dotted lines correspond to graph edges that are not in $\Phi(G)$.

Theorem [Gessel & Sagan, 1996]: $C_n = \sum_{\tau \in G} 2^{\alpha(\tau)}$, where $\alpha(\tau)$ is the number of *cane paths* in τ, defined as follows:



Key observation: *The number of graphs G with a given NFS search tree τ, is equal to $2^{\alpha(\tau)}$.*

Remarks: See also [Gilbert, 1959] and [Kreveras, 1980]

Proof idea: an explicit triangulation into orthoschemes

Conjecture [Hadwiger, 1956]

Every convex polytope in $\mathbb{R}^d$ can be dissected into a finite number of orthoschemes.

Remark: Suffices to prove for simplices. Known for $d \leq 6$.

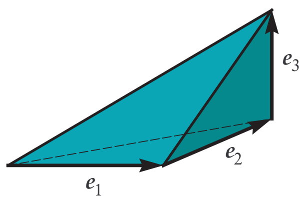
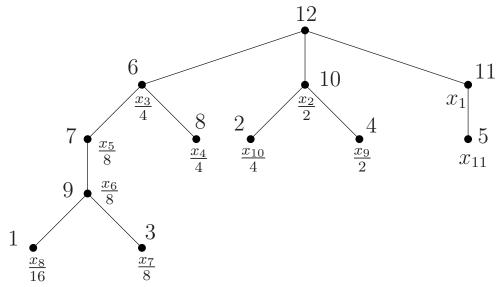


FIGURE 1. An example of an orthoscheme (path-simplex).

Triangulation Construction:



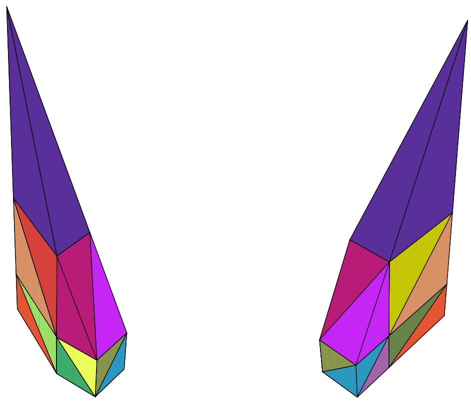
The simplex $\mathbf{S}_\tau \in \mathbb{R}^{11}$ corresponding to a labeled tree $\tau \in K_{11}$ is given by

$$1 \leq \frac{x_8}{16} \leq \frac{x_{10}}{4} \leq \frac{x_7}{8} \leq \frac{x_9}{2} \leq x_{11} \leq \frac{x_3}{4} \leq \frac{x_5}{8} \leq \frac{x_4}{4} \leq \frac{x_6}{8} \leq \frac{x_2}{2} \leq x_1 \leq 2.$$

We have $\alpha(\tau) = 21$, and $\mathrm{vol}(\mathbf{S}_\tau) = 2^{21}/11!$.

Rules: Label nodes according to NFS. Take $x_i/2^{k_i}$ to be the coordinate corresponding to $v \in \tau$, where k_i is the number of cane paths in τ that start in v. For inequalities, use the original ordering of labels in τ.

Example: Our triangulation of Cayley polytope $\mathbf{C}_3$ from two different angles:



Note: There are 16 orthoschemes in the triangulation, each of volume $2^k/3!$, where k varies. In general, there are $(n+1)^{n-1}$ orthoschemes (*Cayley's formula*).

Sequel: extension to other values of the Tutte polynomial

$C_n \;=\; T_{K_n}(1,2)$, where $T_G(x,y)$ is the *Tutte polynomial* of graph G :

$$\mathrm{T}_G(x,y) \;=\; \sum_{H \subseteq G} (x-1)^{k(H)-k(G)}(y-1)^{e(H)-|V|+k(H)}\,,$$

where $k(H)$ is the number of connected components in H. Also:

$$\mathrm{T}_G(x,y) \;=\; \sum_{\tau \in G} x^{ia(\tau)}\,y^{ea(\tau)}\,,$$

where the summation is over all spanning trees τ in G,
ia(τ) is the number of *internally active* edges in τ,
ea(τ) is the number of *externally active* edges in τ.

Tutte polytope

For every $0 < q \le 1$ and $t > 0$, define *Tutte polytope* $\mathbf{T}_n(q,t) \subset \mathbb{R}^n$ by inequalities:

$$x_n \ge 1 - q, \quad \text{and}$$

$$x_i \;\le\; (1+t)x_{i-1} \;-\; \frac{t(1-q)}{q}(1-x_{j-1}), \text{ where } 1 \le j \le i \le n \text{ and } x_0 = 1.$$

Theorem: *Tutte polytopes $\mathbf{T}_n(q,t)$ have 2^n vertices.*

Example: Compare the vertex coordinates of $\mathbf{C}_3$ and $\mathbf{T}_3(q,t)$:

2	4	8		$1+t$	$(1+t)^2$	$(1+t)^3$
2	4	1		$1+t$	$(1+t)^2$	$1-q$
2	1	2		$1+t$	1	$1+t$
2	1	1		$1+t$	$1-q$	$1-q$
1	2	4		1	$1+t$	$(1+t)^2$
1	2	1		1	$1+t$	$1-q$
1	1	2		1	1	$1+t$
1	1	1		$1-q$	$1-q$	$1-q$

Main Theorem

Let $\mathbf{T}_n(q,t) \subset \mathbb{R}^n$ *be the Tutte polytope defined above,* $0 < q \leq 1$*,* $t > 0$*. Then:*

$\mathrm{vol}\,\mathbf{T}_n(q,t) \; = \; t^n\,\mathrm{T}_{K_{n+1}}(1+q/t, 1+t)/n!,$

where $\mathrm{T}_H(x,y)$ *denotes the* Tutte polynomial *of graph* H*.*

Remark: Cayley polytopes are limits of Tutte polytopes:

$\lim_{q\to 0+} \, \mathbf{T}_n(q,1) \; = \; \mathbf{C}_n.$

This follows from the explicit form of vertex coordinates.

Since $\mathrm{T}_{K_n}(1,2) = C_n$, Main Theorem implies Braun's Conjecture.

Proof of Main Theorem:

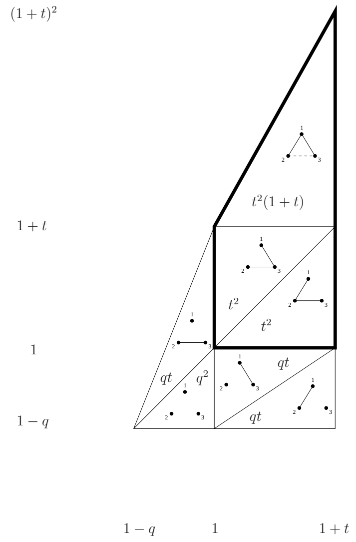
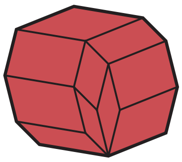
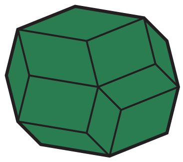


FIGURE 2. A triangulation of the Tutte polytope $\mathrm{T}_2(q,t)$.

Second Story





First prequel: geometric partition identities

Theorem [Andrews, Paule & Riese, 2001]:
Let $\pi(n)$ be the number of *convex partitions* of n, defined by the inequalities $\lambda_1 - \lambda_2 \geq \lambda_2 - \lambda_3 \geq \lambda_3 - \lambda_4 \geq \ldots$ Then:

$$1 + \sum_{n=1}^{\infty} \pi(n) t^n = \prod_{k=2}^{\infty} \frac{1}{1 - t^{\binom{k}{2}}}.$$

Bijective proof [Corteel & Savage, 2004] (see also [P., 2004])

- Think of convex partitions as an integer cone with "triangular basis":

  $\psi : (m_1, m_2, m_3, \ldots) \to m_1 \cdot (1, 0, 0, \ldots) + m_2 \cdot (2, 1, 0, \ldots) + m_3 \cdot (3, 2, 1, \ldots) + \ldots$

- Check that ψ is a bijection.

N.B. Zeilberger called ψ, "brilliant", "human generated" (Aug 98), "extremely elegant" (Mar '01).

Second prequel: combinatorics of LR-coefficients

Theorem [Berenstein & Zelevinsky, 1988], [Knutson & Tao, 1999]
The number of *LR-tableaux* $LR(\lambda, \mu, \nu)$ is equal to the number of *Hives* $H(\lambda, \mu, \nu)$ and the number of *BZ-triangles* $BZ(\lambda, \mu, \nu)$ $(= c^{\nu}_{\lambda,\mu})$.

Hard bijective proofs: [Carré, 1991], [Fulton, 1997]
Using RSK, fundamental symmetry, etc.

Easy, natural bijective proofs: [P. & Vallejo, 2005]

1) Write each set as set of integer points in certain polyhedra.
2) Compute is a unique affine linear map between polyhedra.
3) Observe that these maps map integer points into integer points.

Further results in this direction (almost all non-bijective!)

◇ [Bousquet-Mélou & Eriksson, 1997,'99], [Yee, 2001,'02]
("lecture hall" partition identities via integer points in polytopes)

◇ [Corteel, Savage, & Wilf, 2005], [Corteel, Lee & Savage, 2007]
(enumeration of partitions and compositions defined by inequalities)

◇ [Andrews, Paule, Riese & Strehl, 2001] ("MacMahon's Omega" series V)
Cayley theorem revived and reproved (first new proof in 144 years)

◇ [Beck, Braun & Le, 2011] (Braun's conjecture is stated, among other things)

Cayley's Theorem

The number of integer sequences $(a_1, \ldots, a_n)$ *such that*

$$1 \le a_1 \le 2\,, \quad \textit{and} \quad 1 \le a_{i+1} \le 2a_i \quad \textit{for} \;\; 1 \le i < n\,,$$

is equal to the total number of partitions of integers $\le 2^n - 1$ *into parts* $1, 2, 4, \ldots, 2^{n-1}$.

These are Cayley compositions $\mathcal{A}_n$ and Cayley partitions $\mathcal{B}_n$.

$\lll$ Now think of $\mathcal{A}_n, \mathcal{B}_n \subset \mathbb{R}^n$. $\ggg$

Example: $n = 2$, $|\mathcal{A}_2| = |\mathcal{B}_2| = 6$.

$\mathcal{A}_2 = \big\{ (1,1), (1,2), (2,1), (2,2), (2,3), (2,4) \big\}$,

$\mathcal{B}_2 = \big\{ (1,1), (0,3), (1,0), (0,2), (0,1), (0,0) \big\}$.

Polytope of Cayley partitions

Observe: $\mathcal{B}_n$ is the set of integer points in simplex $\mathbf{Q}_n$:

$$y_1, \ldots, y_n \geq 0, \quad 2^{n-1} y_1 + \ldots + 2y_{n-1} + y_n \leq 2^n - 1$$

Theorem 2. *Let $\mathbf{P}_n \subset \mathbb{R}^n$ be the convex hull of $\mathcal{B}_n$.*
Then $\mathrm{vol}\,\mathbf{P}_n = \mathrm{vol}\,\mathbf{C}_n$ (and thus $= C_{n+1}/n!$).

Sketch of proof: Define $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$\varphi : (a_1, a_2, a_3, \ldots) \to (2 - a_1, 2a_1 - a_2, 2a_2 - a_3, \ldots).$$

Observe that φ is volume-preserving. Now check that $\varphi : \mathbf{C}_n \to \mathbf{P}_n$. $\square$

First ever bijective proof of Cayley's theorem:

Proof. Observe that $\varphi : \mathcal{A}_n \to \mathcal{B}_n$ is a *bijection.* $\square$

Example: Bijection $\varphi : \mathcal{A}_2 \to \mathcal{B}_2$ is then as follows:

$$(1,1) \to (1,1) = \mathbf{21},\ (1,2) \to (1,0) = \mathbf{2},\ (2,1) \to (0,3) = \mathbf{1^3},$$
$$(2,2) \to (0,2) = \mathbf{1^2},\ (2,3) \to (0,1) = \mathbf{1},\ (2,4) \to (0,0) = \varnothing.$$

Corollary: *The number of Cayley partitions of m in $\mathcal{B}_n$ is equal to the number of Cayley compositions $(a_1, \ldots, a_n) \in \mathcal{A}_n$, such that $a_n = 2^n - m$.*

The merge of two stories

Parking functions polytope [Stanley & Pitman, 2002]

Let $\Pi_n\big(\theta_1,\ldots,\theta_n\big)$ be defined by the inequalities:

$$\Pi_n\big(\theta_1,\ldots,\theta_n\big) \;=\; \big\{(x_1,\ldots,x_n) : x_i \geq 0,\, x_1+\ldots+x_i \leq \theta_1+\ldots+\theta_i,\, \forall i\big\}.$$

Theorem [Stanley & Pitman, 2002]

$$\operatorname{vol}\Pi_n\big(\theta_1,\ldots,\theta_n\big) \;=\; \frac{1}{n!} \sum_{(a_1,\ldots,a_n)\in \mathrm{Park(n)}} \theta_{a_1}\cdots\theta_{a_n}$$

Corollary: $\operatorname{vol}\Pi_n\big(1,q,q^2\ldots,q^{n-1}\big) \;=\; \dfrac{1}{n!\,2^{\binom{n}{2}}}\cdot T_n(1,1/q)$

Second proof of Braun's Conjecture [KP, 2011+]

Corollary: $\operatorname{vol}\Pi_n\left(1, \frac{1}{2}, \frac{1}{4}, \ldots\right) = \frac{q^{\binom{n}{2}}}{n! C_n}$

Proof of Braun's Conjecture:

♡ Use map φ to write explicit inequalities for $\mathcal{B}_n$.

♡ Check that $\mathcal{B}_n$ is a scaled version of $\Pi_n\left(1, \frac{1}{2}, \frac{1}{4}, \ldots\right)$. □

Application to asymptotic combinatorics

Idea: Use the nice and explicit geometric structure of $\mathbf{P}_n \subset \mathbf{Q}_n$ to estimate $\mathrm{vol}\mathcal{B}_n$ from above and below.

Theorem: *The number C_n of connected labeled graphs on n nodes satisfies*

$$C_n = 2^{n(n-1)/2}\left(1 - \frac{n}{2^{n-1}} + \frac{O(n^2)}{2^{2n}}\right).$$

This slightly improves the classical bound in [Gilbert, 1959]

$$C_n = 2^{n(n-1)/2}\left(1 - \frac{n}{2^{n-1}} + \frac{O(n^2)}{2^{3n/2}}\right).$$

Thank you!