Acute triangulations of polytopes and the space

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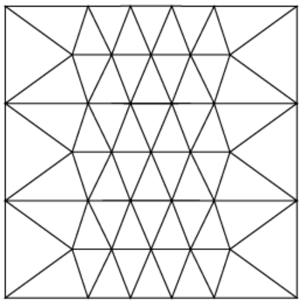


The Problem:

A simplex $\Delta \subset \mathbb{R}^d$ is *acute* if all its dihedral angles are $< \pi/2$.
An *acute triangulation* is a finite subdivision into acute simplices.

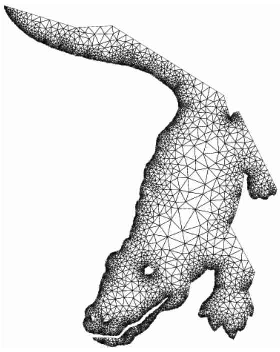
1. For a convex polytope $P \subset \mathbb{R}^d$, find an acute triangulation.

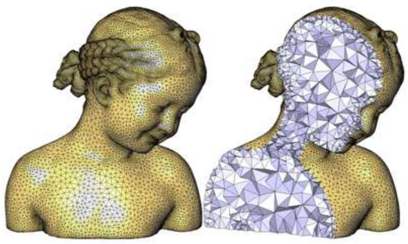
2. Find an acute partition of $\mathbb{R}^d$.

FIGURE 1. An acute triangulation and an acute dissection of a square.

Why acute triangulations?

- Classical geometric problem.
- Finite element method.
- Large recreational literature.





Acute triangulations in the plane

Theorem [BZ] Every polygon in the plane has an acute triangulation.

- ▷ Proposed by Martin Garner (*Scientific American*, 1960)
- ▷ Resolved independently by Burago–Zalgaller (1960)
  Existence only, no bounds follows from the proof.
- ▷ Easy to do in practice (Delaunay triangulations).
- ▷ Beginning 1980's heavily studied in the DCG community
  Optimal result: Bern–Mitchell–Ruppert (1995) + Maehara (2002)
  give a linear size acute triangulation of a polygon

Dimensionality curse

Philosophy: *the higher the dimension, the harder it is to make acute triangulations (both theoretically and practically).*

$d = 2$ – relatively easy

$d = 3$ – possible sometimes; perhaps, always

$d = 4$ – impossible sometimes; perhaps, very rarely

$d \geq 5$ – always impossible

Main Corollary. The d-dimensional cube has an acute triangulation if and only if $d \leq 3$.

Observation: Faces of an acute d-simplex are also acute simplices. Thus, acute triangulation of a d-cube contains acute triangulations of all n-cubes, for $n < d$.

$d = 3$ case: the beginning of a beautiful friendship

♡ Eppstein–Sullivan–Üngör (2004) construct a periodic acute partition of $\mathbb{R}^3$ (the proof is based on Sommerville's space–filling tetrahedra).

♡ VanderZee–Hirani–Zharnitsky–Guoy, Kopczyński–P.–Przytycki (2009) :

- ⋄ acute triangulation of a cube (VHZG: 1370, KPP: 2715 tetrahedra)
- ⋄ acute triangulation of all Platonic solids (KPP)
- ⋄ VHZG proof uses advanced simulation (mesh-improving technique)
- ⋄ KPP proof uses the 600–cell (regular polytope in $\mathbb{R}^4$)

Conjecture: Every convex polytope in $\mathbb{R}^3$ has an acute triangulation.

The 600-cell:

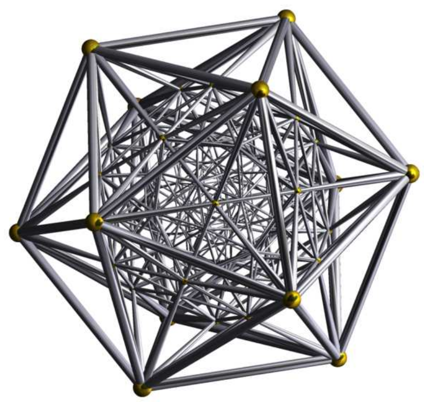


FIGURE 2. Graph drawn in the perspective projection of the 600-cell.

$d \geq 5$ case: completely impossible

Theorem* A point in $\mathbb{R}^5$ cannot be surrounded with acute simplices.

Proof steps:

1) A triangulation of a d-manifold M is *rich* if every codim 2 face is surrounded with at least 5 simplices.
2) Use the generalized Dehn–Sommerville equations to show that for every rich 4-manifold M, we have:

$$\#\text{ of points in } M \ \leq\ \chi(M).$$

3) For $d = 5$, take simplices containing a given point. They form a rich triangulation of a 4-sphere, a contradiction.
4) The $d = 5$ case implies all $d > 5$ (Křížek).

* Křížek (2006) gave an erroneous proof of the theorem.
Kalai (1990) proved a strongly related theorem.

D–S equations for simplicial manifolds:

Theorem [Klee (1964), Macdonald (1971)]

Let M be a compact m–dimensional triangulated manifold with boundary. For $k = 0, \ldots, m$, we have:

$$f_k(M) - f_k(\partial M) = \sum_{i=k}^{m} (-1)^{i+m} \binom{i+1}{k+1} f_i(M).$$

$d = 4$ case: the tipping point

1) The 4-cube does not have an acute triangulation. (KPP)
2) There is no periodic acute partition of $\mathbb{R}^4$. (KPP)

Main Theorem [KPP] There is no partition of $\mathbb{R}^4$ into simplices with all dihedral angles $< \pi - \varepsilon$, for every $\varepsilon > 0$.

Conjecture*: Space $\mathbb{R}^4$ has an acute partition.

* Brandts–Korotov–Křížek–Šolc (2009) make the opposite conjecture.

Proofs of two basic $d = 4$ results.

2) Acute triangulation of a 4–cube can be repeatedly reflected to make a periodic acute partition of the whole space $\mathbb{R}^4$.

1) A periodic acute partition of $\mathbb{R}^4$ gives a *rich triangulation* of a 4–torus, a contradiction.

Note: Although the regular cross-polytope tiles $\mathbb{R}^4$ periodically, this does not extend to its triangulations, and the above argument fails. Still, we conjecture that the regular cross-polytope does not have acute triangulations.

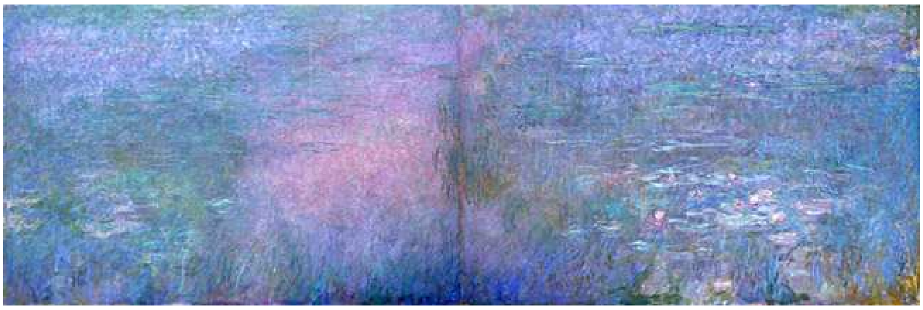
Proof of the main theorem:

(i) Check that if all dihedral angles are $< \pi - \varepsilon$, then the ratio of the edge lengths in every tetrahedron is bounded.

(ii) Use generalized D–S equations for simplicial (homology) manifolds to derive the 4-parabolicity of graphs of rich such partitions of $\mathbb{R}^4$.
(via quasi-conformal mappings, after Bonk–Kleiner, 2002).

(iii) Get a contradiction with the Benjamini–Curien (2009) isoperimetric inequality (an extension of the Benjamini–Schramm inequality which they used to establish impossibility of certain kissing sphere configurations in higher dimensions).

Thank you!



Finite element applications:

Input: triangulated surface $S \subset \mathbb{R}^3$.

Goal: find a *good** triangulation of the interior of S.

* *good* means all tetrahedra are as close to regular as possible.