Oda's strong factorization conjecture on stellar subdivisions of triangulations

Joint Algebra & Combinatorics Seminar, UCLA
(joint talk with Joaquín Moraga)
[joint work with Karim Adiprasito]
Triangulations in the plane
Stellar subdivisions in the plane
Main question: Do every two triangulations of the same polygon have a common (iterated) stellar subdivision?
Common stellar subdivisions

A

B

C
Proof of Oda’s conjecture in the plane

**Theorem** [strong factorization for convex polygons in the plane]

Suppose triangulations $T, T'$ of a convex polygon $Q$ have at most $n$ vertices. Then there is a triangulation $S \in \mathcal{T}(Q)$ which can be obtained by a sequence of at most $30n^3$ stellar subdivisions from both $T$ and $T'$.

[Danilov’83], [Włodarczyk’97]: connectivity by stellar and inverse stellar subdivisions

[Ewald’86]: ad hoc proof of the plane case (?, in German)

[Morelli’96], false proof in full generality
Step 1: scaling a fin of triangulation $A \rightarrow A'$
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Step 2: Stellar subdivision $B \rightarrow B'$ s.t. $A' \subset B'$
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Step 2: Stellar subdivision $B \rightarrow B'$ s.t. $A' \subset B'$
Step 3: Stellar subdivision $A \rightarrow B'$ (when $A$ is a stripe)

Key idea: *shedding process*
Step 3: Stellar subdivision $A \rightarrow B'$ (when $A$ is a stripe)

Shedding
Step 3: Stellar subdivision $A \rightarrow B'$ (when $A$ is a stripe)

Stellar subdivision
Step 3: Stellar subdivision $A' \rightarrow B'$ (general case)

Sketch:
1. In Step 1 add to $A$ vertices from $B$ (making a fan)
2. Step 2 is the same (no vertices not on the fan are added)
3. Step 3 is the same (some stellar subdivisions can now be skipped)
1. Bing’s lemma’83 reduces to the case when $A$ is a simplex
2. Variation on shellability called *relative shellability* w.r.t a vertex (by Adiprasito-Benedetti’17)
3. Brugesser–Mani trick’71 (shelling of a polyhedron)

**Lemma 5.1** (Bing’s extension lemma, [Bing83, §I.2]). Let $X \subset \Delta$ be a geometric complex embedded in a simplex. Then there is a triangulation of $\Delta$ that contains $X$ as a subcomplex.
Shellability

An ordering $C_1, C_2, \ldots$ of the maximal simplices of $\Delta$ is a shelling if the complex

$$B_k := \left( \bigcup_{i=1}^{k-1} C_i \right) \cap C_k$$

is pure and of dimension $\dim C_k - 1$ for all $k = 2, 3, \ldots$.

Shellability is NP-complete

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- A shellable complex is homotopy equivalent to a wedge sum of spheres
Shellability

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Topological result

Theorem [former Alexander’s conjecture’30]
Every two PL homeomorphic simplicial complexes have combinatorially isomorphic stellar subdivisions.

Observation [Anderson-Mnëv’06]
Oda’s conjecture implies Alexander’s conjecture.
Thank you!