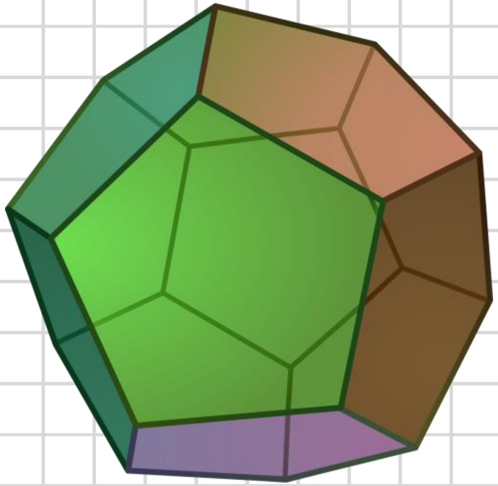
Sep 1 , 2022

Colloquium, UIUC

Combinatorial inequalities

Igor Pak, UCLA



.pdf file of the paper

Plan of the talk:

1) Overview of combinatorial inequalities and their proofs
2) Recent results

Main thing to remember:

Good inequalities deserve good proofs!

Binomial coefficients

$$\binom{n}{0} \leq \binom{n}{1} \leq \binom{n}{2} \leq \cdots \leq \binom{n}{\lfloor n/2 \rfloor}$$

(1) *Direct calculation*

$$\binom{n}{k} - \binom{n}{k-1} = \frac{n!}{k!(n-k+1)!}\left((n-k)-k\right) \geq 0$$

Binomial coefficients

$$\binom{n}{0} \leq \binom{n}{1} \leq \binom{n}{2} \leq \dots \leq \binom{n}{\lfloor n/2 \rfloor}$$

(2) *Real roots* $\Rightarrow$ *log-concavity*

Newton (1707)

$$\prod_{i=1}^{n}(x + c_i) = \sum_{k=0}^{n} a_k x^k \quad \Longrightarrow \quad a_k^2 \geq a_{k-1}a_{k+1}$$

$$(e_k)^2 \geq e_{k-1}e_{k+1}$$

log-concavity $\Rightarrow$ *unimodality*

$$a_k^2 \geq a_{k-1}a_{k+1} \quad \Longrightarrow \quad (a_1, a_2, \dots, a_n) \text{ is unimodal}$$

$$(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Binomial coefficients

$$\binom{n}{0} \leq \binom{n}{1} \leq \binom{n}{2} \leq \ldots \leq \binom{n}{\lfloor n/2 \rfloor}$$

(3) *Combinatorial interpretation*

Bertrand's ballot theorem (1887)

$$\binom{n}{k} - \binom{n}{k-1} = \text{number of ballot sequences of length } n$$

ballot sequences := 0/1 sequences with $(n-k)$ 0s, with k 1s,
and #0's $\geq$ #1's in every prefix

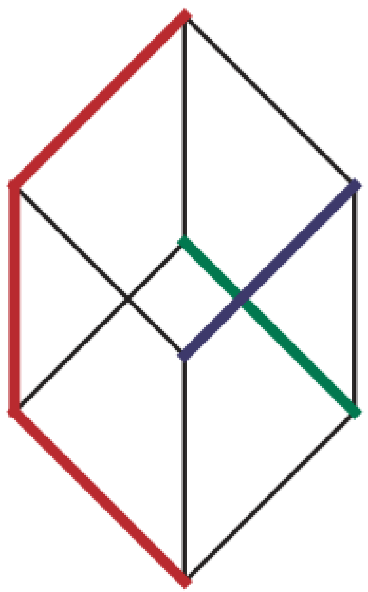
0100101011

Catalan Numbers

Binomial coefficients

$$\binom{n}{0} \leq \binom{n}{1} \leq \binom{n}{2} \leq \cdots \leq \binom{n}{\lfloor n/2 \rfloor}$$

(4) *Symmetric saturated chain decomposition*



[De Bruijn, Tengbergen, Kruyswijk, 1951]

[Greene, Kleitman, 1976]

Binomial coefficients

$$\binom{n}{0} \leq \binom{n}{1} \leq \binom{n}{2} \leq \ldots \leq \binom{n}{\lfloor n/2 \rfloor}$$

(5) *Linear algebra*

Exterior algebra $\Lambda = \mathbb{C}\langle \xi_1, \ldots, \xi_n \rangle, \quad \xi_i \xi_j = -\xi_j \xi_i, \forall i, j$

Linear map $\Phi : f \to f \cdot (\xi_1 + \ldots + \xi_n), \quad \Phi : \Lambda^{k-1} \to \Lambda^k$

Observation: Φ *is injective for* $1 \leq k \leq n/2$

Binomial coefficients

$$\binom{n}{0} \leq \binom{n}{1} \leq \binom{n}{2} \leq \cdots \leq \binom{n}{\lfloor n/2 \rfloor}$$

(1) *Direct calculation*

(2) *Real roots* $\Rightarrow$ *log-concavity*

(3) *Combinatorial interpretation*

(4) *Symmetric saturated chain decomposition*

(5) *Linear algebra*

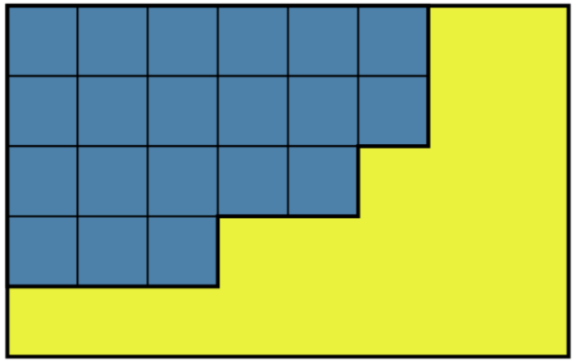
(6) *Hard Lefschetz theorem*

[Stanley, 1980]

Gaussian coefficients

***Unimodality of Gaussian coefficients*:**

$$p_{ab}(0) \le p_{ab}(1) \le \ldots \le p_{ab}(\lfloor ab/2 \rfloor)$$



$$\binom{a+b}{a}_q = \frac{(q^{a+1}-1)\cdots(q^{a+b}-1)}{(q-1)\cdots(q^b-1)} = \sum_{n=0}^{ab} p_{ab}(n)\, q^n$$

$p_{ab}(n)$ = number of partitions of n which fit rectangle $[a \times b]$

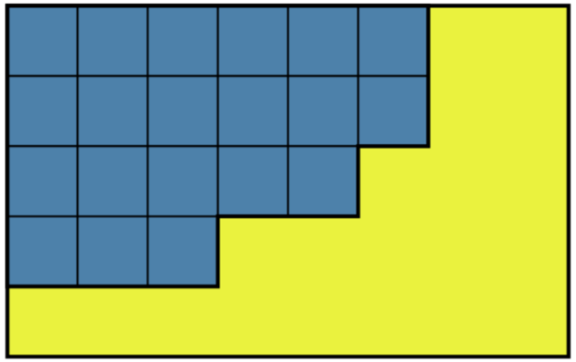
$$\binom{6}{3}_q = 1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + q^8 + q^9$$

More examples

(6) Unimodality of *Gaussian coefficients* $p_{ab}(0) \le p_{ab}(1) \le \ldots \le p_{ab}(\lfloor ab/2 \rfloor)$

$$\binom{a+b}{a}_q = \frac{(q^{a+1}-1)\cdots(q^{a+b}-1)}{(q-1)\cdots(q^b-1)} = \sum_{n=0}^{ab} p_{ab}(n)\, q^n$$

$p_{ab}(n)$ = number of partitions of n which fit rectangle $[a \times b]$



Conjectured: [Cayley, 1856]

[Sylvester, 1878] (*invariant theory*)

[Stanley, 1980] (*hard Lefschetz theorem*)

[Proctor, 1982] (*linear algebra*)

[O'Hara, 1990] (*combinatorial proof*, not injective!)

[P.–Panova, 2013] (*Kronecker coefficients*, strict)

$$g\big(a^b, a^b, (ab-k, k)\big) = p_{ab}(k) - p_{ab}(k-1)$$

$$\Rightarrow \; p_{ab}(k) - p_{ab}(k-1) \ge 1 \quad \forall a, b \ge 8$$

More examples

(6) Unimodality of *Gaussian coefficients* $p_{ab}(0) \le p_{ab}(1) \le \ldots \le p_{ab}(\lfloor ab/2 \rfloor)$

$$\binom{a+b}{a}_q = \frac{(q^{a+1}-1)\cdots(q^{a+b}-1)}{(q-1)\cdots(q^b-1)} = \sum_{n=0}^{ab} p_{ab}(n)\,q^n$$

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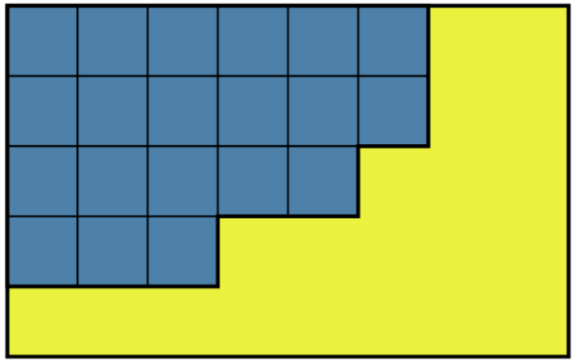

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[O'Hara, 1990] (*combinatorial proof*, not injective!)

[P.–Panova, 2013] (*Kronecker coefficients*, strict)

A SYMMETRIC CHAIN DECOMPOSITION OF L(4,n)

by

Douglas B. West

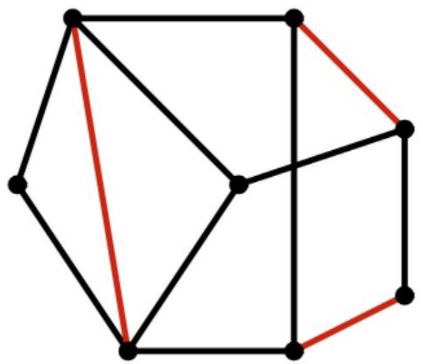
Computer Science Department
Stanford University

August 1979

Matching numbers

Log-concavity of the *matching numbers*: $m_k(G)^2 \geq m_{k+1}(G)\, m_{k-1}(G)$

$m_k(G) := \#\ k\text{-matchings in } G = (V, E)$



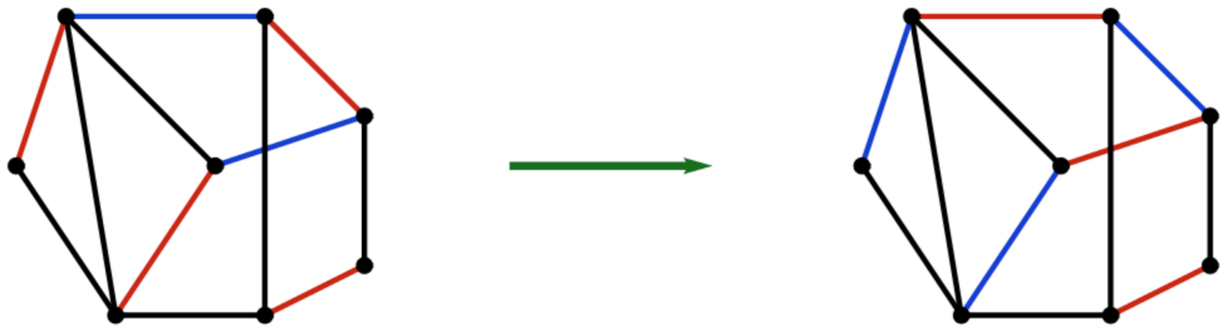
[Heilmann–Lieb, 1972] (*interlacing of eigenvalues*)

[Krattenthaler, 1996] (*injective proof*)

Theory of monomer-dimer systems

OJ Heilmann, EH Lieb - Statistical Mechanics, 1972 - Springer

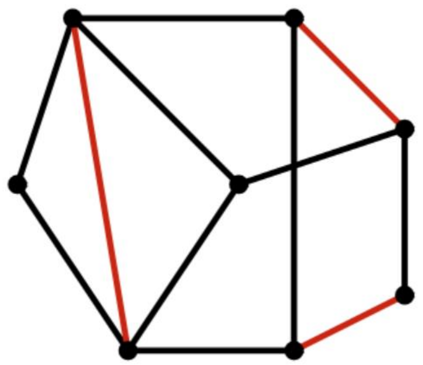
We investigate the general monomer-dimer partition function, P(x), which is a polynomial in the monomer activity, x, with coefficients depending on the dimer activities. Our main result is …



Matching numbers

Log-concavity of the *matching numbers*: $m_k(G)^2 \geq m_{k+1}(G)\, m_{k-1}(G)$

$m_k(G) := \#\ k\text{-matchings in } G = (V, E)$



[Heilmann–Lieb, 1972] (*interlacing of eigenvalues*)

[Krattenthaler, 1996] (*injective proof*)

MATHEMATICS

'Outsiders' Crack 50-Year-Old Math Problem

Three computer scientists have solved a problem central to a dozen far-flung mathematical fields.

[Submitted on 17 Jun 2013 (v1), last revised 14 Apr 2014 (this version, v4)]

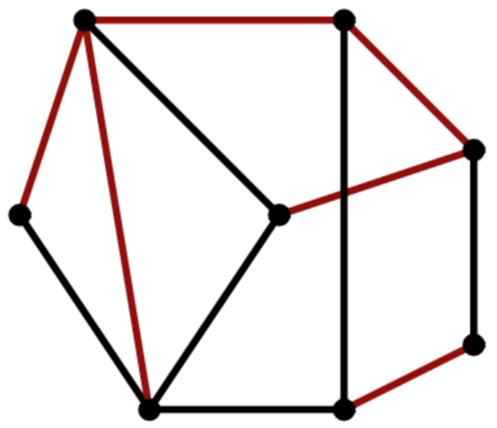
Interlacing Families II: Mixed Characteristic Polynomials and the Kadison-Singer Problem

Adam Marcus, Daniel A Spielman, Nikhil Srivastava

Forest numbers

Log-concavity of the *forest numbers*: $f_k(G)^2 \geq f_{k+1}(G)\, f_{k-1}(G)$

$f_k(G) := \#$ spanning k-forests in $G = (V, E)$



Conjectured: [Mason, 1972], [Welsh, 1976]

[Adiprasito–Huh–Katz, 2018] (*Hodge theory*)

[Brändén–Huh, 2020], [Anari et. al, 2018] (*Lorentzian polynomials*)

[Chan–P., 2021] (*linear algebra*)

Open Problem:

Find a combinatorial interpretation for $\rho_k(G) := f_k(G)^2 - f_{k+1}(G)\, f_{k-1}(G)$

More precisely, is $\rho_k(G) \in \#\mathrm{P}$ *?*

Note: Computing $f_k(G)$ is #P-complete.

Forest numbers

Log-concavity of the *forest numbers*: $f_k(G)^2 \geq f_{k+1}(G)\, f_{k-1}(G)$

$f_k(G) :=$ # spanning k-forests in $G = (V, E)$

Positivity Problems and Conjectures in Algebraic Combinatorics

Richard P. Stanley[1]

24 September 1999

Problem 25. *Are the sequences below unimodal or log-concave?*

(a) *The absolute value of the coefficients of the chromatic polynomial of a graph, or more generally, the characteristic polynomial of a matroid.*

(b) *The number of i-edge spanning forests of a graph, or more generally, the number of i-element independent sets of a matroid.*

Our own feeling is that these questions have negative answers, but that the counterexamples will be huge and difficult to construct.

Counting subgraphs

Kleitman's inequality [Kleitman, 1966] (*induction*)

Example:

$$\mathbb{P}\big[H \text{ is Hamiltonian}\big] \geq \mathbb{P}\big[H \text{ is Hamiltonian} \,\big|\, H \text{ is planar}\big]$$

H is a random subgraph of a fixed $G = (V, E)$

Why works: *planarity* is closed down, *Hamiltonicity* is closed up, so they have *negative correlation*.

Kleitman's inequality generalizes to

- the *FKG inequality* (Fortuin—Kasteleyn–Ginibre, 1971)
- the *four functions inequality* (Ahlswede–Daykin, 1978)

[Submitted on 27 Apr 2022]

What is in #P and what is not?

Christian Ikenmeyer, Igor Pak

arXiv.org > math > arXiv:2110.10740

Mathematics > Combinatorics

[Submitted on 20 Oct 2021]

Log-concave poset inequalities

Swee Hong Chan, Igor Pak

Comments: 71 pages, 4 figures

[Submitted on 3 Mar 2022]

Introduction to the combinatorial atlas

Swee Hong Chan, Igor Pak

Stanley's inequality

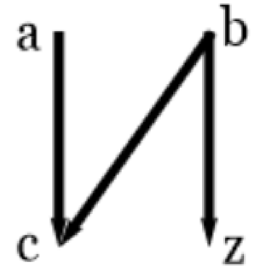
Let $\mathcal{P} := (X, \prec)$ be a poset on $n := |X|$ elements. Fix $z \in X$.

A *linear extension* of $\mathcal{P}$ is a bijection $L : X \to \{1, \ldots, n\}$, such that $L(x) < L(y)$ for all $x \prec y$.

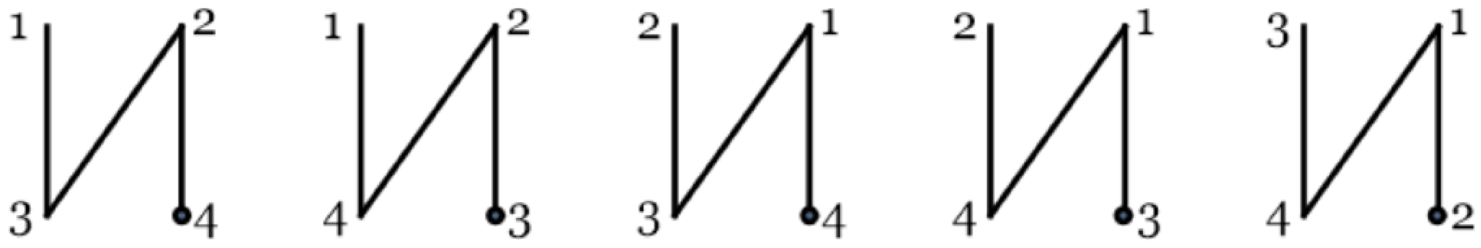
Denote by $\mathcal{E} := \mathcal{E}(P)$ the set of linear extensions of $\mathcal{P}$.

Let $\mathcal{E}_k := \{L \in \mathcal{E} : L(z) = k\}$, $\mathrm{N}(k) := |\,\mathcal{E}_k\,|$.

Theorem [Stanley, 1981]: $\mathrm{N}(k)^2 \geq \mathrm{N}(k-1)\mathrm{N}(k+1)$ for all $1 < k < n$.



a<c, b<c, b<z



$\mathrm{N}(2) = 1$, $\mathrm{N}(3) = 2$, $\mathrm{N}(4) = 2$

Weighted Stanley inequality

Let $\omega : X \to \mathbb{R}_{>0}$ be *weight function* on X. We say that ω is *order-reversing* if:

$$x \preccurlyeq y \quad \Rightarrow \quad \omega(x) \geq \omega(y).$$

Fix $z \in X$. Define $\omega : \mathcal{E} \to \mathbb{R}_{>0}$ by

$$\omega(L) := \prod_{x \,:\, L(x) < L(z)} \omega(x),$$

and

$$\mathrm{N}_\omega(k) := \sum_{L \in \mathcal{E}_k} \omega(L), \quad \text{for all } 1 \leq k \leq n.$$

Theorem [Chan–P.'21]: $\mathrm{N}_\omega(k)^2 \geq \mathrm{N}_\omega(k-1)\,\mathrm{N}_\omega(k+1)$ for all $1 < k < n$.

Note: Our proof uses a completely novel technology of *combinatorial atlas*.

Alexandrov-Fenchel inequalities

Theorem [Alexandrov'37, Fenchel'36] $K_1, \ldots, K_n \subset \mathbb{R}^n$ convex polytopes. Define:

$$V(K_1, \ldots, K_n) \;:=\; [\lambda_1 \cdots \lambda_n] \operatorname{vol}(\lambda_1 K_1 + \ldots + \lambda_n K_n)$$

Then:

$$V(K_1, K_2, K_3, \ldots, K_n)^2 \;\geq\; V(K_1, K_1, K_3, \ldots, K_n)\, V(K_2, K_2, K_3, \ldots, K_n)$$

Corollary: Sequence $\{V_k\}$ is log-concave, where $V_k := V(\underbrace{P, \ldots, P}_{k}, \underbrace{Q, \ldots, Q}_{n-k})$ for every $P, Q \subset \mathbb{R}^n$ convex polytopes.

The van der Waerden Conjecture: Two Proofs in One Year

J. H. van Lint (1980)

Note: AF is super powerful! For example, for *boxes* $K_i = [a_{i1} \times \ldots \times a_{in}]$ we have:

$$V(K_1, \ldots, K_n) \;=\; \operatorname{Per}(A), \quad \text{where} \quad A = (a_{ij})_{1 \leq i,j \leq n}$$

Now AF implies identity for the permanents which in turn easily implies *Van der Waerden Conjecture*

Proof of Stanley's inequality

JOURNAL OF COMBINATORIAL THEORY, Series A 31, 56–65 (1981)

Two Combinatorial Applications of the Aleksandrov–Fenchel Inequalities*

RICHARD P. STANLEY

$$V(xK + yL) = \sum_{i=0}^{n} \binom{n}{i} V_i(K, L) x^{n-i} y^i,$$

THEOREM 4 (The Aleksandrov–Fenchel inequalities): For any convex bodies K, L in $\mathbb{R}^n$, the sequence

$$V_0(K, L), V_1(K, L), \ldots, V_n(K, L) \tag{9}$$

is log-concave (with no internal zeros).

Sketch of proof: Let $P = \{v_1, \ldots, v_{n-1}, v\}$. Let K be the set of all points $(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}$ satisfying:

(a) $0 \le t_i \le 1$,
(b) if $v_i \le v_j$ in P, then $t_i \le t_j$,
(c) if $v_i < v$, then $t_i = 0$.

Similarly define $L \subset \mathbb{R}^{n-1}$ by (a), (b), and:

(c′) if $v_i > v$, then $t_i = 1$.

Then K and L are convex polytopes. By an explicit decomposition of $xK + yL$ into products of simplices, it can be computed that $V_i(K, L) = N_{i+1}/(n-1)!$. The proof follows from Theorem 4. □

Log-Concave and Unimodal Sequences in Algebra, Combinatorics, and Geometry[a]

RICHARD P. STANLEY

Proving AF inequalities

CHAPTER IV

TO THE THEORY OF MIXED VOLUMES
OF CONVEX BODIES
PART II[1]

MATEMATICHESKIĬ SBORNIK, VOL. 2 (44), No. 6, 1205–1238 (1937).

CLASSICS OF SOVIET MATHEMATICS
VOLUME 4

A.D. ALEXANDROV
SELECTED WORKS

PART I

Selected Scientific Papers

CRC Press

ELSEVIER

Journal of Functional Analysis

Volume 274, Issue 7, 1 April 2018, Pages 2061-2088

A remark on the Alexandrov–Fenchel inequality

Xu Wang

ELSEVIER

Comptes Rendus Mathematique

Volume 357, Issue 8, August 2019, Pages 676-680

Functional analysis/Geometry

One more proof of the Alexandrov–Fenchel inequality

Une autre preuve de l'inégalité d'Alexandrov–Fenchel

Presented by Gilles Pisier

Dario Cordero-Erausquin [a], Bo'az Klartag [b], Quentin Merigot [c], Filippo Santambrogio [d]

Grundlehren der mathematischen Wissenschaften 285

A Series of Comprehensive Studies in Mathematics

Yu. D. Burago
V. A. Zalgaller

Geometric Inequalities

Springer-Verlag Berlin Heidelberg GmbH

BONNESEN-TYPE INEQUALITIES IN ALGEBRAIC GEOMETRY, I: INTRODUCTION TO THE PROBLEM

From the book Seminar on Differential Geometry. (AM-102), Volume 102

B. Teissier

Annals of Mathematics **176** (2012), 925–978
http://dx.doi.org/10.4007/annals.2012.176.2.5

Newton-Okounkov bodies, semigroups of integral points, graded algebras and intersection theory

By KIUMARS KAVEH and A. G. KHOVANSKII

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 147, Number 12, December 2019, Pages 5385–5402
https://doi.org/10.1090/proc/14651
Article electronically published on June 10, 2019

MIXED VOLUMES AND THE BOCHNER METHOD

YAIR SHENFELD AND RAMON VAN HANDEL

Does an elementary proof of AF inequality give an elementary proof of Stanley's inequality?

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 147, Number 12, December 2019, Pages 5385–5402
https://doi.org/10.1090/proc/14651
Article electronically published on June 10, 2019

MIXED VOLUMES AND THE BOCHNER METHOD

YAIR SHENFELD AND RAMON VAN HANDEL

Answer: Yes. This is what we did!
Along the way we introduces new linear algebraic setting which proved useful for other log-concave inequalities.

How to start:

Definition: $d \times d$ symmetric real $\mathbf{M}$ is *hyperbolic*:

(Hyp) $\langle \mathbf{v}, \mathbf{M}\,\mathbf{w}\rangle^2 \geq \langle \mathbf{v}, \mathbf{M}\,\mathbf{v}\rangle\langle \mathbf{w}, \mathbf{M}\,\mathbf{w}\rangle$ for every $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$, such that $\langle \mathbf{w}, \mathbf{M}\,\mathbf{w}\rangle > 0$.

$$\begin{pmatrix} a_{i+1} & a_i \\ a_i & a_{i-1} \end{pmatrix}$$

has non-positive determinant,

Lemma: (Hyp) $\Leftrightarrow$ $\mathbf{M}$ has at most one positive eigenvalue.
(counting multiplicity)

Note: (Hyp) is used to imply log-concavity,
it is established by an elaborate induction,
(OPE) is used to establish (Hyp) in base cases.

How the induction works

Atlas $\mathbb{A}$ construction:

Acyclic digraph $\Gamma := (\Omega, \Theta)$, $d := 2(n-1)$, and

symmetric (nonnegative) $d \times d$ matrix $\mathbf{M}_v$ for every $v \in \Omega$,

nonnegative vector $\mathbf{h}_v \in \mathbb{R}^d$ for every $v \in \Omega$,

map $\mathbf{T} : \mathbb{R}^d \to \mathbb{R}^d$ for every edge $(v, w) \in \Theta$.

Theorem 5.2 (*local–global principle*). *Let $\mathbb{A}$ be a combinatorial atlas that satisfies properties* (Inh) *and* (Pull), *and let $v \in \Omega^+$ be a non-sink regular vertex of Γ. Suppose every out-neighbor of v is hyperbolic. Then v is also hyperbolic.*

In the base cases, (Hyp) is proved by direct calculation in all posets on 3 elements. Conditions on ω are exactly those which work for the base cases, and cannot be improved for general posets.

What works for Stanley's inequality

$v = (\alpha, \beta, k, t) \in \Omega$, $\mathbf{h}_v \in \mathbb{R}^d$ defined to have coordinates

$$\mathrm{h}_x \ := \begin{cases} t & \text{if} \quad x \in Z_{\text{down}}\,, \\ 1-t & \text{if} \quad x \in Z_{\text{up}}\,. \end{cases}$$

$$\mathbf{M}_v \ := \ t\,\mathbf{C}(\alpha,\beta,k+1) \ + \ (1-t)\,\mathbf{C}(\alpha,\beta,k).$$

$\mathbf{T}^{\langle x\rangle} : \mathbb{R}^d \to \mathbb{R}^d$ associated to the edge $(v, v^{\langle x\rangle})$

$$\left(\mathbf{T}^{\langle x\rangle}\,\mathbf{v}\right)_y \ := \begin{cases} \mathrm{v}_y & \text{if} \ \ y \in \operatorname{supp}(\mathbf{M}), \\ \mathrm{v}_x & \text{if} \ \ y \in Z \setminus \operatorname{supp}(\mathbf{M}). \end{cases}$$

$$\mathrm{q}(\alpha) \ := \begin{cases} \omega(\alpha) & \text{for } \alpha \in \mathcal{E}, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathrm{q}_{\alpha,\beta}(\gamma) \ := \ \mathrm{q}(\alpha\gamma\beta)$$

$$\mathrm{q}_{\alpha,\beta}(A) \ := \ \sum_{\gamma\in A} \mathrm{q}(\alpha\gamma\beta)$$

$$\mathrm{C}_{xy} \ := \ \mathrm{C}_{yx} \ := \sum_{\gamma \in \mathrm{Comp}_{k-1}(\alpha x, y\beta)} \mathrm{q}_{\alpha,\beta}(x\gamma y) \qquad \text{for} \quad x \in Z_{\text{down}}, \quad y \in Z_{\text{up}}$$

$$\mathrm{C}_{xy} \ := \sum_{\gamma \in \mathrm{Comp}_{k-1}(\alpha xy, \beta)} \mathrm{q}_{\alpha,\beta}(xy\gamma) \qquad \text{for} \quad x \,||\, y, \quad x,\, y \in Z_{\text{down}}$$

$$\mathrm{C}_{xy} \ := \ \mathrm{C}_{yx} \ := \ 0 \quad \text{for} \quad x \prec y, \quad x,\, y \in Z_{\text{down}}$$

$$\mathrm{C}_{xy} \ := \sum_{\gamma \in \mathrm{Comp}_{k-1}(\alpha, xy\beta)} \mathrm{q}_{\alpha,\beta}(\gamma xy) \qquad \text{for} \quad x \,||\, y, \quad x,\, y \in Z_{\text{up}}$$

$$\mathrm{C}_{xy} \ := \ \mathrm{C}_{yx} \ := \ 0 \qquad \text{for} \quad x \prec y, \quad x,\, y \in Z_{\text{up}}$$

$$\mathrm{C}_{xx} \ := \sum_{y \succ x} \ \sum_{\gamma \in \mathrm{Comp}_{k-1}(\alpha xy, \beta)} \mathrm{q}_{\alpha,\beta}(xy\gamma) \qquad \text{for} \quad x \in Z_{\text{down}}$$

$$\mathrm{C}_{xx} \ := \sum_{y \prec x} \ \sum_{\gamma \in \mathrm{Comp}_{k-1}(\alpha, yx\beta)} \mathrm{q}_{\alpha,\beta}(\gamma yx) \qquad \text{for} \quad x \in Z_{\text{up}}$$

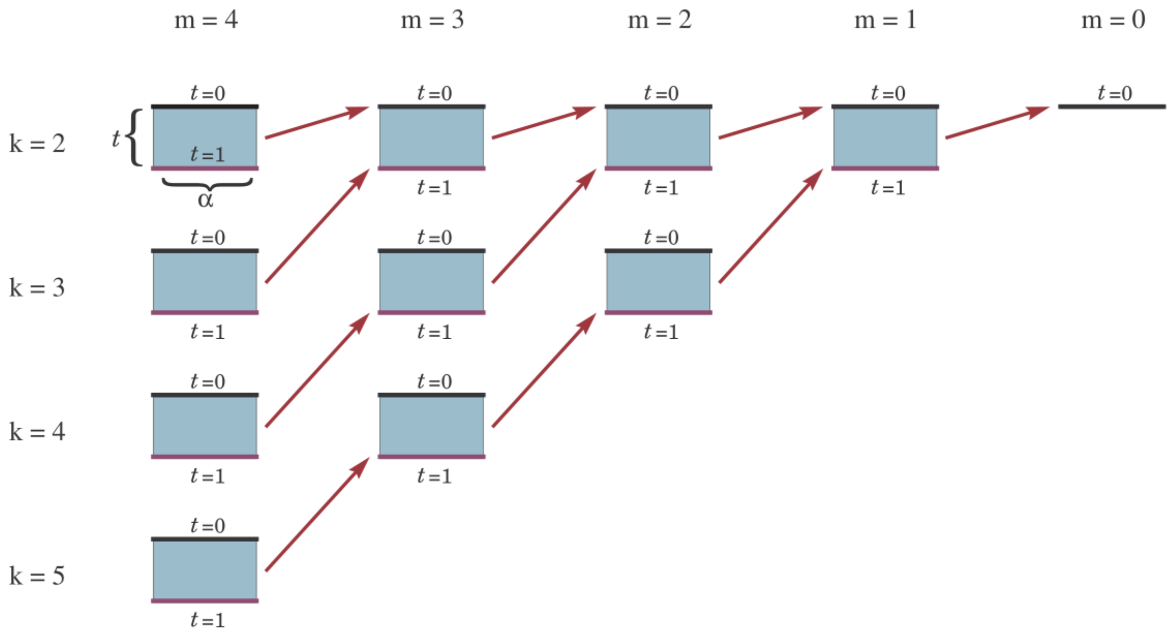


FIGURE 14.1. Graph Γ of the combinatorial atlas $\mathbb{A}$ for linear extensions of $\mathcal{P}$.

Observations on the proof

1) Stanley's inequality corresponds to *t*=0 case.

2) This limit is mild enough to allow reversing the graph and obtaining the equality conditions.

3) For general AF inequalities for general convex polytopes, the SvH proof works by induction on the dimension for combinatorially equivalent polytopes with equal normals. There is no way to avoid taking nontrivial limits in this case.

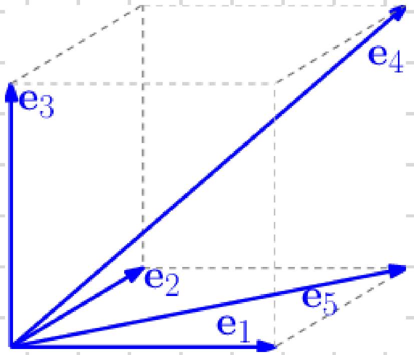
4) The proof of Stanley's inequality is *substantially harder* than the proofs of *Mason inequalities* and their refined versions, including their equality conditions which uses the same setup of combinatorial atlas, but much simpler matrix construction and case by case analysis.

Matroids

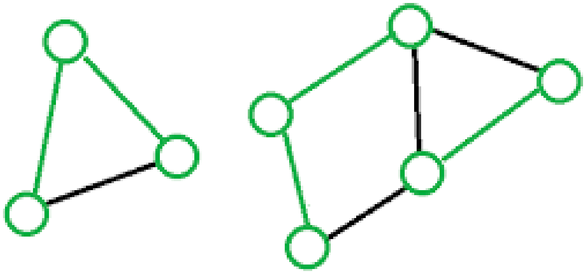
Definition [H. Whitney, 1933] A *matroid* $\mathcal{M}$ is a pair $(X, \mathcal{I})$ of a *ground set* X, and a nonempty collection of *independent sets* $\mathcal{I} \subseteq 2^X$ that satisfies the following:

- (*hereditary property*) $S \subset T,\ T \in \mathcal{I} \ \Rightarrow\ S \in \mathcal{I}$, and
- (*exchange property*) $S, T \in \mathcal{I},\ |S| < |T| \ \Rightarrow\ \exists x \in T \setminus S$ s.t. $S + x \in \mathcal{I}$.

Main Example: A *linear matroid* $\mathcal{M} = (X, \mathcal{I})$, where
$X \subset \mathbb{K}^d$ vector space over field $\mathbb{K}$, and
$\mathcal{I}$ is a collection of linearly independent subsets of X.



Simpler Example: A *graphic matroid* $\mathcal{M} = (E, \mathcal{I})$, where
E is the set of edges of a graph $G = (V, E)$, and
$\mathcal{I}$ is a collection of forests in G (subsets of edges with no cycles).



Mason inequalities

K. Adiprasito, J. Huh and E. Katz,

Theorem 1.1 (*Log-concavity for matroids*, [AHK18, Thm 9.9 (3)], formerly *Welsh–Mason conjecture*). *For a matroid $\mathcal{M} = (X, \mathcal{I})$ and integer $1 \le k < \operatorname{rk}(\mathcal{M})$, we have:*

$$\mathrm{I}(k)^2 \ \ge\ \mathrm{I}(k-1) \cdot \mathrm{I}(k+1)\,. \tag{1.1}$$

Here $\mathcal{I}_k := \{S \in \mathcal{I},\, |S| = k\}$, are *independent sets* in $\mathcal{M}$ of size k, $\mathrm{I}(k) = |\mathcal{I}_k|$, $0 \le k \le \operatorname{rk}(\mathcal{M})$.

Theorem 1.2 (*One-sided ultra-log-concavity for matroids*, [HSW21, Cor. 9], formerly *weak Mason conjecture*). *For a matroid $\mathcal{M} = (X, \mathcal{I})$ and integer $1 \le k < \operatorname{rk}(\mathcal{M})$, we have:*

$$\mathrm{I}(k)^2 \ \ge\ \left(1 + \frac{1}{k}\right) \mathrm{I}(k-1)\ \mathrm{I}(k+1)\,. \tag{1.2}$$

J. Huh, B. Schröter and B. Wang

Theorem 1.3 (*Ultra-log-concavity for matroids*, [ALOV18, Thm 1.2] and [BH20, Thm 4.14], formerly *strong Mason conjecture*). *For a matroid $\mathcal{M} = (X, \mathcal{I})$, $|X| = n$, and integer $1 \le k < \operatorname{rk}(\mathcal{M})$, we have:*

$$\mathrm{I}(k)^2 \ \ge\ \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) \mathrm{I}(k-1)\ \mathrm{I}(k+1)\,. \tag{1.3}$$

P. Brändén and J. Huh

N. Anari, K. Liu, S. Oveis Gharan and C. Vinzant

Refined Mason inequalities

For an independent set $S \in \mathcal{I}$ of a matroid $\mathcal{M} = (X, \mathcal{I})$, denote by

$$\mathrm{Cont}(S) := \{x \in X \setminus S \ : \ S + x \in \mathcal{I}\}$$

the set of *continuations* of S.

Let $x \sim_S y$, $x, y \in \mathrm{Cont}(S)$, when $S + x + y \notin \mathcal{I}$. Note that "$\sim_S$" is an equivalence relation.

We call an equivalence class of the relation $\sim_S$ a *parallel class* of S.

Denote by $\mathrm{Par}(S)$ the set of parallel classes of S. Define:

$$\mathrm{p}(k) := \max\{ |\mathrm{Par}(S)| \ : \ S \in \mathcal{I}_k \}.$$

Clearly, $\mathrm{p}(k) \leq n - k$.

Theorem 1.4 (*Refined log-concavity for matroids*). *For a matroid $\mathcal{M} = (X, \mathcal{I})$ and integer $1 \leq k < \mathrm{rk}(\mathcal{M})$, we have:*

$$\mathrm{I}(k)^2 \geq \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{\mathrm{p}(k-1) - 1}\right) \mathrm{I}(k-1)\ \mathrm{I}(k+1). \tag{1.6}$$

Example: graphical matroid

Old notation:
$\mathrm{I}(\mathrm{N}) = f_{\mathrm{N}}(G)$

Let $G = (V, E)$ be a connected graph on $|V| = \mathrm{N}$ vertices.

Let $k = \mathrm{N} - 2$. Observe that $\mathrm{p}(\mathrm{N} - 3) \leq 3$ since $T - e - e'$ can have at most three connected components, for every spanning tree T in G and edges $e, e' \in E$. Then:

$$\frac{\mathrm{I}(\mathrm{N}-2)^2}{\mathrm{I}(\mathrm{N}-3) \cdot \mathrm{I}(\mathrm{N}-1)} \geq \frac{3}{2}\left(1 + \frac{1}{\mathrm{N}-2}\right)$$

Refined Mason inequality [Chan-P.]

$$\frac{\mathrm{I}(\mathrm{N}-2)^2}{\mathrm{I}(\mathrm{N}-3) \cdot \mathrm{I}(n-1)} \geq_{(1.3)} \left(1 + \frac{1}{|E| - \mathrm{N} + 2}\right)\left(1 + \frac{1}{\mathrm{N}-2}\right)$$

Strong Mason inequality

Note: The refined inequality is sharp and holds if and only if G is a cycle.

Thank you!