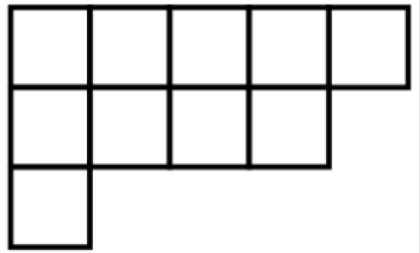
Combinatorial Inequalities

Igor Pak, UCLA

Colloquium, Yale University (May 5, 2021)

Maximal dimension of S_n representations

$$f^\lambda \;:=\; \chi^\lambda(1) \;=\; |\mathrm{SYT}(\lambda)|$$



1	2	4	7	8
3	5	6	9	
10				

$$D_n \;:=\; \max\{\, f^\lambda \;:\; \lambda \vdash n \,\}$$

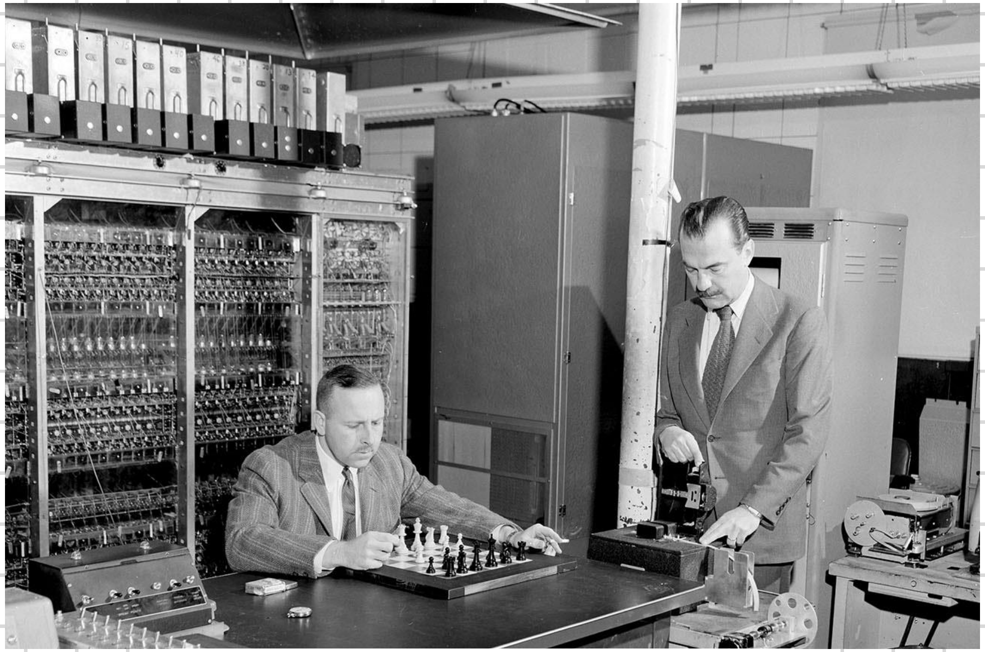
Question: $D_n \;=\; ???$

Robert L. Bivins
N. Metropolis
Paul R. Stein
Mark B. Wells

Characters of the Symmetric Groups of Degree 15 and 16

Mathematical Tables and Other Aids to Computation, Oct., 1954

Accordingly, we set up and carried out the calculation of the complete character tables of the symmetric groups of degree $n = 15$ and 16 using the Los Alamos Electronic Computer (MANIAC).



Paul Stein and Nick Metropolis

MANIAC I

From Wikipedia, the free encyclopedia

(***M****athematical* ***A****nalyzer* ***N****umerical* ***I****ntegrator and* ***A****utomatic* ***C****omputer Model* ***I***)

MANIAC weighed about 1,000 pounds

[9] The following question is of some practical interest: given n, for what partition (λ) does (9) take its largest value, and how does this value vary with n? The authors have been unable to find any discussion of this problem in the literature.

Maximal dimension of S_n representations

Burnside formula: $$\sum_{\lambda \vdash n} \left(f^{\lambda}\right)^2 = n!$$

$D_n \le \sqrt{n!}, \quad D_n \ge \sqrt{n!/p(n)},$

where $p(n) = e^{O(\sqrt{n})}$ is the number of partitions.

Corollary:

$$\log D_n = \frac{1}{2} n \log n - \frac{1}{2} n - O(\sqrt{n})$$

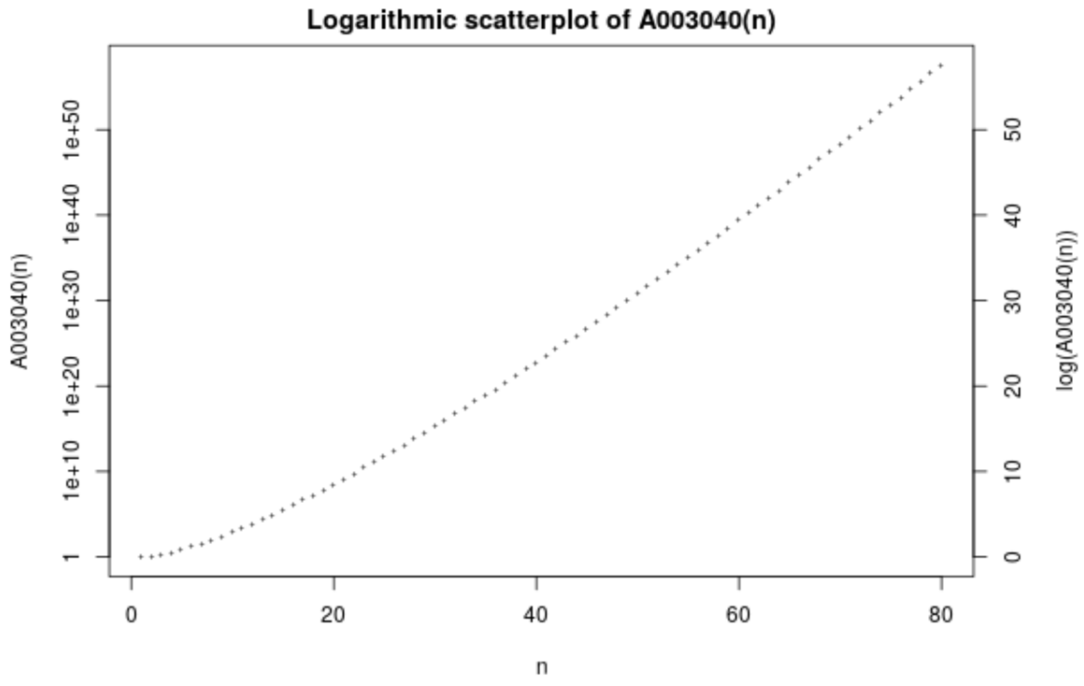
Vershik–Kerov–Pass Conjecture:

$$\log D_n = \frac{1}{2} n \log n - \frac{1}{2} n - c\sqrt{n} + o(\sqrt{n})$$

Partition $\lambda \vdash n$ with maximal dimension

A003040 Highest degree of an irreducible representation of symmetric group S_n of degree n.

1, 1, 2, 3, 6, 16, 35, 90, 216, 768, 2310, 7700, 21450, 69498, 292864, 1153152, 4873050, 16336320, 64664600, 249420600, 1118939184, 5462865408, 28542158568, 117487079424, 547591590000,



$$p(81) = 18,004,327$$

$$(15, 12, 10, 9, 7, 6, 5, 4, 3^2, 2^2, 1^3)$$

hook-length formula

$$f^\lambda = n! \prod_{(i,j)\in\lambda} \frac{1}{h_{ij}}$$

PDP-8

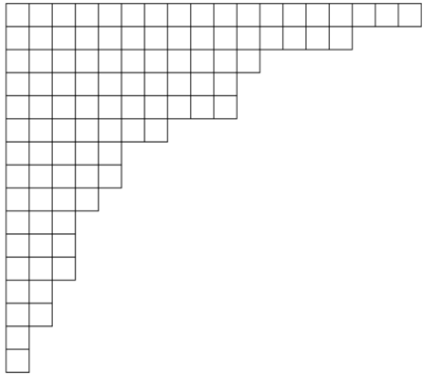
John McKay

Math. Comp. **30** (1976), 624-631

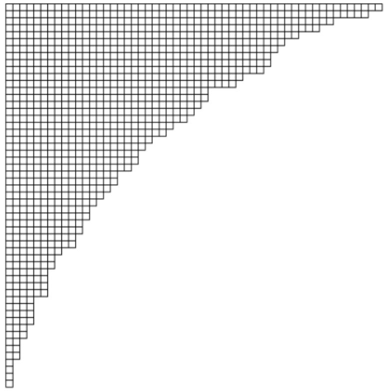
12-bit minicomputer

Digital Equipment Corporation

Partition $\lambda \vdash n$ with maximal dimension



$n = 100$

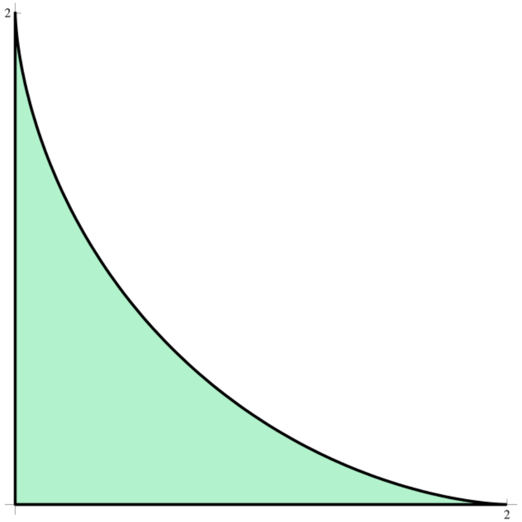


$n = 1000$

RSK : random partitions w.r.t.

$$\frac{(f^{\lambda})^2}{n!}$$

Robinson–Schensted correspondence



Vershik–Kerov–Logan–Shepp (VKLS) *shape*

$$\omega = (\omega_x, \omega_y) : [-\pi/2, \pi/2] \to \mathbb{R}^2$$

$$\omega_x(\theta) = \left(\frac{2\theta}{\pi} + 1\right)\sin\theta + \frac{2}{\pi}\cos\theta,$$

$$\omega_y(\theta) = \left(\frac{2\theta}{\pi} - 1\right)\sin\theta + \frac{2}{\pi}\cos\theta.$$

Longest increasing subsequences

B. F. Logan and L. A. Shepp (1977)

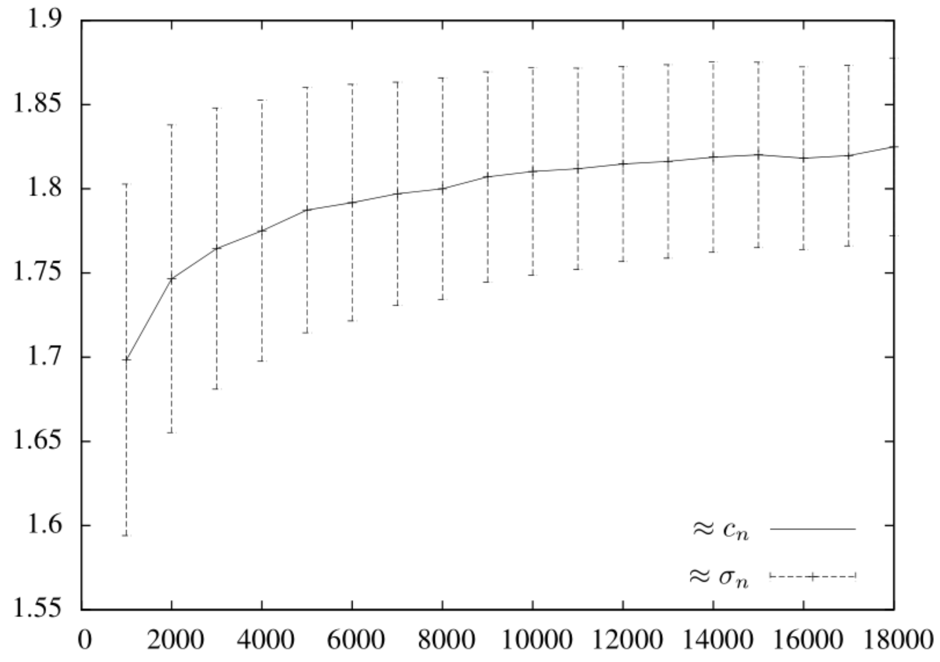
A. M. Vershik and S. V. Kerov (1981)

Partition $\lambda \vdash n$ with maximal dimension

Theorem (Vershik–Kerov, 1985)

$$\sqrt{n!}\, e^{-d\sqrt{n}(1+o(1))} \;\leq\; D_n \;\leq\; \sqrt{n!}\, e^{-c\sqrt{n}(1+o(1))},$$

where $d = \pi\sqrt{\frac{1}{6}} \approx 1.2825$ and $c = \frac{\pi - 2}{\pi^2} \approx 0.1157$



$$c \approx 0.91 \ (??)$$

Numerical experiments in problems of asymptotic representation theory

A. Vershik & D. Pavlov

Littlewood–Richardson coefficients $c^{\lambda}_{\mu,\nu}$

structure constants in the ring of characters of $\mathrm{GL}_N(\mathbb{C})$

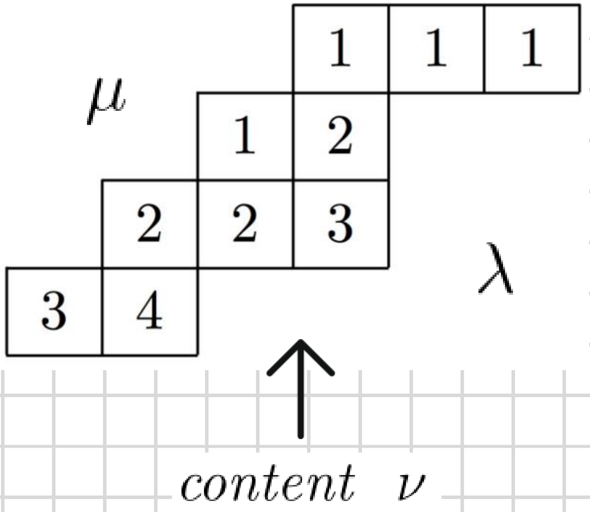
$$s_\mu \cdot s_\nu \;=\; \sum_{\lambda \vdash n} c^{\lambda}_{\mu,\nu}\, s_\lambda \qquad |\lambda| \;=\; |\mu| + |\nu| \;=\; n$$

Schur functions

Theorem (Littlewood–Richardson, 1934)

$$c^{\lambda}_{\mu,\nu} \;=\; \big|\mathrm{LR}(\lambda/\mu, \nu)\big|$$

Littlewood–Richardson tableaux



$\lambda = (6, 4, 4, 2)$

$\mu = (3, 2, 1)$

$\nu = (4, 3, 2, 1)$

Maximal Littlewood–Richardson coefficients

Theorem (Stanley, 2016)

$$\max_{0\le k\le n}\ \max_{\lambda\vdash n}\ \max_{\mu\vdash k}\ \max_{\nu\vdash n-k}\ c^{\lambda}_{\mu,\nu} \;=\; 2^{n/2\,-\,O(\sqrt{n})}$$

Harris–Willenbring identity (2014)

$$\sum_{n\ge 0}\ \sum_{\lambda,\mu,\nu\vdash n} \big(c^{\lambda}_{\mu,\nu}\big)^2\, q^{|\mu|}\, t^{|\nu|} \;=\; \prod_{i=1}^{\infty} \frac{1}{1-q^i-t^i}$$

$$\mathbf{C}(n,k) \;:=\; \max_{\lambda\vdash n}\ \max_{\mu\vdash k}\ \max_{\nu\vdash n-k}\ c^{\lambda}_{\mu,\nu}$$

Theorem (P.–Panova–Yeliussizov, 2019)

$$\sqrt{\binom{n}{k}}\, e^{O(\sqrt{n})} \;\le\; \mathbf{C}(n,k) \;\le\; \sqrt{\binom{n}{k}}$$

$$\sum_{\lambda\vdash n}\ \sum_{\mu\vdash k,\,\nu\vdash n-k} \big(c^{\lambda}_{\mu,\nu}\big)^2 \;\ge\; \binom{n}{k} \qquad \forall\, k$$

$$\sum_{\lambda\vdash n} \big(c^{\lambda}_{\mu,\nu}\big)^2 \;\le\; \binom{n}{k} \qquad \forall\, \mu\vdash k,\ \nu\vdash n-k$$

Maximal Littlewood–Richardson coefficients

Question: What are the shapes which achieve the max?

Answer: VKLS shapes!

Formally, $\forall \mu, \nu$ VKLS shape $\exists \lambda$ LSVK shape ... [PPY, 2019]

Note: Major difference between *max-dim* and *max-LR* problems

max-dim $\Longleftrightarrow$ VKLS shape

max-LR $\Longrightarrow$ VKLS shapes for λ, μ, ν $\not\Longleftarrow$ (??)

Conjecture (PPY, 2019) $\exists\, \lambda \vdash n,\ \mu, \nu \vdash n/2$ s.t. $\underbrace{\quad}_{\text{VKLS shapes}}$ $c^{\lambda}_{\mu,\nu} = 2^{n/2 - \omega(\sqrt{n})}$

VKLS shapes

Maximal LR–coefficient with ℓ rows

$$\mathbf{C}_\ell(n) := \max_{\lambda \vdash n,\, \ell(\lambda)=\ell} \; \max_{\mu \vdash k} \; \max_{\nu \vdash n-k} c^{\lambda}_{\mu,\nu}$$

Theorem (P.–Panova–Yeliussizov, 2019)

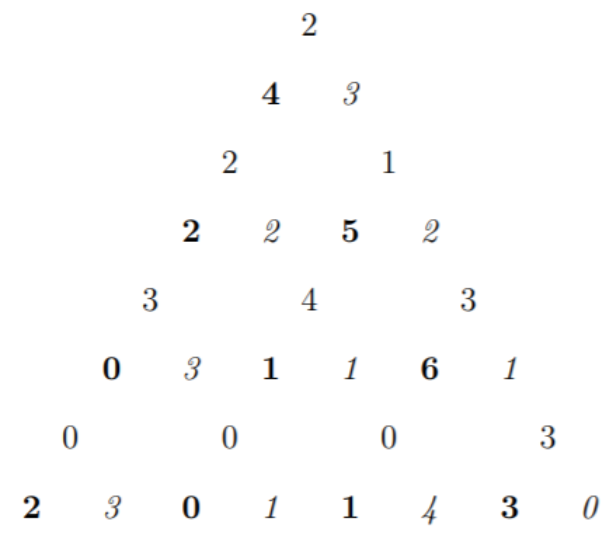
$$\log \mathbf{C}_{\ell_n}(n) \sim \frac{1}{2} (\ell_n)^2 \log n$$

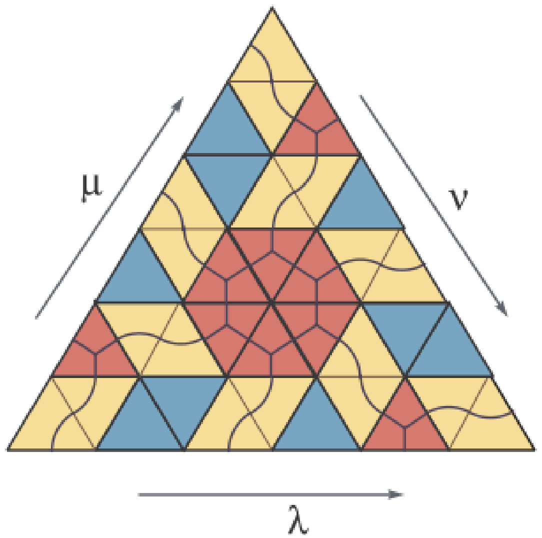
Lower bound:

$$c^{\lambda}_{\mu,\nu} = \#\textit{Knutson–Tao puzzles}$$
with margins (λ, μ, ν)

Upper bound: $\mathbf{C}_\ell(n) \le (n+1)^{(\ell-1)(\ell-2)/2}$

$$c^{\lambda}_{\mu,\nu} = \#\textit{Berenstein–Zelevinsky patterns}$$
with margins (λ, μ, ν)





Moral: *Need combinatorial interpretations!*

Kronecker coefficients

structure constants in the ring of characters of S_n

$\lambda, \mu, \nu \vdash n$

$$\chi^\mu \cdot \chi^\mu = \sum_{\lambda \vdash n} g(\lambda, \mu, \nu)\, \chi^\lambda$$

$$g(\lambda, \mu, \nu) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\chi^\mu(\sigma)\chi^\nu(\sigma) \quad \Longrightarrow \quad g(\lambda, \mu, \nu) = g(\lambda, \nu, \mu) = g(\mu, \lambda, \nu) = \ldots$$

Open Problem (Murnaghan, 1938)

Give a combinatorial interpretation for $g(\lambda, \mu, \nu)$.

Conjecture:

$g(\lambda, \mu, \nu) \notin \#\mathsf{P}$

Note: This is known in a few (very) special cases

Maximal Kronecker coefficient

Theorem (Stanley, 2016)

$$\max_{\lambda\vdash n}\max_{\mu\vdash n}\max_{\nu\vdash n} g(\lambda,\mu,\nu) = \sqrt{n!}\, e^{-O(\sqrt{n})}$$

Proposition $g(\lambda,\mu,\nu) \le \min\{f^\lambda, f^\mu, f^\nu\}$ $\Longrightarrow K_n \le D_n$

Proof $g(\lambda,\mu,\nu) \le \dfrac{f^\lambda f^\mu}{f^\nu} \le f^\lambda$, for all $f^\lambda \le f^\mu \le f^\nu$

Theorem (P.–Panova–Yeliussizov, 2019)

$$\sum_{\lambda,\mu,\nu\vdash n} g(\lambda,\mu,\nu)^2 = n!\left(1+\frac{2}{n^2}+O\left(\frac{1}{n^3}\right)\right)$$

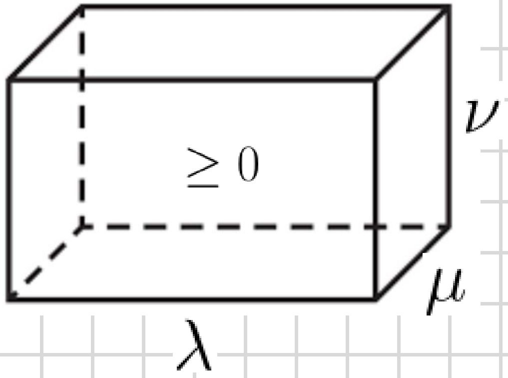
max-Kron $\Longrightarrow$ VKLS shapes for λ, μ, ν $\not\Leftarrow$ (??)

Bounds on Kronecker coefficients

Theorem (P.–Panova, 2020)

$$g(\lambda,\mu,\nu) \;\le\; \mathrm{CT}(\lambda,\mu,\nu)$$

where $\mathrm{CT}(\lambda,\mu,\nu) \;=\; \#\big[$3-dim contingency tables with marginals $\lambda,\mu,\nu\big]$.



Theorem (P.–Panova, 2020)

Let $\ell(\lambda)=\ell,\ \ell(\mu)=m,$ and $\ell(\nu)=r.$ Then:

$$g(\lambda,\mu,\nu) \;\le\; \left(1+\frac{\ell m r}{n}\right)^{n}\left(1+\frac{n}{\ell m r}\right)^{\ell m r}$$

Proof uses [Barvinok'09] and *majorization* over reals.

Example: $\lambda=\mu=\nu=(\ell^2)^\ell.$

Prop. gives $g(\lambda,\mu,\nu) \;\le\; f^\lambda \;\approx\; e^{\frac{1}{3}n\log n - O(n)}$

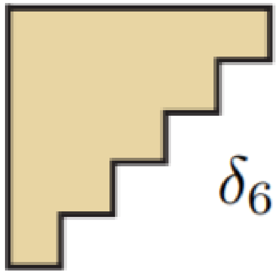
Thm gives $g(\lambda,\mu,\nu) \le 4^n.$

We conjecture: $g(\lambda,\mu,\nu) \;=\; 4^{n-o(n)}.$

Issue: no combinatorial interpretation

Conjectural lower bounds

Staircase shape $\delta_k := (k-1, k-2, \ldots, 2, 1) \vdash n = \binom{k}{2}$



Conjecture: $g(\delta_k, \delta_k, \delta_k) \; = \; \sqrt{n!}\, e^{-O(n)}$

Best known lower bound: $g(\delta_k, \delta_k, \delta_k) \; \geq \; 1$ [Bessenrodt–Behns'04]

Saxl Conjecture: $g(\delta_k, \delta_k, \mu) \; > \; 0$ for all $\mu \vdash \binom{k}{2}$. Remains open (2012)

[Ikenmeyer'15], [P.–Panova–Vallejo'16] $\leftarrow$ various families of μ

[Luo–Sellke'17] $\leftarrow$ random $\mu \vdash \binom{k}{2}$

[Bessenrodt–Bowman–Sutton] $\leftarrow$ $\mu \vdash \binom{k}{2}$ s.t. f^μ is odd

Complexity aspects

$f^\lambda \in \mathsf{FP} \quad \leftarrow \quad$ can be computed in poly-time (via the hook-length formula)

$c^\lambda_{\mu,\nu} \in \#\mathsf{P}$-complete $\quad \leftarrow \quad$ probably not (for input in binary, open in unary)

Theorem [Ikenmeyer–Mulmuley–Walter, 2017]

$g(\lambda,\mu,\nu) \in \#\mathsf{P}$-hard $\quad \leftarrow \quad$ (for input in unary)

$g(\lambda,\mu,\nu) >^? 0 \in \mathsf{NP}$-hard $\quad \leftarrow \quad$ (for input in unary)

Theorem [Bürgisser–Ikenmeyer, 2008]: $g(\lambda,\mu,\nu) \in \mathsf{GapP} := \#\mathsf{P} - \#\mathsf{P}$

$$g(\lambda,\mu,\nu) = \sum_{\omega\in S_\ell} \sum_{\pi\in S_m} \sum_{\tau\in S_r} \operatorname{sign}(\omega\pi\tau)\cdot \mathrm{CT}\big(\lambda + 1_\ell - \omega,\ \mu + 1_m - \pi,\ \lambda + 1_r - \tau\big)$$

[P.-Panova, 2017]

Complexity aspects

Theorem [Hepler, 1994]

$\chi^\lambda(\mu) \in \#\mathsf{P}$-hard $\leftarrow$ (for input in unary)

Murnaghan–Nakayama rule $\Longrightarrow$ $\chi^\lambda(\mu) \in \mathsf{GapP} = \#\mathsf{P} - \#\mathsf{P}$

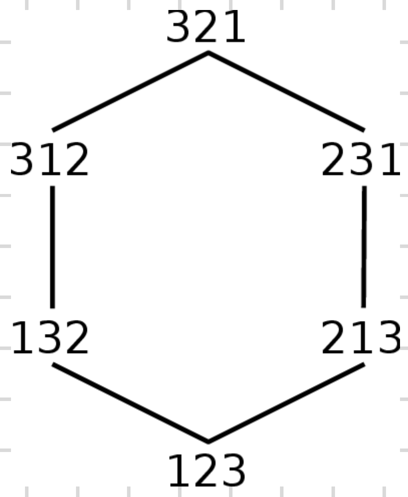
Theorem [Ikenmeyer–P.–Panova, 2021+]

$\chi^\lambda(\mu)^2 =^? 0 \in \mathsf{C_=P}$-hard $\leftarrow$ as hard as any $\mathsf{GapP}^2 =^? 0$

$\Longrightarrow$ There is probably NO combinatorial interpretation for $\chi^\lambda(\mu)^2$

(unless PH collapses)

Weak Bruhat order and bubble sorting



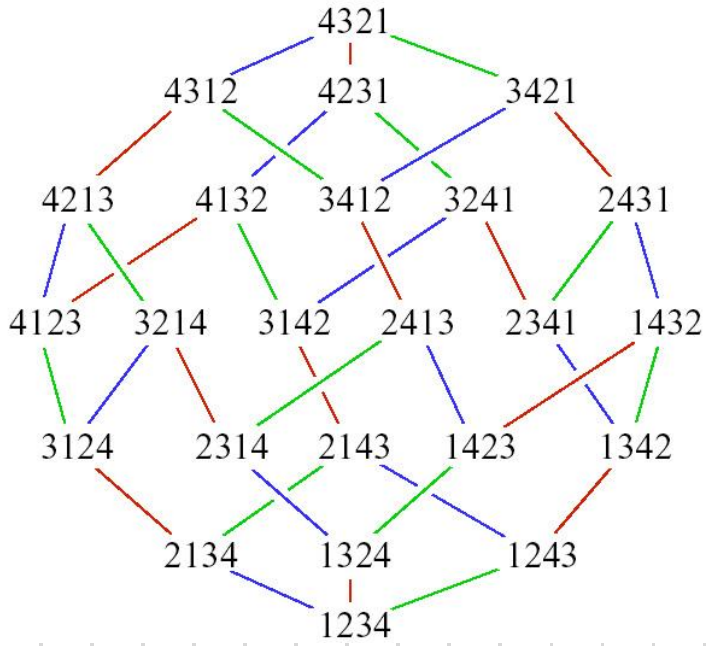
Definition: $B_n = (S_n, \prec)$

$\tau \preccurlyeq \sigma$ if τ can be obtained from σ by a *bubble sorting*

$$\Updownarrow$$

$$\sigma = \tau \cdot (i_1, i_1 + 1) \cdots (i_\ell, i_\ell + 1), \quad \text{where} \quad \mathrm{inv}(\sigma) = \mathrm{inv}(\tau) + \ell.$$

$$e(\sigma) := \#\{\tau \in S_n,\ \tau \preccurlyeq \sigma\}$$



Theorem [Dittmer–Pak, 2018+]

$e(\sigma) \in$ #P-complete

Sidorenko inequality

$$\overline{\sigma} := \big(\sigma(n), \ldots, \sigma(1)\big)$$

Theorem [Sidorenko, 1991]

$$e(\sigma) \cdot e(\overline{\sigma}) \geq n!$$

Original proof uses Combinatorial Optimization.

Alternative proof and generalization to other root systems by [Gaetz–Gao, 2020]

Theorem [P., 2021+]

$$e(\sigma) \cdot e(\overline{\sigma}) - n! \in \#\mathsf{P} \quad \Longrightarrow$$

The *defect* in Sidorenko's inequality has a combinatorial interpretation.

Based on a new proof of Sidorenko's inequality with a computable injection RHS $\longrightarrow$ LHS

Thank you!