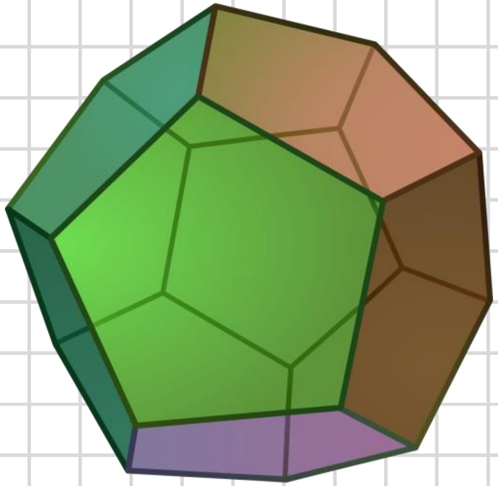
Igor Pak, UCLA

May 4 , 2021

Combinatorial inequalities

The Vinberg Lecture



.pdf file of the paper

Plan of the talk:

1) Overview of combinatorial inequalities and their proofs
2) Recent results

Main thing to remember:

good inequalities deserve a good proof!

First examples

(1) $\mathrm{Fib}(n+1) \leq 2^n$ *Definition:* $\mathrm{Fib}(n+1) = \#\ 0/1$ sequences of length n with no 11

First examples

(1) $\mathrm{Fib}(n+1) \le 2^n$ *Definition*

(2) $\binom{n}{0} \le \binom{n}{1} \le \binom{n}{2} \le \ldots \le \binom{n}{\lfloor n/2 \rfloor}$ *Direct calculation*

(3) $\mathrm{Cat}(n) \le \mathrm{Bell}(n)$ *Induction* [Kuznetsov–P.–Postnikov'94]

1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796

1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975

(4) $p(n) \le p(n+1)$ *Injection* $p(n)$ = number of integer partitions of n

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231

$p(n+1) - p(n)$ = number of partitions of $(n+1)$ with no parts 1

First examples

Note: inequalities (1) – (4) can be proved by direct injection, while (5) $\leftarrow$ unlikely!

(1) $\mathrm{Fib}(n+1) \le 2^n$ *Definition*

(2) $\binom{n}{0} \le \binom{n}{1} \le \binom{n}{2} \le \ldots \le \binom{n}{\lfloor n/2 \rfloor}$ *Direct calculation*

$\binom{n}{k} - \binom{n}{k-1} = \#$ ballot sequences [Bertrand, 1887]

(3) $\mathrm{Cat}(n) \le \mathrm{Bell}(n)$ *Induction* [Kuznetsov–P.–Postnikov'94]

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(4) $p(n) \le p(n+1)$ *Injection* $p(n) =$ number of integer partitions of n

1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, 101, 135, 176, 231

(5) $p(n)^2 \ge p(n+1)p(n-1)$ $n > 25$ *Asymptotics* [DeSalvo–P.'15]

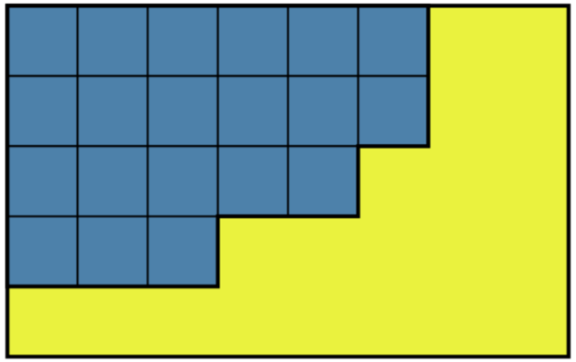
$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

More examples

(6) Unimodality of *Gaussian coefficients* $p_{ab}(0) \leq p_{ab}(1) \leq \ldots \leq p_{ab}(\lfloor ab/2 \rfloor)$

$$\binom{a+b}{a}_q = \frac{(q^{a+1}-1)\cdots(q^{a+b}-1)}{(q-1)\cdots(q^b-1)} = \sum_{n=0}^{ab} p_{ab}(n)\, q^n$$

$p_{ab}(n)$ = number of partitions of n which fit rectangle $[a \times b]$



$$\binom{6}{3}_q = 1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + q^8 + q^9$$

More examples

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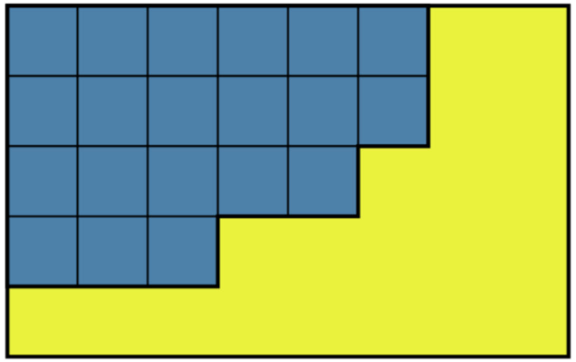


Conjectured: [Cayley, 1856]

[Sylvester, 1878] (*invariant theory*)

[Stanley, 1980] (*hard Lefschetz theorem*)

[Proctor, 1982] (*linear algebra*)

[O'Hara, 1990] (*combinatorial proof*, not injective!)

[P.–Panova, 2013] (*Kronecker coefficients*, strict)

Want: Combinatorial interpretation for $p_{ab}(n) - p_{ab}(n-1)$

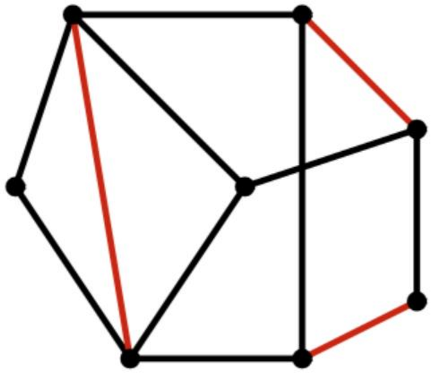
(none "nice" are known, cf. [P.–Panova'15])

More examples

Want: Combinatorial interpretation for the difference

(7) Log-concavity of the *matching numbers*: $m_k(G)^2 \geq m_{k+1}(G)\, m_{k-1}(G)$

$m_k(G) := \#\ k\text{-matchings in } G = (V, E)$



[Heilmann–Lieb, 1972] (*interlacing of eigenvalues*)

[Krattenthaler, 1996] (*injective proof*)

Theory of monomer-dimer systems

OJ Heilmann, EH Lieb - Statistical Mechanics, 1972 - Springer

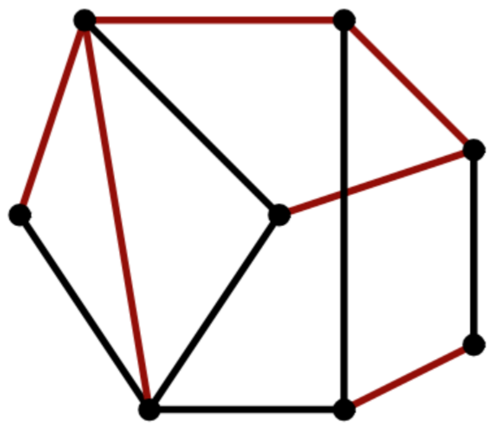
We investigate the general monomer-dimer partition function, P(x), which is a polynomial in the monomer activity, x, with coefficients depending on the dimer activities. Our main result is …

More examples

Want: Combinatorial interpretation for the difference

(8) Log-concavity of the *forest numbers*: $f_k(G)^2 \geq f_{k+1}(G)\, f_{k-1}(G)$

$f_k(G) := \#$ spanning k-forests in $G = (V, E)$



Conjectured: [Mason, 1972], [Welsh, 1976]

[Adiprasito–Huh–Katz, 2018] (*Hodge theory*)

[Brändén–Huh, 2020], [Anari et. al, 2018] (*Lorentzian polynomials*)

[Chan–P., 2021] (*linear algebra*)

Open Problem:

Find a combinatorial interpretation for $\rho_k(G) := f_k(G)^2 - f_{k+1}(G)\, f_{k-1}(G)$

More precisely, is $\rho_k(G) \in \#\mathrm{P}$ *?*

Note: Computing $f_k(G)$ is #P-complete.

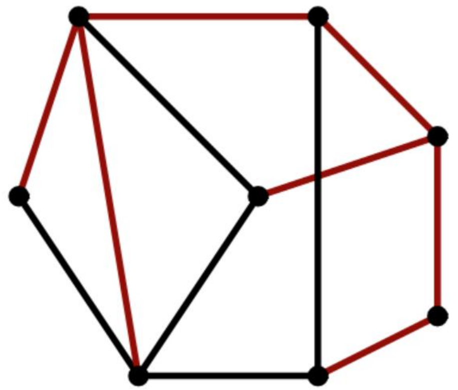
More examples

Want: Combinatorial interpretation for the difference

(9) *Grimmett's inequality* [Grimmett, 1976] (*linear algebra*)

$$t(G) \leq \frac{1}{n}\left(\frac{2m}{n-1}\right)^{n-1}$$

$t(G) =$ *number of spanning trees* in simple $G = (V, E)$, $|V| = n$ and $|E| = m$



Observation:

$\tau(G) \in \mathrm{FP}$, i.e. can be computed in polynomial time (by *matrix-tree theorem*)

$$\tau(G) := (2m)^{n-1} - n(n-1)^{n-1} t(G) \geq 0$$

<u>Therefore</u>, $\tau(G) \in \#\mathrm{P}$, i.e. $\tau(G)$ has a combinatorial interpretation (from Computational Complexity POV).

More examples

Want: Combinatorial interpretation for the difference

(10) *Kleitman's inequality* [Kleitman, 1966] (*induction*)

Example:

$$\mathbb{P}\big[H \text{ is Hamiltonian}\big] \geq \mathbb{P}\big[H \text{ is Hamiltonian} \,\big|\, H \text{ is planar}\big]$$

H is a random subgraph of a fixed $G = (V, E)$

Why works: *planarity* is closed down, *Hamiltonicity* is closed up, so they have *negative correlation*.

Kleitman's inequality generalizes to

- the *FKG inequality* (Fortuin—Kasteleyn–Ginibre, 1971)
- the *four functions inequality* (Ahlswede–Daykin, 1978)

Where are we?

Computer Science > Computational Complexity

[Submitted on 27 Apr 2022]

What is in #P and what is not?

Christian Ikenmeyer, Igor Pak 82 pp.

Theorem [Ikenmeyer–P., 2022]

The four functions inequality is not in #P.

Mid talk summary:

- Combinatorial inequalities can be easy, hard and very hard.
- The way to judge an inequality is by its proof.
- Combinatorial proofs are *the best*, but not all have them.
- Sometimes linear algebra proofs come to the rescue. Look for those!

Note: From this point on, we consider only *log-concave poset inequalities*.

arXiv.org > math > arXiv:2110.10740

Mathematics > Combinatorics

[Submitted on 20 Oct 2021]

Log-concave poset inequalities

Swee Hong Chan, Igor Pak

Comments: 71 pages, 4 figures

Stanley's inequality

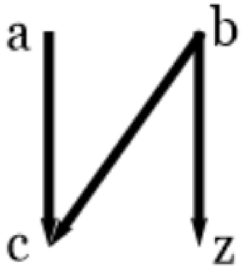
Let $\mathcal{P} := (X, \prec)$ be a poset on $n := |X|$ elements. Fix $z \in X$.

A *linear extension* of $\mathcal{P}$ is a bijection $L : X \to \{1, \ldots, n\}$, such that $L(x) < L(y)$ for all $x \prec y$.

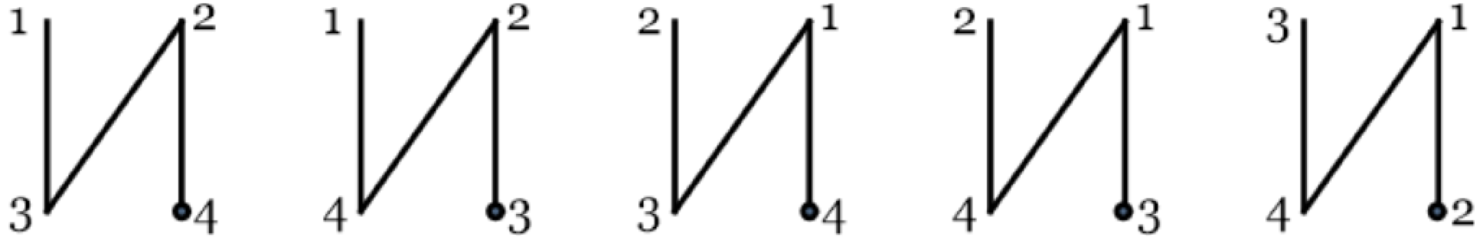
Denote by $\mathcal{E} := \mathcal{E}(P)$ the set of linear extensions of $\mathcal{P}$.

Let $\mathcal{E}_k := \{L \in \mathcal{E} : L(z) = k\}$, $\mathrm{N}(k) := |\,\mathcal{E}_k\,|$.

Theorem [Stanley, 1981]: $\mathrm{N}(k)^2 \geq \mathrm{N}(k-1)\mathrm{N}(k+1)$ for all $1 < k < n$.



a<c, b<c, b<z



$\mathrm{N}(2) = 1, \;\; \mathrm{N}(3) = 2, \;\; \mathrm{N}(4) = 2$

Weighted Stanley inequality

Let $\omega : X \to \mathbb{R}_{>0}$ be *weight function* on X. We say that ω is *order-reversing* if:

$$x \preccurlyeq y \quad \Rightarrow \quad \omega(x) \geq \omega(y).$$

Fix $z \in X$. Define $\omega : \mathcal{E} \to \mathbb{R}_{>0}$ by

$$\omega(L) := \prod_{x\,:\,L(x)<L(z)} \omega(x),$$

and

$$\mathrm{N}_\omega(k) := \sum_{L \in \mathcal{E}_k} \omega(L), \quad \text{for all } 1 \leq k \leq n.$$

Theorem [Chan–P.'21]: $\mathrm{N}_\omega(k)^2 \geq \mathrm{N}_\omega(k-1)\,\mathrm{N}_\omega(k+1)$ for all $1 < k < n$.

Note: Our proof uses a completely novel technology of *combinatorial atlas*.

Example: Bruhat orders

Let $\sigma \in S_n$. Define the *permutation poset* $\mathcal{P}_\sigma = ([n], \prec)$, $[n] = \{1, \ldots, n\}$ by:

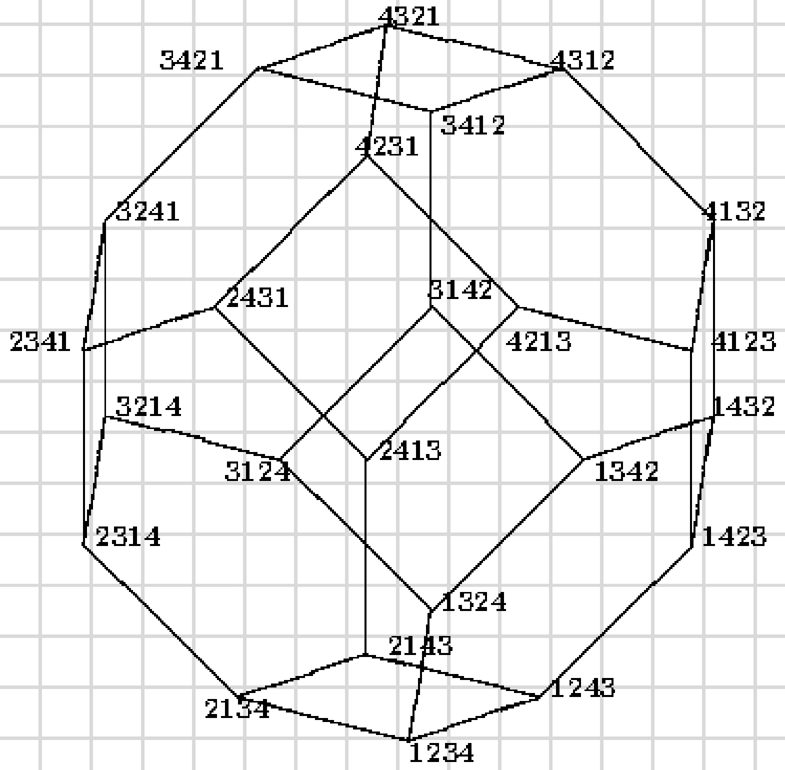
$$i \preccurlyeq j \quad \Leftrightarrow \quad i \leq j \text{ and } \sigma(i) \leq \sigma(j).$$

Then $\mathcal{E}(\mathcal{P}_\sigma) \subseteq S_n$ is the lower ideal of σ in the (weak) *Bruhat order* $\mathcal{B}_n = (S_n, \lhd)$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Fix $z \in [n]$. Then $\mathrm{N}(k) = \big|\{\nu \in S_n : \nu(z) = k, \nu \trianglelefteq \sigma\}\big|$.

Stanley's inequality: $\mathrm{N}(k)^2 \geq \mathrm{N}(k-1)\mathrm{N}(k+1)$



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Stanley's inequality: $\mathrm{N}(k)^2 \geq \mathrm{N}(k-1)\,\mathrm{N}(k+1)$

Fix $0 < q < 1$, and let $\omega(i) := q^i$, so ω is order-reversing. Then: $\omega(\nu) = q^{\beta(\nu)}$, where

$$\beta(\nu) := \sum_{i=1}^{z-1} i \cdot \chi\big(k - \nu(i)\big) \quad \text{and} \quad \chi(t) := \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

Weighted Stanley inequality: $\mathrm{N}_\omega(k)^2 \geq \mathrm{N}_\omega(k-1)\,\mathrm{N}_\omega(k+1)$, where

$$\mathrm{N}_\omega(k) = \sum_{\nu \in S_n : \nu \trianglelefteq \sigma, \nu(z) = k} q^{\beta(\nu)}.$$

Proof of Stanley's inequality

$$V(xK + yL) = \sum_{i=0}^{n} \binom{n}{i} V_i(K, L) x^{n-i} y^i,$$

THEOREM 4 (The Aleksandrov–Fenchel inequalities): For any convex bodies K, L in $\mathbb{R}^n$, the sequence

$$V_0(K, L), V_1(K, L), \ldots, V_n(K, L) \tag{9}$$

is log-concave (with no internal zeros).

Sketch of proof: Let $P = \{v_1, \ldots, v_{n-1}, v\}$. Let K be the set of all points $(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}$ satisfying:

(a) $0 \le t_i \le 1$,
(b) if $v_i \le v_j$ in P, then $t_i \le t_j$,
(c) if $v_i < v$, then $t_i = 0$.

Similarly define $L \subset \mathbb{R}^{n-1}$ by (a), (b), and:

(c′) if $v_i > v$, then $t_i = 1$.

Then K and L are convex polytopes. By an explicit decomposition of $xK + yL$ into products of simplices, it can be computed that $V_i(K, L) = N_{i+1}/(n-1)!$. The proof follows from Theorem 4. □

JOURNAL OF COMBINATORIAL THEORY, Series A 31, 56–65 (1981)

Two Combinatorial Applications of the Aleksandrov–Fenchel Inequalities*

RICHARD P. STANLEY

Log-Concave and Unimodal Sequences in Algebra, Combinatorics, and Geometry[a]

RICHARD P. STANLEY

Alexandrov-Fenchel inequalities

Theorem [Alexandrov'37, Fenchel'36] $K_1, \ldots, K_n \subset \mathbb{R}^n$ convex polytopes. Define:

$$V(K_1, \ldots, K_n) \;:=\; [\lambda_1 \cdots \lambda_n]\,\mathrm{vol}\big(\lambda_1 K_1 + \ldots + \lambda_n K_n\big)$$

Then:

$$V(K_1, K_2, K_3, \ldots, K_n)^2 \;\geq\; V(K_1, K_1, K_3, \ldots, K_n)\, V(K_2, K_2, K_3, \ldots, K_n)$$

Corollary: Sequence $\{V_k\}$ is log-concave, where $V_k := V(\underbrace{P, \ldots, P}_{k}, \underbrace{Q, \ldots, Q}_{n-k})$ for every $P, Q \subset \mathbb{R}^n$ convex polytopes.

The van der Waerden Conjecture: Two Proofs in One Year

J. H. van Lint

Note: AF is super powerful! For example, for *boxes* $K_i = [a_{i1} \times \ldots \times a_{in}]$ we have:

$$V(K_1, \ldots, K_n) \;=\; \mathrm{Per}(A), \quad \text{where} \quad A = (a_{ij})_{1 \leq i,j \leq n}$$

Now AF implies identity for the permanents which in turn easily implies *Van der Waerden Conjecture*

Equality conditions

The equality conditions for AF inequalities is a well known open problem.

For convex polytopes, it was resolved by Shenfeld and van Handel (2020).

In the special case of *order polytopes* in Stanley's proof, they obtained:

Theorem: [Shenfeld – van Handel'20] Suppose $\mathrm{N}(k) > 0$. TFAE:

(a) $\mathrm{N}(k)^2 = \mathrm{N}(k-1) \cdot \mathrm{N}(k+1)$,
(b) $\mathrm{N}(k+1) = \mathrm{N}(k) = \mathrm{N}(k-1)$,
(c) we have $f(x) > k$ for all $x \succ z$, and $g(x) > n-k+1$ for all $x \prec z$,

where $f(x) := \big|\{y \in X \, : \, y \prec x\}\big|$ and $g(x) := \big|\{y \in X \, : \, y \succ x\}\big|$ are sizes of lower and upper ideals of x, excluding x.

arXiv.org > math > arXiv:2011.04059

Mathematics > Metric Geometry

[Submitted on 8 Nov 2020]

The Extremals of the Alexandrov-Fenchel Inequality for Convex Polytopes

Yair Shenfeld, Ramon van Handel

Comments: 82 pages, 4 figures

Equality conditions

Theorem: [Shenfeld – van Handel'20] Suppose $\mathrm{N}(k) > 0$. TFAE:

(a) $\mathrm{N}(k)^2 = \mathrm{N}(k-1) \cdot \mathrm{N}(k+1)$,
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(c) we have $f(x) > k$ for all $x \succ z$, and $g(x) > n-k+1$ for all $x \prec z$,

where $f(x) := \big|\{y \in X \, : \, y \prec x\}\big|$ and $g(x) := \big|\{y \in X \, : \, y \succ x\}\big|$ are sizes of lower and upper ideals of x, excluding x.

Theorem [Chan–P.'21]: Suppose that $\mathrm{N}_\omega(k) > 0$. TFAE:

(a) $\mathrm{N}_\omega(k)^2 = \mathrm{N}_\omega(k-1) \cdot \mathrm{N}_\omega(k+1)$,
(b) there exists $\mathrm{s} = \mathrm{s}(k, z) > 0$, s.t.

$$\mathrm{N}_\omega(k+1) = s\mathrm{N}_\omega(k) = s^2\mathrm{N}_\omega(k-1),$$

(c) there exists $\mathrm{s} = \mathrm{s}(k, z) > 0$, s.t. $f(x) > k$ for all $x \succ z$, $g(x) > n-k+1$ for all $x \prec z$, and

$$\omega\big(L^{-1}(k-1)\big) = \omega\big(L^{-1}(k+1)\big) = \mathrm{s}, \quad \text{for all } \; L \in \mathcal{E}_k \; .$$

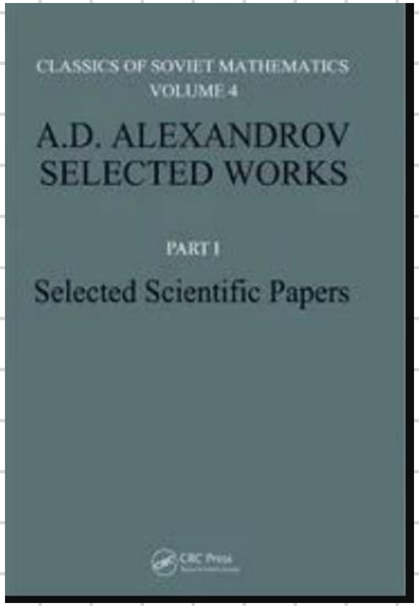
Note: Our proof again uses *combinatorial atlas* and avoids geometry altogether.

Proving AF inequalities

CHAPTER IV

TO THE THEORY OF MIXED VOLUMES OF CONVEX BODIES PART II[1]

MATEMATICHESKIĬ SBORNIK, VOL. 2 (44), No. 6, 1205–1238 (1937).



Journal of Functional Analysis
Volume 274, Issue 7, 1 April 2018, Pages 2061-2088

ELSEVIER

A remark on the Alexandrov–Fenchel inequality

Xu Wang

Comptes Rendus Mathematique
Volume 357, Issue 8, August 2019, Pages 676-680

ELSEVIER

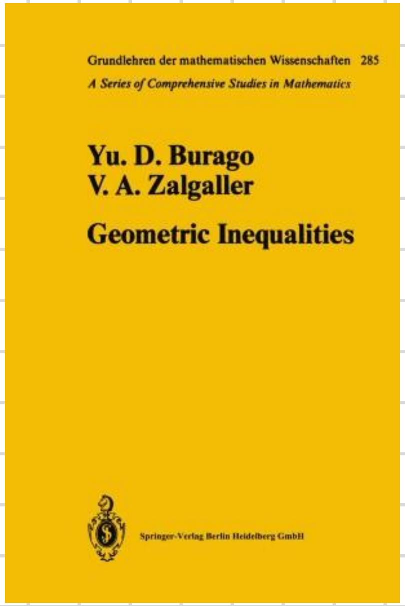
Functional analysis/Geometry

One more proof of the Alexandrov–Fenchel inequality

Une autre preuve de l'inégalité d'Alexandrov–Fenchel

Presented by Gilles Pisier

Dario Cordero-Erausquin [a], Bo'az Klartag [b], Quentin Merigot [c], Filippo Santambrogio [d]



BONNESEN-TYPE INEQUALITIES IN ALGEBRAIC GEOMETRY, I: INTRODUCTION TO THE PROBLEM

From the book Seminar on Differential Geometry. (AM-102), Volume 102

B. Teissier

Annals of Mathematics **176** (2012), 925–978
http://dx.doi.org/10.4007/annals.2012.176.2.5

Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory

By Kiumars Kaveh and A. G. Khovanskii

PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY
Volume 147, Number 12, December 2019, Pages 5385–5402
https://doi.org/10.1090/proc/14651
Article electronically published on June 10, 2019

MIXED VOLUMES AND THE BOCHNER METHOD

YAIR SHENFELD AND RAMON VAN HANDEL

Does an elementary proof of AF inequality give an elementary proof of Stanley's inequality?

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 147, Number 12, December 2019, Pages 5385–5402
https://doi.org/10.1090/proc/14651
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MIXED VOLUMES AND THE BOCHNER METHOD

YAIR SHENFELD AND RAMON VAN HANDEL

Answer: Yes. This is what we did!
Along the way we introduces new linear algebraic setting which proved useful for other log-concave inequalities.

Note: Ironically, [SvH'20] doesn't actually use [SvH'19]. Our proof uses ideas from [SvH'19] to obtain re-rederive and generalize equality conditions for Stanley's inequality in [SvH'20]

"While we originally developed the Bochner method in the hope that it would shed light on AF equality cases, *this was a complete failure*. It turns out the Bochner method says *nothing* new about AF equality." – Ramon van Handel (Oct 15, 2021)

How to start:

Interview with Karim Adiprasito

Toufik Mansour

The idea is quite simple: log-concavity of sequences a_i can be restated as saying that a certain matrix, the matrix

$$\begin{pmatrix} a_{i+1} & a_i \\ a_i & a_{i-1} \end{pmatrix}$$

has non-positive determinant, or equivalently, it cannot be definite. To prove that, one needs to establish that the matrix arises as a bilinear form that has a geometric meaning, in our case, the Hodge-Riemann relations. Proving them is the major feat of our joint work, as we had to reprove a classical algebraic geometry result in a much larger generality than previously known. The limits of the latter are the most interesting to me and remain to be explored.

How to start:

Definition: $d \times d$ symmetric real $\mathbf{M}$ is *hyperbolic*:

(Hyp) $\langle \mathbf{v}, \mathbf{M}\,\mathbf{w} \rangle^2 \geq \langle \mathbf{v}, \mathbf{M}\,\mathbf{v} \rangle \langle \mathbf{w}, \mathbf{M}\,\mathbf{w} \rangle$ for every $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$, such that $\langle \mathbf{w}, \mathbf{M}\,\mathbf{w} \rangle > 0$.

$$\begin{pmatrix} a_{i+1} & a_i \\ a_i & a_{i-1} \end{pmatrix}$$

has non-positive determinant,

Lemma: (Hyp) $\Leftrightarrow$ $\mathbf{M}$ has at most one positive eigenvalue.
(counting multiplicity)

Note: (Hyp) is used to imply log-concavity,
it is established by an elaborate induction,
(OPE) is used to establish (Hyp) in base cases.

How the induction works

Atlas $\mathbb{A}$ construction:

Acyclic digraph $\Gamma := (\Omega, \Theta)$, $d := 2(n-1)$, and

symmetric (nonnegative) $d \times d$ matrix $\mathbf{M}_v$ for every $v \in \Omega$,

nonnegative vector $\mathbf{h}_v \in \mathbb{R}^d$ for every $v \in \Omega$,

map $\mathbf{T} : \mathbb{R}^d \to \mathbb{R}^d$ for every edge $(v, w) \in \Theta$.

Theorem 5.2 (*local–global principle*). *Let $\mathbb{A}$ be a combinatorial atlas that satisfies properties* (Inh) *and* (Pull), *and let $v \in \Omega^+$ be a non-sink regular vertex of Γ. Suppose every out-neighbor of v is hyperbolic. Then v is also hyperbolic.*

In the base cases, (Hyp) is proved by direct calculation in all posets on 3 elements. Conditions on ω are exactly those which work for the base cases, and cannot be improved for general posets.

What works for Stanley's inequality

$v = (\alpha, \beta, k, t) \in \Omega,$ $\mathbf{h}_v \in \mathbb{R}^d$ defined to have coordinates

$$\mathrm{h}_x \;:=\; \begin{cases} t & \text{if} \quad x \in Z_{\text{down}}\,, \\ 1-t & \text{if} \quad x \in Z_{\text{up}}\,. \end{cases}$$

$$\mathbf{M}_v \;:=\; t\,\mathbf{C}(\alpha,\beta,k+1) \;+\; (1-t)\,\mathbf{C}(\alpha,\beta,k).$$

$\mathbf{T}^{\langle x\rangle} : \mathbb{R}^d \to \mathbb{R}^d$ associated to the edge $(v, v^{\langle x\rangle})$

$$\left(\mathbf{T}^{\langle x\rangle}\,\mathbf{v}\right)_y \;:=\; \begin{cases} \mathrm{v}_y & \text{if} \;\; y \in \operatorname{supp}(\mathbf{M}), \\ \mathrm{v}_x & \text{if} \;\; y \in Z \setminus \operatorname{supp}(\mathbf{M}). \end{cases}$$

$$\mathrm{q}(\alpha) \;:=\; \begin{cases} \omega(\alpha) & \text{for } \alpha \in \mathcal{E}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{q}_{\alpha,\beta}(\gamma) \;:=\; \mathrm{q}(\alpha\gamma\beta)$$

$$\mathrm{q}_{\alpha,\beta}(A) \;:=\; \sum_{\gamma\in A} \mathrm{q}(\alpha\gamma\beta)$$

$$\mathrm{C}_{xy} \;:=\; \mathrm{C}_{yx} \;:=\; \sum_{\gamma\in \mathrm{Comp}_{k-1}(\alpha x,\, y\beta)} \mathrm{q}_{\alpha,\beta}(x\gamma y) \qquad \text{for} \quad x \in Z_{\text{down}}, \quad y \in Z_{\text{up}}$$

$$\mathrm{C}_{xy} \;:=\; \sum_{\gamma\in \mathrm{Comp}_{k-1}(\alpha x y,\, \beta)} \mathrm{q}_{\alpha,\beta}(xy\gamma) \qquad \text{for} \quad x \,||\, y, \quad x,\, y \in Z_{\text{down}}$$

$$\mathrm{C}_{xy} \;:=\; \mathrm{C}_{yx} \;:=\; 0 \;\; \text{for} \;\; x \prec y, \quad x,\, y \in Z_{\text{down}}$$

$$\mathrm{C}_{xy} \;:=\; \sum_{\gamma\in \mathrm{Comp}_{k-1}(\alpha,\, xy\beta)} \mathrm{q}_{\alpha,\beta}(\gamma xy) \qquad \text{for} \quad x \,||\, y, \quad x,\, y \in Z_{\text{up}}$$

$$\mathrm{C}_{xy} \;:=\; \mathrm{C}_{yx} \;:=\; 0 \qquad \text{for} \quad x \prec y, \quad x,\, y \in Z_{\text{up}}$$

$$\mathrm{C}_{xx} \;:=\; \sum_{y \succ x} \; \sum_{\gamma\in \mathrm{Comp}_{k-1}(\alpha x y,\, \beta)} \mathrm{q}_{\alpha,\beta}(xy\gamma) \qquad \text{for} \quad x \in Z_{\text{down}}$$

$$\mathrm{C}_{xx} \;:=\; \sum_{y \prec x} \; \sum_{\gamma\in \mathrm{Comp}_{k-1}(\alpha,\, yx\beta)} \mathrm{q}_{\alpha,\beta}(\gamma yx) \qquad \text{for} \quad x \in Z_{\text{up}}$$

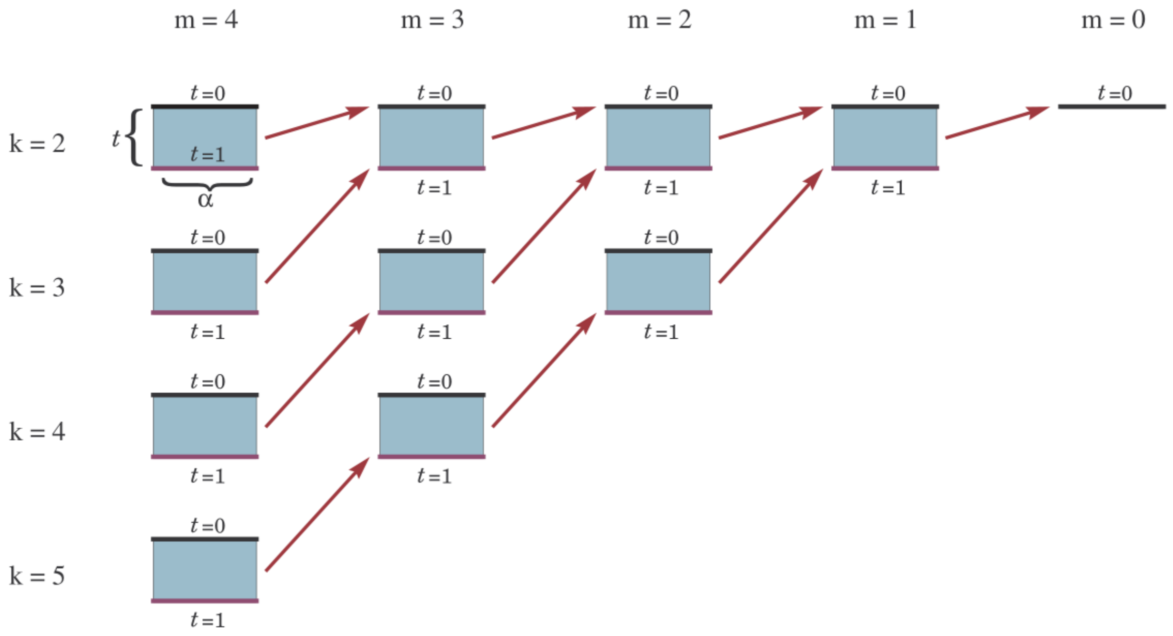


FIGURE 14.1. Graph Γ of the combinatorial atlas $\mathbb{A}$ for linear extensions of $\mathcal{P}$.

Observations on the proof

1) Stanley's inequality corresponds to $t=0$ case.

2) This limit is mild enough to allow reversing the graph and obtaining the equality conditions.

3) For general AF inequalities for general convex polytopes, the SvH proof works by induction on the dimension for combinatorially equivalent polytopes with equal normals. There is no way to avoid taking nontrivial limits in this case.

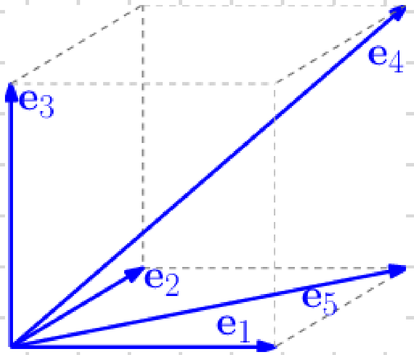
4) The proof of Stanley's inequality is *substantially harder* than the proofs of *Mason inequalities* and their refined versions, including their equality conditions which uses the same setup of combinatorial atlas, but much simpler matrix construction and case by case analysis.

Matroids

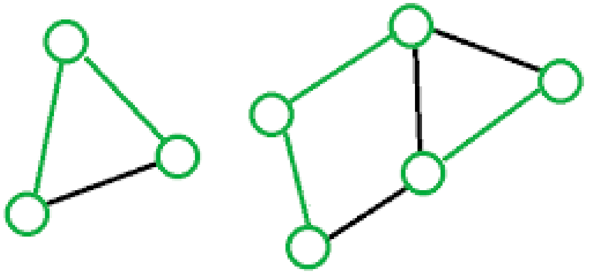
Definition [H. Whitney, 1933] A *matroid* $\mathcal{M}$ is a pair $(X, \mathcal{I})$ of a *ground set* X, and a nonempty collection of *independent sets* $\mathcal{I} \subseteq 2^X$ that satisfies the following:

- (*hereditary property*) $S \subset T,\ T \in \mathcal{I} \ \Rightarrow\ S \in \mathcal{I}$, and
- (*exchange property*) $S, T \in \mathcal{I},\ |S| < |T| \ \Rightarrow\ \exists x \in T \setminus S$ s.t. $S + x \in \mathcal{I}$.

Main Example: A *linear matroid* $\mathcal{M} = (X, \mathcal{I})$, where
$X \subset \mathbb{K}^d$ vector space over field $\mathbb{K}$, and
$\mathcal{I}$ is a collection of linearly independent subsets of X.



Simpler Example: A *graphic matroid* $\mathcal{M} = (E, \mathcal{I})$, where
E is the set of edges of a graph $G = (V, E)$, and
$\mathcal{I}$ is a collection of forests in G (subsets of edges with no cycles).



Mason inequalities

K. Adiprasito, J. Huh and E. Katz,

Theorem 1.1 (*Log-concavity for matroids*, [AHK18, Thm 9.9 (3)], formerly *Welsh–Mason conjecture*). *For a matroid* $\mathcal{M} = (X, \mathcal{I})$ *and integer* $1 \le k < \mathrm{rk}(\mathcal{M})$, *we have:*

$$\mathrm{I}(k)^2 \ \ge \ \mathrm{I}(k-1) \cdot \mathrm{I}(k+1). \tag{1.1}$$

Here $\mathcal{I}_k := \{S \in \mathcal{I}, |S| = k\}$, are *independent sets* in $\mathcal{M}$ of size k, $\mathrm{I}(k) = |\mathcal{I}_k|$, $0 \le k \le \mathrm{rk}(\mathcal{M})$.

Theorem 1.2 (*One-sided ultra-log-concavity for matroids*, [HSW21, Cor. 9], formerly *weak Mason conjecture*). *For a matroid* $\mathcal{M} = (X, \mathcal{I})$ *and integer* $1 \le k < \mathrm{rk}(\mathcal{M})$, *we have:*

$$\mathrm{I}(k)^2 \ \ge \ \left(1 + \frac{1}{k}\right) \mathrm{I}(k-1)\ \mathrm{I}(k+1). \tag{1.2}$$

J. Huh, B. Schröter and B. Wang

Theorem 1.3 (*Ultra-log-concavity for matroids*, [ALOV18, Thm 1.2] and [BH20, Thm 4.14], formerly *strong Mason conjecture*). *For a matroid* $\mathcal{M} = (X, \mathcal{I})$, $|X| = n$, *and integer* $1 \le k < \mathrm{rk}(\mathcal{M})$, *we have:*

$$\mathrm{I}(k)^2 \ \ge \ \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) \mathrm{I}(k-1)\ \mathrm{I}(k+1). \tag{1.3}$$

N. Anari, K. Liu, S. Oveis Gharan and C. Vinzant

P. Brändén and J. Huh

Refined Mason inequalities

For an independent set $S \in \mathcal{I}$ of a matroid $\mathcal{M} = (X, \mathcal{I})$, denote by

$$\mathrm{Cont}(S) := \{x \in X \setminus S \ : \ S + x \in \mathcal{I}\}$$

the set of *continuations* of S.

Let $x \sim_S y$, $x, y \in \mathrm{Cont}(S)$, when $S + x + y \notin \mathcal{I}$. Note that "$\sim_S$" is an equivalence relation.

We call an equivalence class of the relation $\sim_S$ a *parallel class* of S.

Denote by $\mathrm{Par}(S)$ the set of parallel classes of S. Define:

$$\mathrm{p}(k) := \max\{ |\mathrm{Par}(S)| \ : \ S \in \mathcal{I}_k \}.$$

Clearly, $\mathrm{p}(k) \leq n - k$.

Theorem 1.4 (*Refined log-concavity for matroids*). *For a matroid $\mathcal{M} = (X, \mathcal{I})$ and integer $1 \leq k < \mathrm{rk}(\mathcal{M})$, we have:*

$$\mathrm{I}(k)^2 \geq \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}(k-1) - 1}\right) \mathrm{I}(k-1)\, \mathrm{I}(k+1). \tag{1.6}$$

Example: graphical matroid

Old notation:
$\mathrm{I}(\mathrm{N}) = f_{\mathrm{N}}(G)$

Let $G = (V, E)$ be a connected graph on $|V| = \mathrm{N}$ vertices.

Let $k = \mathrm{N} - 2$. Observe that $\mathrm{p}(\mathrm{N} - 3) \leq 3$ since $T - e - e'$ can have at most three connected components, for every spanning tree T in G and edges $e, e' \in E$. Then:

$$\frac{\mathrm{I}(\mathrm{N}-2)^2}{\mathrm{I}(\mathrm{N}-3) \cdot \mathrm{I}(\mathrm{N}-1)} \geq \frac{3}{2}\left(1 + \frac{1}{\mathrm{N}-2}\right)$$

Refined Mason inequality [Chan-P.]

$$\frac{\mathrm{I}(\mathrm{N}-2)^2}{\mathrm{I}(\mathrm{N}-3) \cdot \mathrm{I}(n-1)} \geq_{(1.3)} \left(1 + \frac{1}{|E| - \mathrm{N} + 2}\right)\left(1 + \frac{1}{\mathrm{N}-2}\right)$$

Strong Mason inequality

Note: The refined inequality is sharp and holds if and only if G is a cycle.

Equality conditions

Theorem 1.8 (*Equality for matroids*, [MNY21, Cor. 1.2]). *Let* $\mathcal{M} = (X, \mathcal{I})$ *be a matroid on* $|X| = n$ *elements, and let* $1 \leq k < \mathrm{rk}(\mathcal{M})$. *Then:*

$$\mathrm{I}(k)^2 = \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) \mathrm{I}(k-1)\,\mathrm{I}(k+1) \tag{1.9}$$

if and only if $\mathrm{girth}(\mathcal{M}) > (k+1)$.

S. Murai, T. Nagaoka and A. Yazawa

Theorem 1.10 (*Refined equality for matroids*). *Let* $\mathcal{M} = (X, \mathcal{I})$ *be a matroid,* $1 \leq k < \mathrm{rk}(\mathcal{M})$, *and let* $\omega : X \to \mathbb{R}_{>0}$ *be a weight function. Then:*

$$\mathrm{I}_\omega(k)^2 = \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}(k-1) - 1}\right) \mathrm{I}_\omega(k-1)\,\mathrm{I}_\omega(k+1) \tag{1.11}$$

if and only if there exists $\mathrm{s}(k-1) > 0$, *such that for every* $S \in \mathcal{I}_{k-1}$ *we have:*

$$\big|\mathrm{Par}(S)\big| = \mathrm{p}(k-1)\,, \quad \textit{and} \tag{ME1}$$

$$\sum_{x \in \mathcal{C}} \omega(x) = \mathrm{s}(k-1) \quad \textit{for every} \quad \mathcal{C} \in \mathrm{Par}(S)\,. \tag{ME2}$$

[Chan-P.'21]

Thank you!