Caged eggs and the rigidity of convex polyhedra

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Theorem [Oded Schramm, "How to cage an egg", 1992]

Let $K \subset \mathbb{R}^3$ be a smooth strictly convex compact body and let $P \subset \mathbb{R}^3$ be a convex polyhedron. Then there exists a combinatorially equivalent polyhedron Q which midscribes *K.*

Here Q *midscribes* K if all edges of Q are tangent to K.

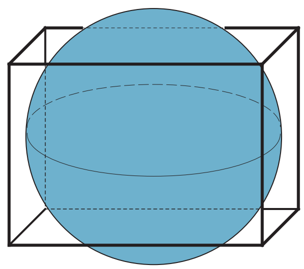

FIGURE 1. A caged sphere.

First motivation

Steinitz (1928): Inscribed and circumscribed spheres don't work.

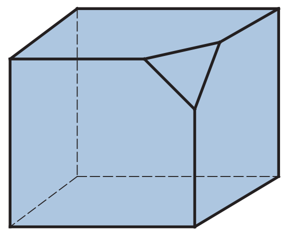

Figure 2. A polyhedron which cannot be inscribed into a sphere.

Igor Rivin resolved this problem using hyperbolic geometry (c. 1990).

Second motivation

Question: [Fejes Tóth, Besicovitch, Eggleston, etc., c. 1960]

What is the smallest total length of a cage which holds a unit sphere?

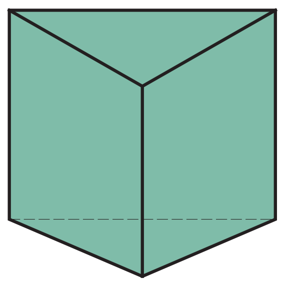

FIGURE 3. Conjectured optimal polyhedron (later proved incorrect).

Third (main) motivation

Circle packing: cut the miscribed sphere with facet planes.
This gives gives the circle packing on a sphere corresponding to the graph dual to P.

Alternatively, draw the circles through the tangent points of edges.

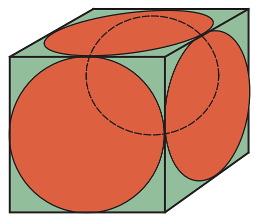

FIGURE 4. Circles in the facet planes.

Variational principle approach:

There is a problem with extending traditional variational principle proofs, even when $P \subset \mathbb{R}^3$ is *simplicial*.

$$\mathcal{F}(P) := \sum_{e \in P} f(e, P)$$

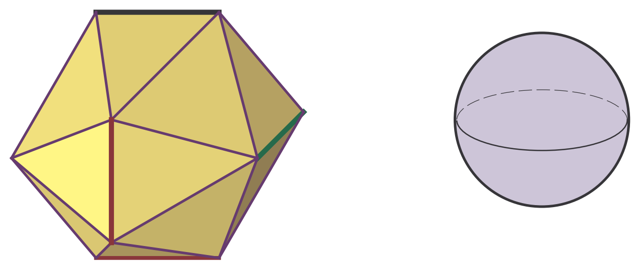

FIGURE 5. Variational principle: summation over edges.

Would need to show that $\mathcal{F}$ is convex and min can be attained only on midscribed polyhedra... (no such proof is known)

Schramm's configurations (for non-simple polyhedra)

Configuration: {points, planes}.

Midscribed polyhedron: incidence + tangent conditions.

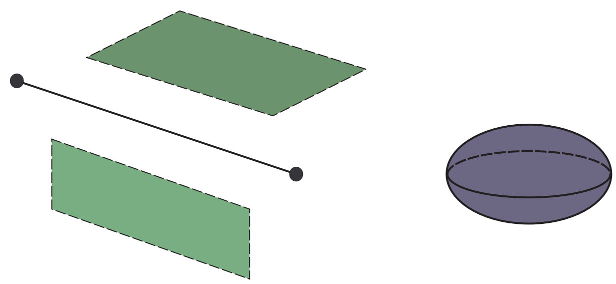

FIGURE 6. Schramm's configurations.

Schramm's proof

Idea: Prove the rigidity result first; interpret it as non-degeneration of the Jacobian of the configuration map and use the inverse function theorem.

Rigidity:

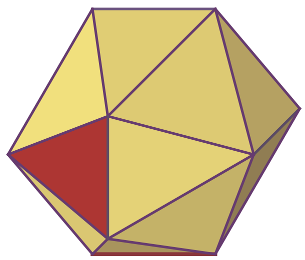

FIGURE 7. Rigidity of midscribed polyhedra.

Topological heart of the proof

The mapping lemma [Alexandrov, 1930's] Let $\mathcal{A}$ and $\mathcal{B}$ be two manifolds of the same dimension. Suppose a map $\varphi : \mathcal{A} \to \mathcal{B}$ satisfies the following conditions:

1) every connected component of $\mathcal{B}$ intersects the image $\varphi(\mathcal{A})$,
2) map φ is injective, i.e. $\varphi(a_1) = \varphi(a_2)$ implies that $a_1 = a_2$,
3) map φ is continuous,
4) map φ is proper.

Then φ is a homeomorphism; in particular, φ is bijective.

Here by a *proper map* φ we mean that for every sequence of points $\{a_i \in \mathcal{A}\}$ and images $\{b_i = \varphi(a_i)\}$, if $b_i \to b \in \mathcal{B}$ as $i \to \infty$, then there exists $a \in \mathcal{A}$, such that $b = \varphi(a) \in \mathcal{B}$, and a is a limit point of $\{a_i \in \mathcal{A}\}$.

An example to play with

Question: Can one describe all possible angle sequences of convex n-gons with vertices on given rays?

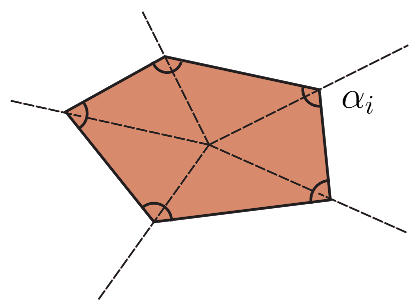

FIGURE 8. Angles of a polygon with vertices on rays.

Answer: Yes, $\sum_i \alpha_i = (n-2)\pi$,
+ some explicit linear inequalities on α_i's.

Key observation: Given angles, there is at most one such polygon (up to expansion).

First touch rule:

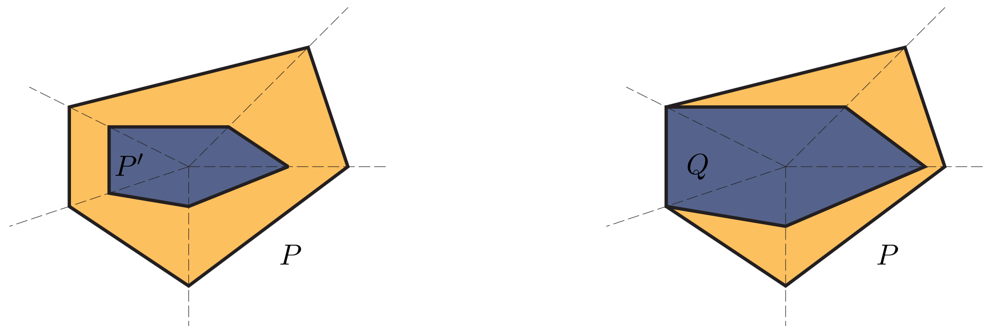

FIGURE 9. Polygons P, P' and the maximal expansion $Q = cP' \subset P$.

Alexandrov's curvature theorem (1943). Suppose $R_1, \ldots, R_n \subset \mathbb{R}^3$ are rays which do not lie in the same half-space, and suppose $\omega_1, \ldots, \omega_n \in \mathbb{R}$ satisfy the following conditions:

1) $\omega_i > 0$ for all $i \in [n]$,
2) $\omega_1 + \ldots + \omega_n = 4\pi$,
3) $\sum_{j \notin I} \omega_j > \omega(C_I)$ for every $I \subset [n]$, where $C_I = \operatorname{conv}\{R_i, i \in I\}$.

Then, up to expansion, there exists a unique convex polytope $P \subset \mathbb{R}^3$ which lies on rays R_i and has curvatures ω_i at vertices $v_i \in R_i$, for all $i \in [n]$. Conversely, all curvatures ω_i of such polytopes P must satisfy conditions 1)–3).

Here:

$$\omega_i \, = \, 2\pi \, - \sum_{F \ni v_i} \alpha(F)\,.$$

Infinitesimal rigidity

Dehn's Rigidity Theorem: *Every simplicial convex polytope in* $\mathbb{R}^3$ *is infinitesimally rigid.*

Here: $\{v_1, \ldots, v_n\}$ – vertices, $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ – velocity vectors

An *infinitesimal motion*:

$$(*) \qquad (\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_i - \boldsymbol{a}_j) = 0, \quad \text{for every edge } (v_i, v_j).$$

Infinitesimal rigidity: every infinitesimal motion is a rigid motion.

Proof [P, 2006] (inspired by Schramm's combinatorial tools)

Weaken ($*$) to allow the following three possibilities:

1. $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_i) = (\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_j) = 0$,
2. $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_i) < 0$ and $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_j) < 0$,
3. $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_i) > 0$ and $(\boldsymbol{v}_i - \boldsymbol{v}_j, \boldsymbol{a}_j) > 0$.

Orient edges (v_i, v_j) depending on the sign.

Fix face $(v_1 v_2 v_3)$ to rid of rigid motions.

Idea: prove that every orientation which satisfies **1,2,3** can have only unoriented edges, so the corresponding infinitesimal motion is zero.

Schramm's inversions:

Two edges in the same face meeting at v_i have:

- *one inversion* if one of them is oriented into and the other out of v_i,
- *zero inversions* if both of them are oriented into v_i or out of v_i,
- a *half-inversion* if exactly one of the edges is oriented,
- *one inversion* if both of them are unoriented and v_i is a live vertex,
- *zero inversions* if both of them are unoriented and v_i is a dead vertex.

Vertex is *live* if adjacent to an oriented edge; *dead* otherwise.

Lemma 1. *Every triangle with at least one live vertex has at least one inversion.*

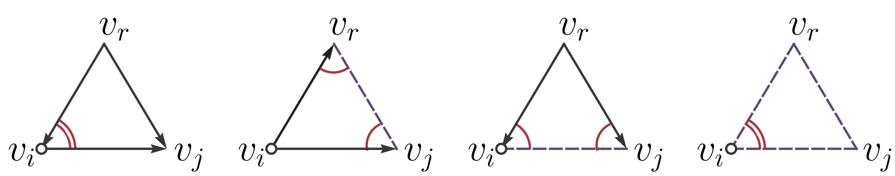

FIGURE 10. Different orientations of $(v_iv_jv_r)$, where vertex v_i is live.

Bound in the other direction:

Lemma 2. *There are at most two inversions around every live vertex.*

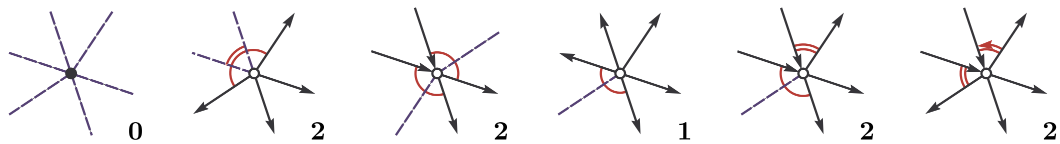

FIGURE 11. The number of inversions around a vertex in different cases.

Proof: Euler's formula + double counting argument.

In other words, Schramm's approach can be used to derive Dehn's rigidity theorem.

Thank you!