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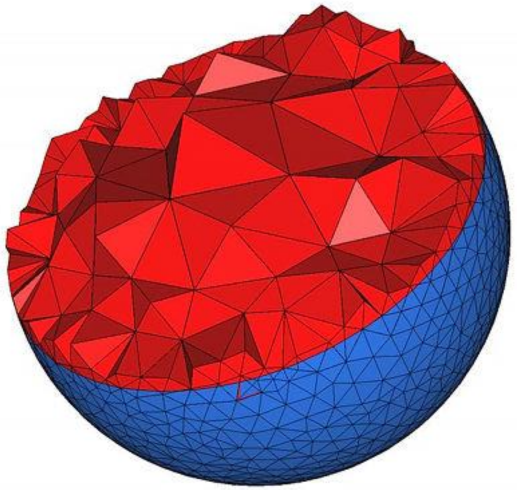
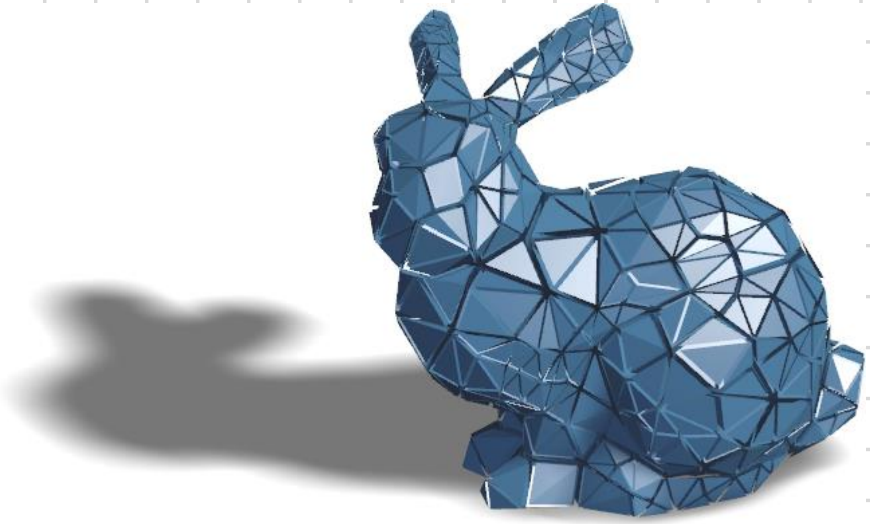
Oda's strong factorization conjecture on stellar subdivisions of triangulations

Joint Algebra & Combinatorics Seminar, UCLA

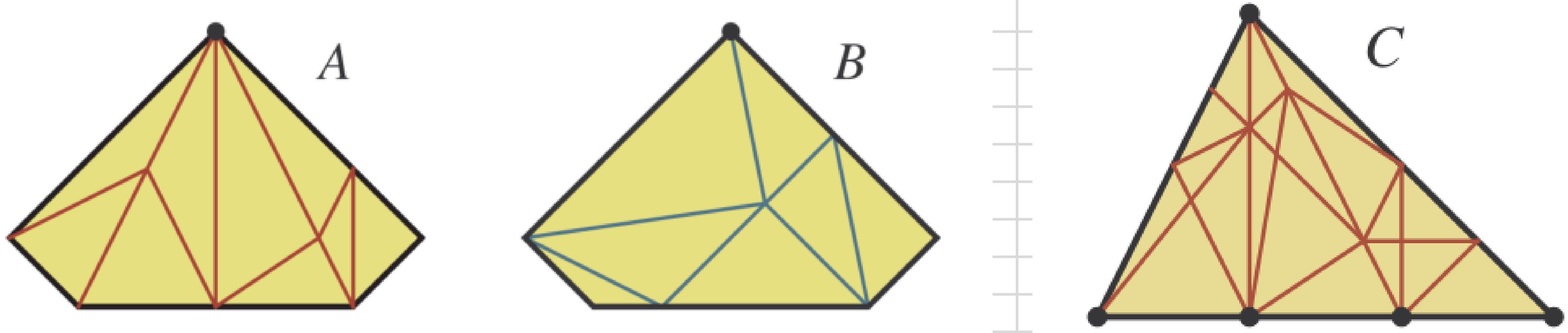
(joint talk with Joaquín Moraga)

[joint work with **Karim Adiprasito**]

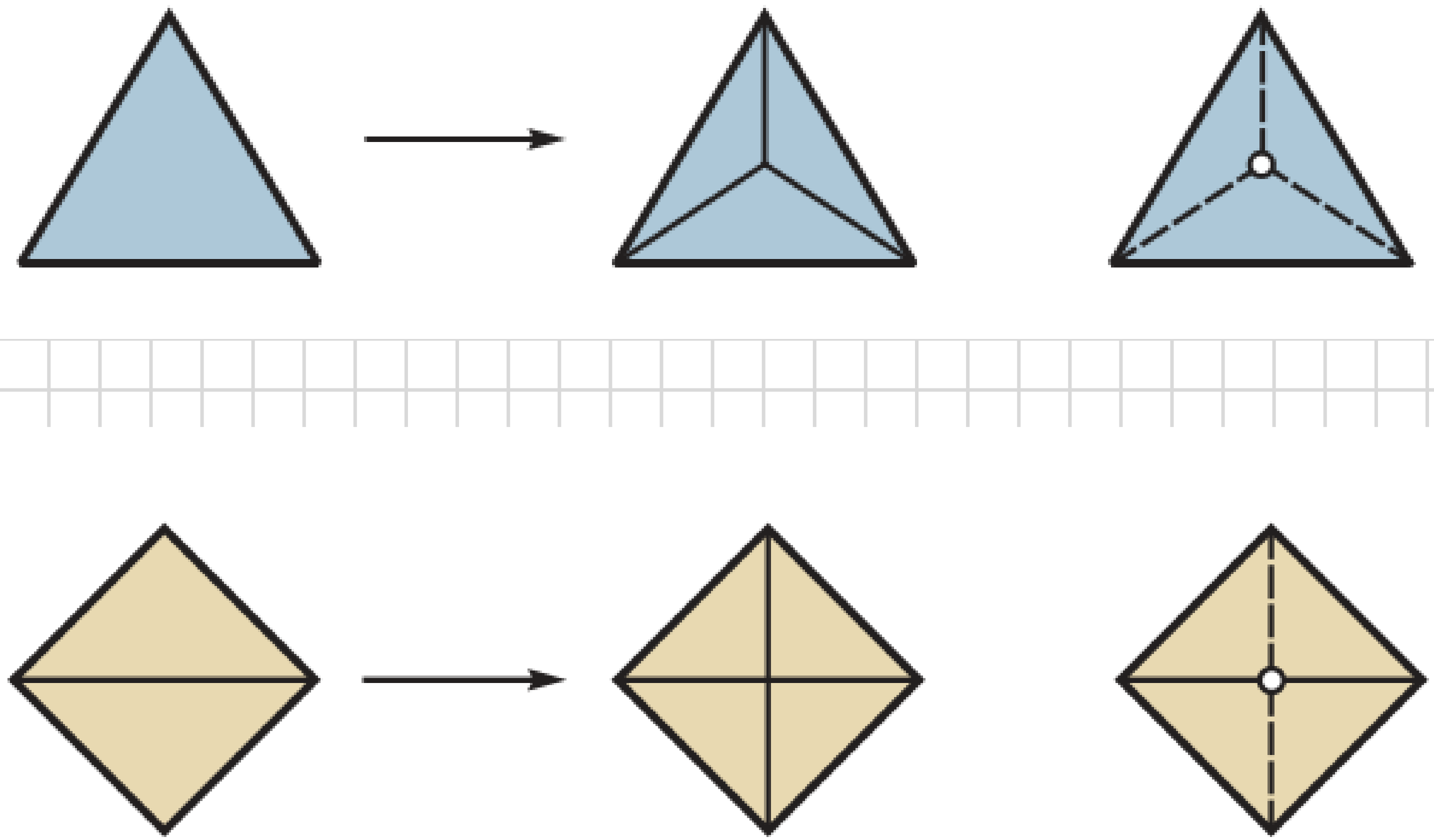




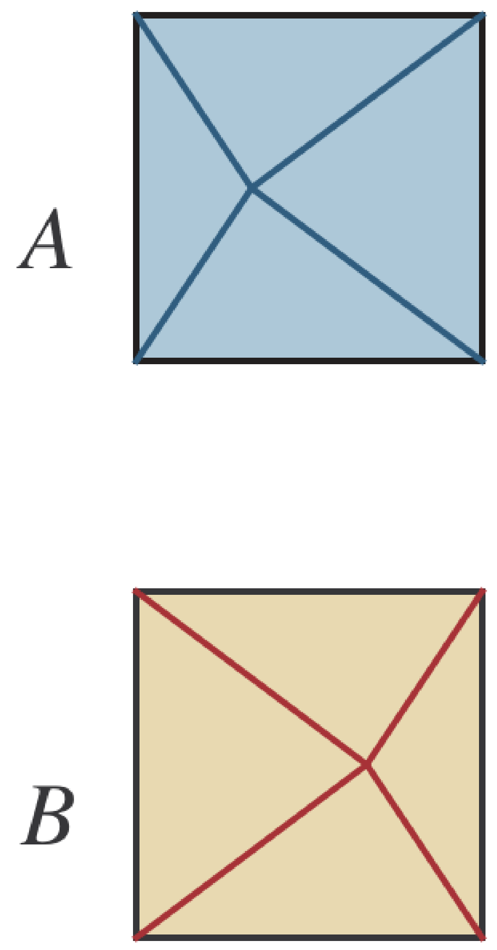
Triangulations in the plane



Stellar subdivisions in the plane

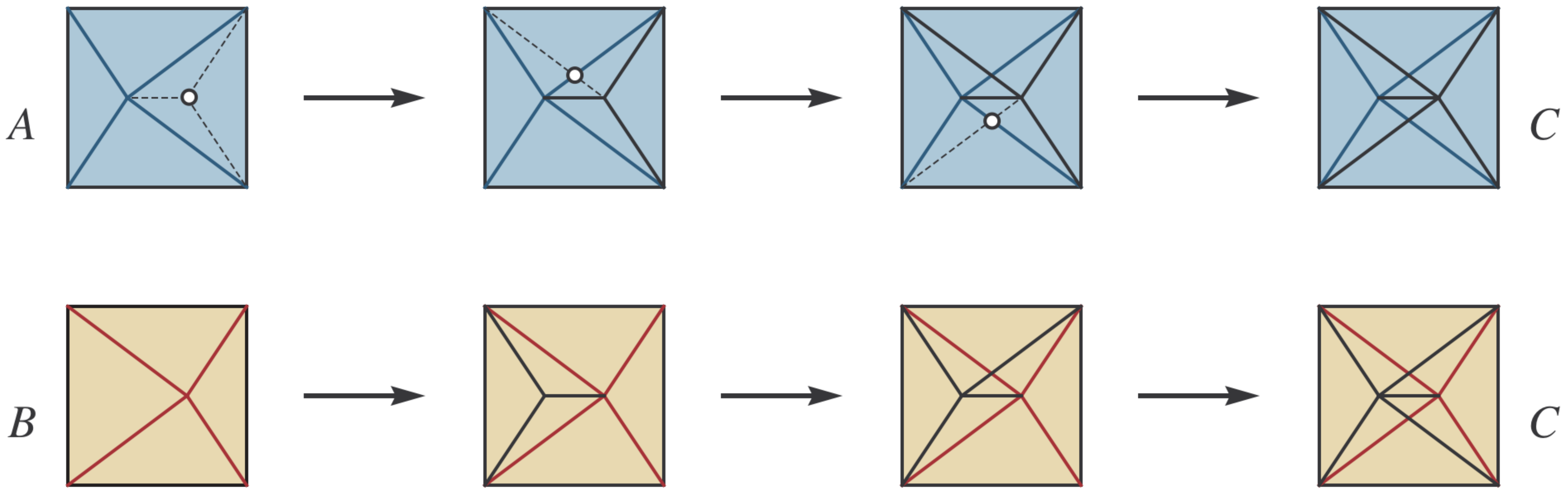


Common stellar subdivisions



Main question: Do every two triangulations of the same polygon have a common (iterated) stellar subdivision?

Common stellar subdivisions

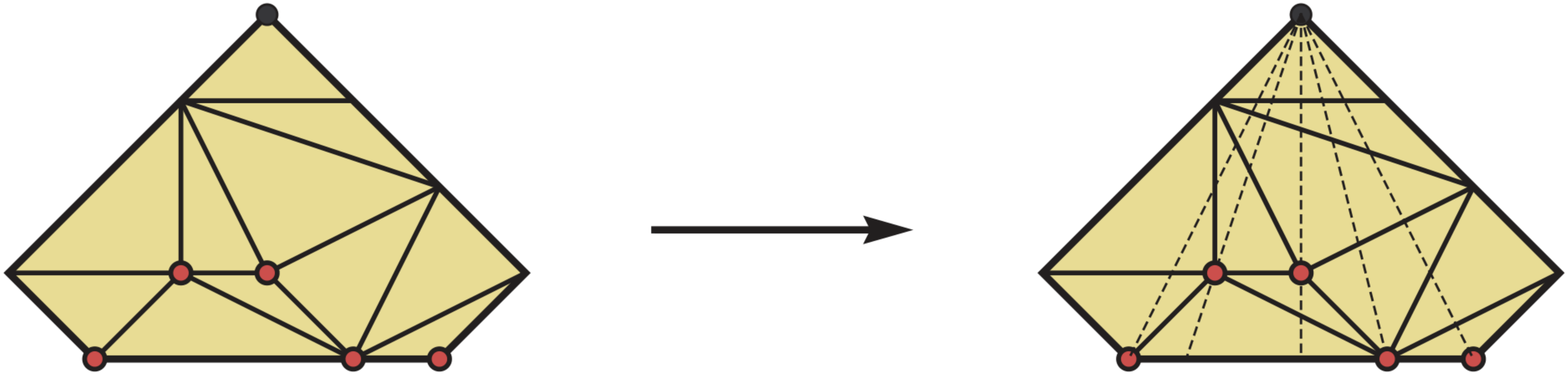


Proof of Oda's conjecture in the plane

Theorem [*strong factorization for convex polygons in the plane*]
Suppose triangulations T, T' of a convex polygon Q have at most n vertices. Then there is a triangulation $S \in \mathcal{T}(Q)$ which can be obtained by a sequence of at most $30n^3$ stellar subdivisions from both T and T'.

[Danilov'83], [Włodarczyk'97]: connectivity by stellar and inverse stellar subdivisions
[Ewald'86]: ad hoc proof of the plane case (?, in German)
[Morelli'96], false proof in full generality

Step 1: scaling a fin of triangulation $A \to A'$



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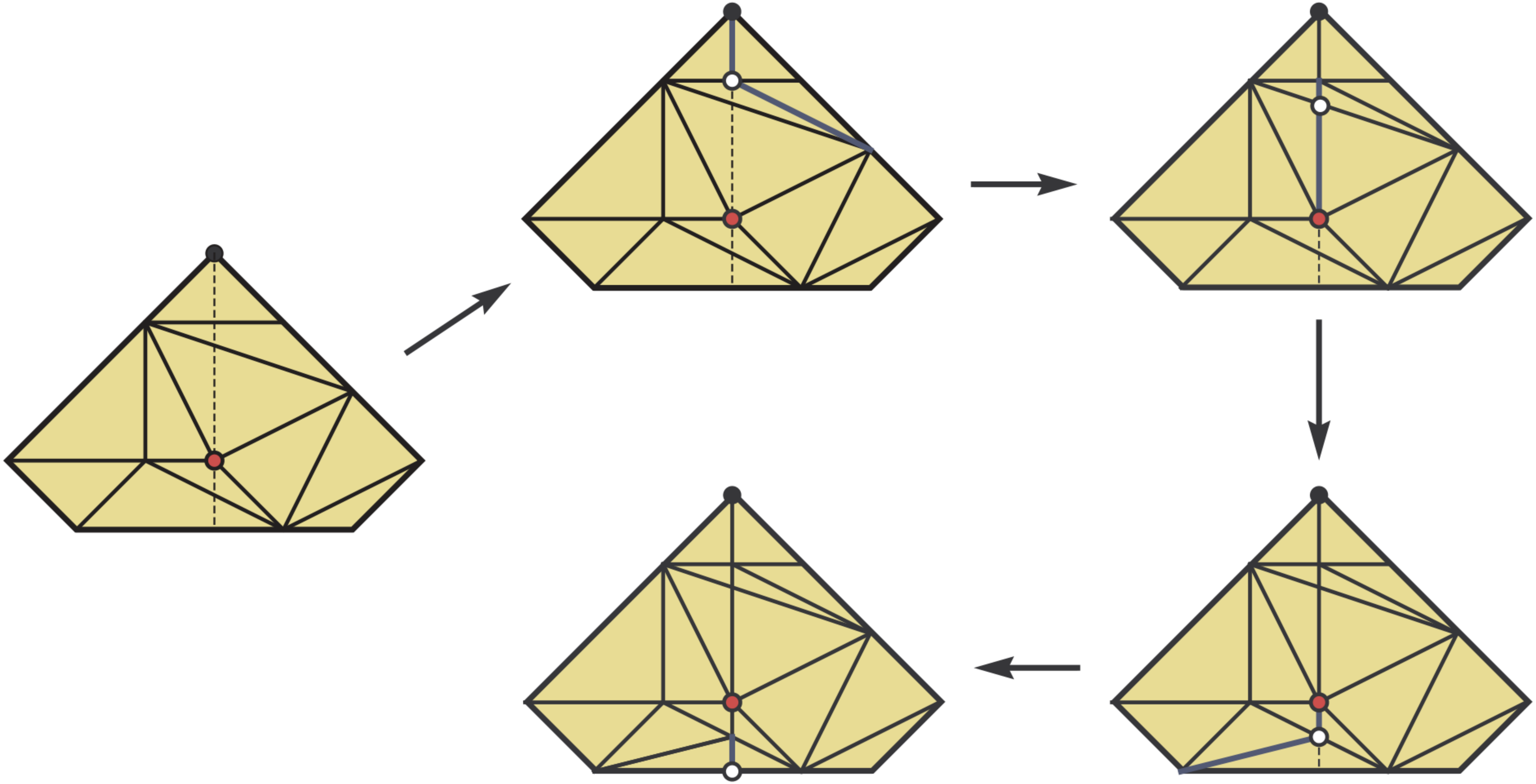
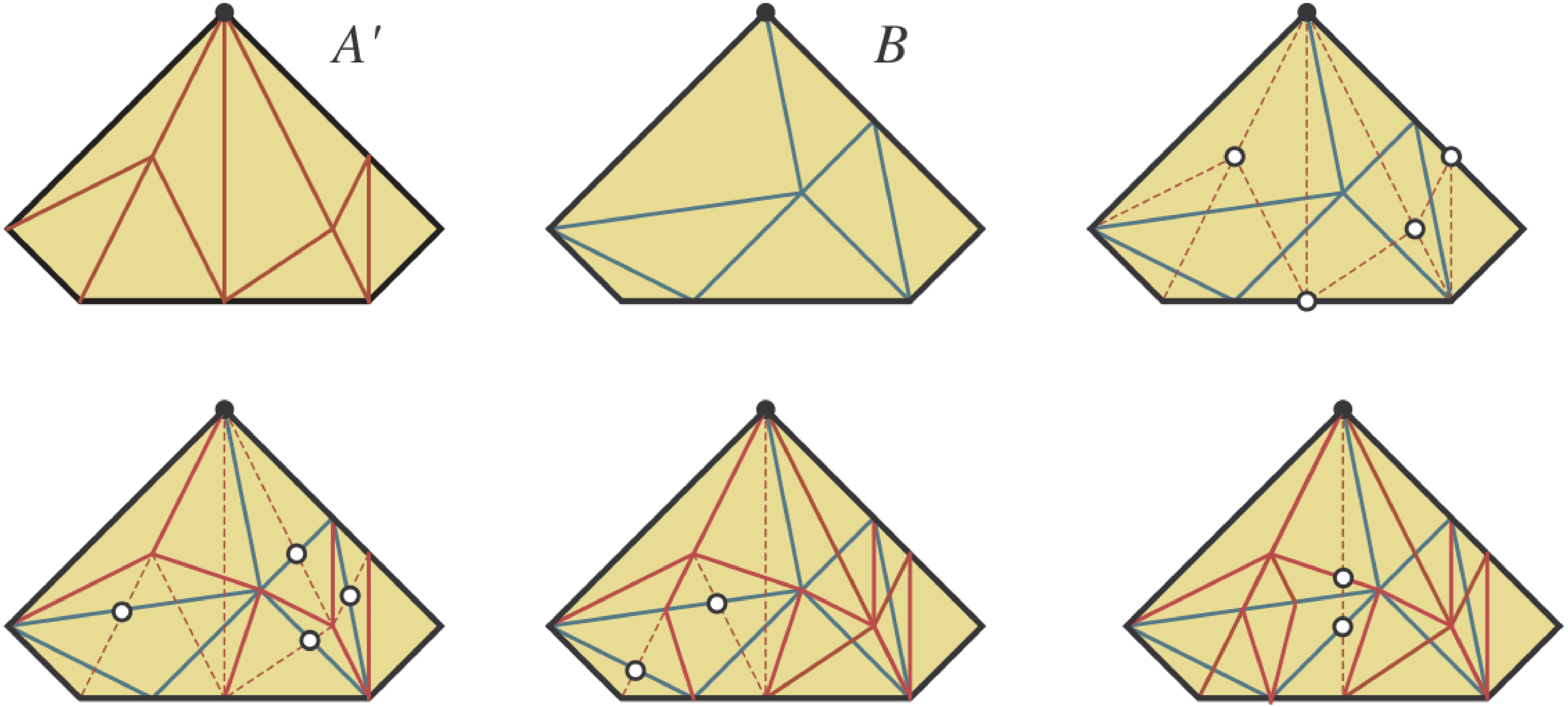


Step 2: Stellar subdivision $B \to B'$ s.t. $A' \subset B'$



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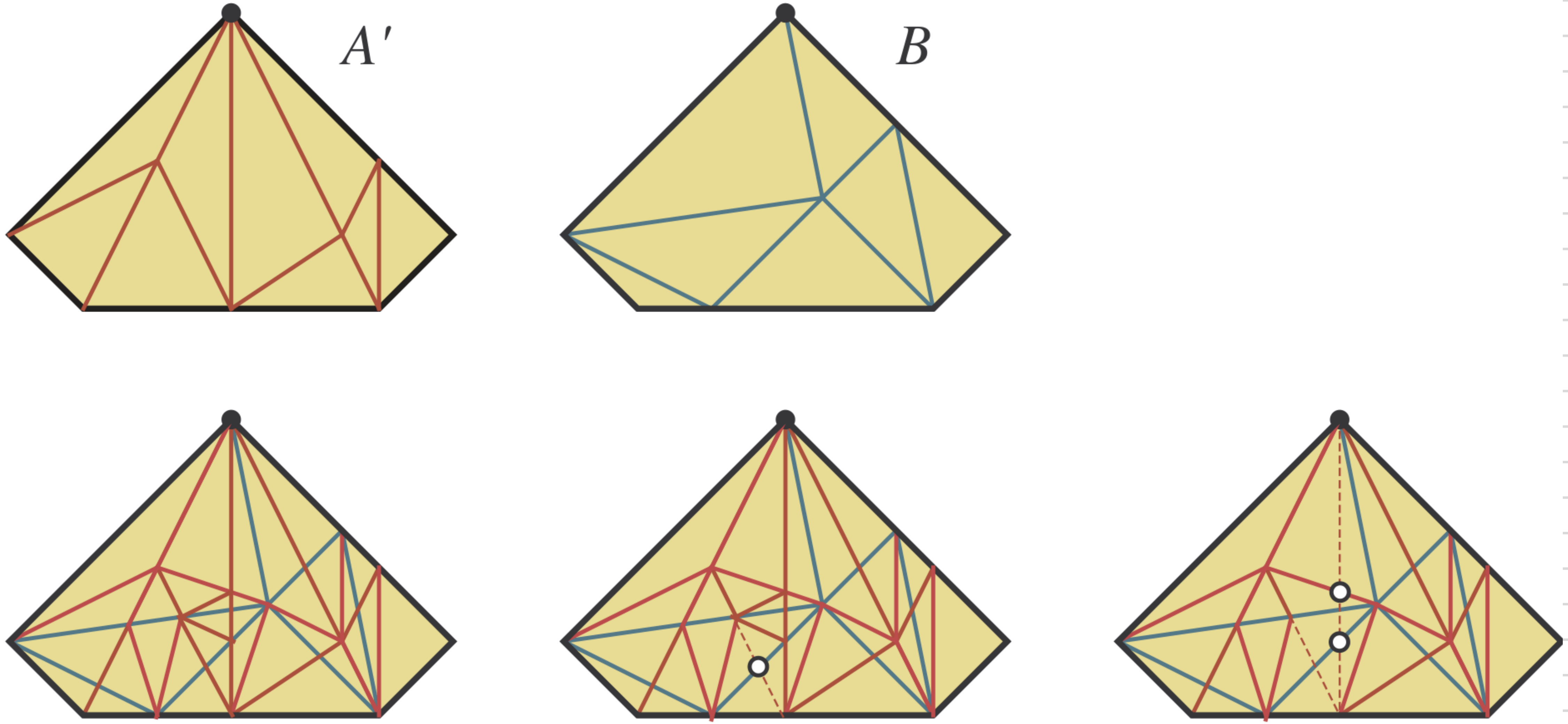

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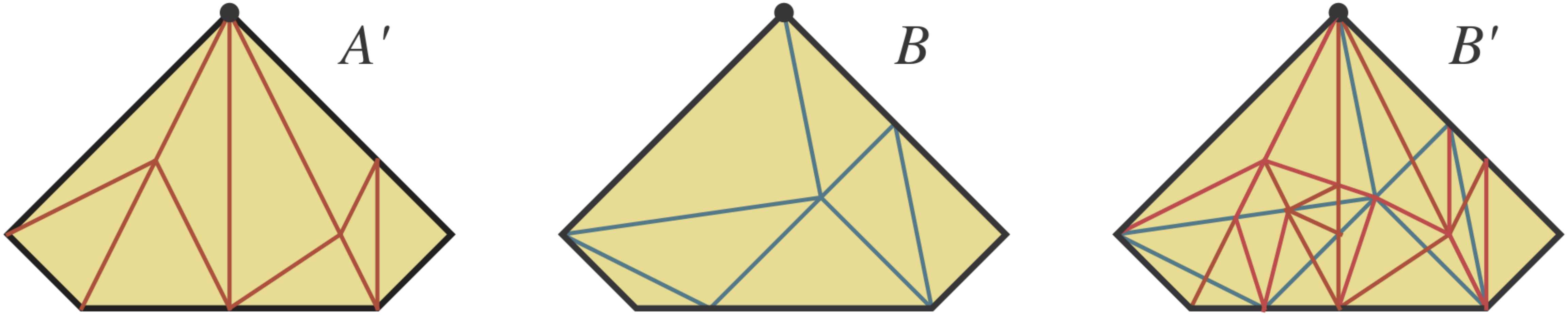
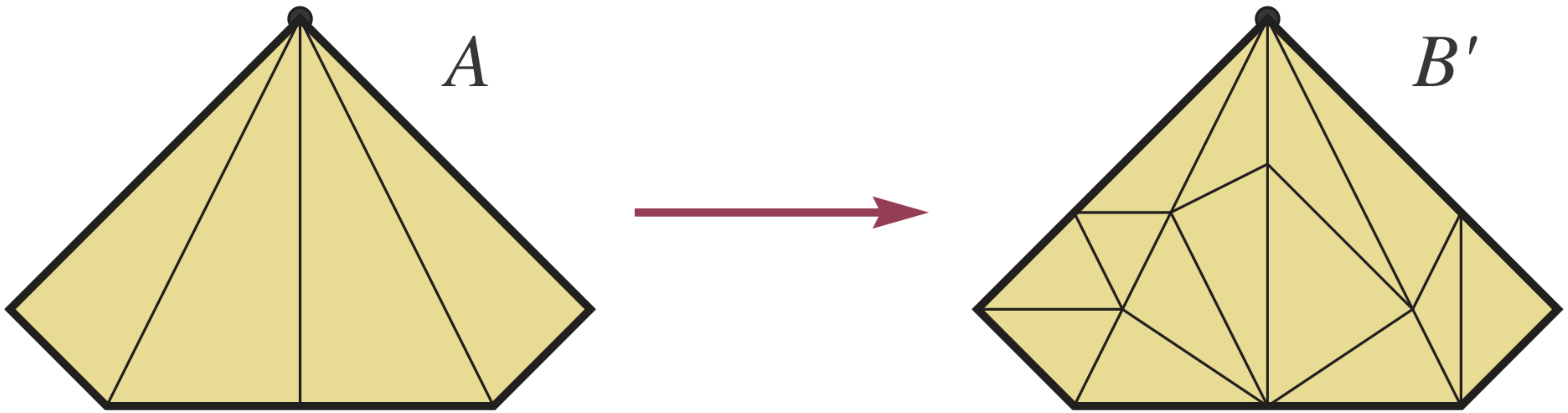
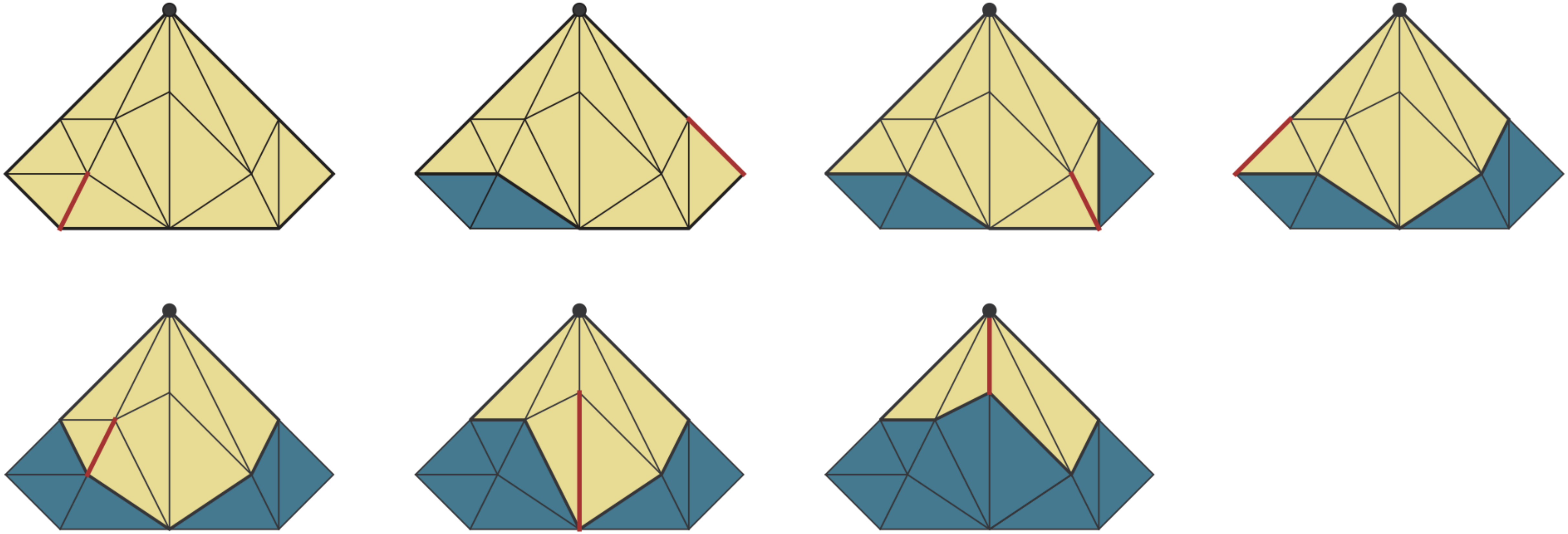


Step 3: Stellar subdivision $A \to B'$ (when A is a stripe)



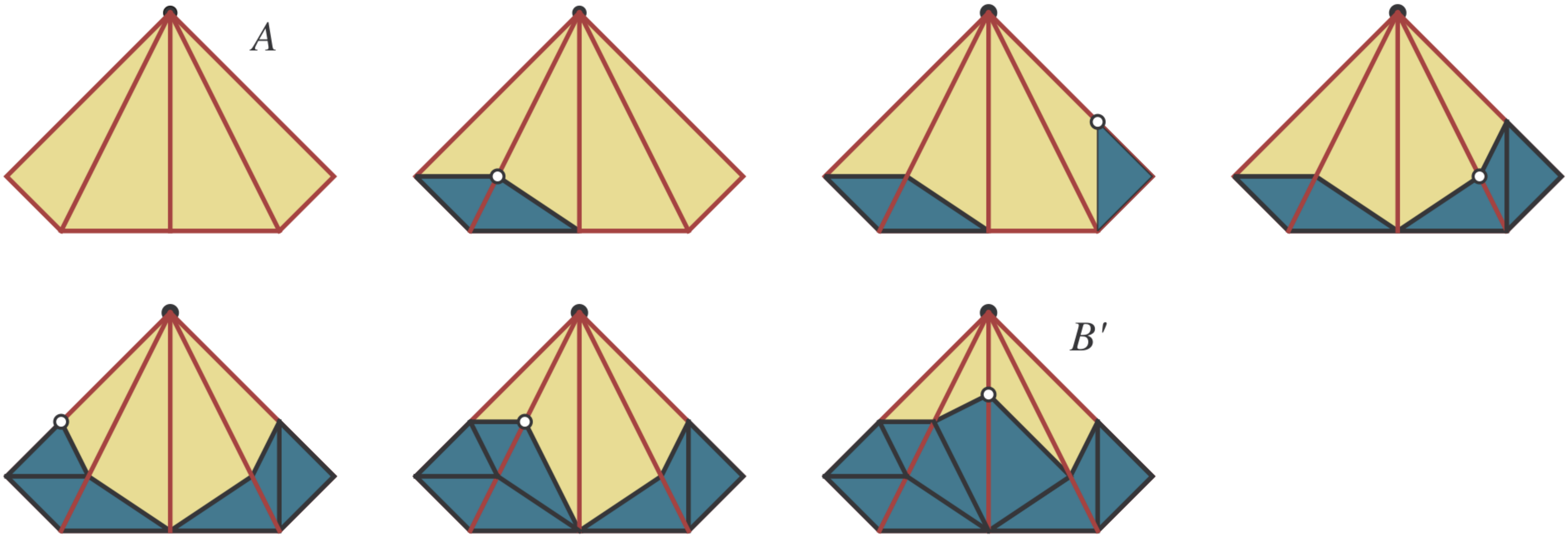
Key idea: *shedding process*

Step 3: Stellar subdivision *A* → *B'* (when *A* is a stripe)



Shedding

Step 3: Stellar subdivision *A* → *B’* (when *A* is a stripe)



Stellar subdivision

Step 3: Stellar subdivision $A' \to B'$ (general case)

Sketch:

1. In Step 1 add to *A* vertices from *B* (making a fan)
2. Step 2 is the same (no vertices not on the fan are added)
3. Step 3 is the same (some stellar subdivisions can now be skipped)

Higher dimension (ideas only)

1. Bing's lemma'83 reduces to the case when *A* is a simplex
2. Variation on shellability called *relative shellability* w.r.t a vertex (by Adiprasito-Benedetti'17)
3. Brugesser–Mani trick'71 (shelling of a polyhedron)

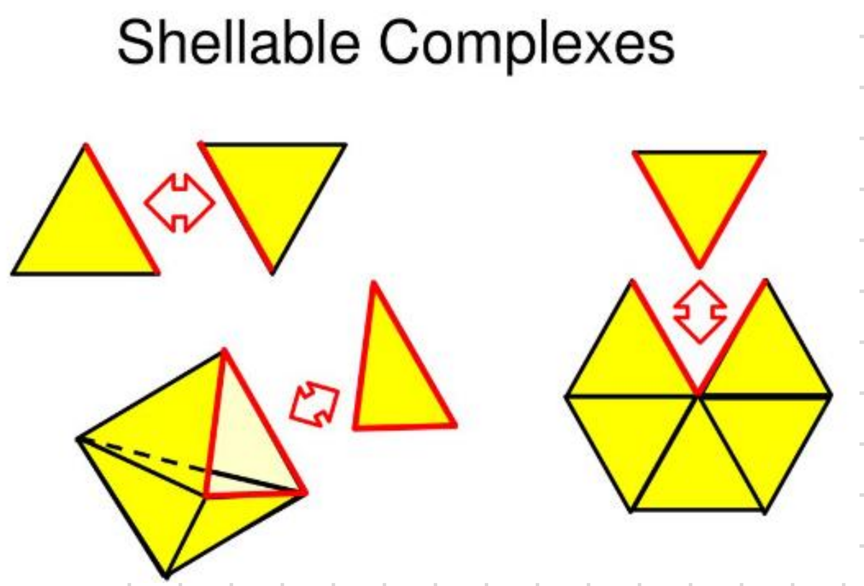
Lemma 5.1 (Bing's extension lemma, [Bing83, §I.2]). *Let $X \subset \Delta$ is a geometric complex embedded in a simplex. Then there is a triangulation of Δ that contains X as a subcomplex.*

Shellability

An ordering $C_1, C_2, \ldots$ of the maximal simplices of Δ is a **shelling** if the complex

$$B_k := \left(\bigcup_{i=1}^{k-1} C_i \right) \cap C_k$$

is pure and of dimension $\dim C_k - 1$ for all $k = 2, 3, \ldots$.



Shellability is NP-complete

Xavier Goaoc, Pavel Paták, Zuzana Patáková, Martin Tancer, Uli Wagner

- A shellable complex is homotopy equivalent to a wedge sum of spheres

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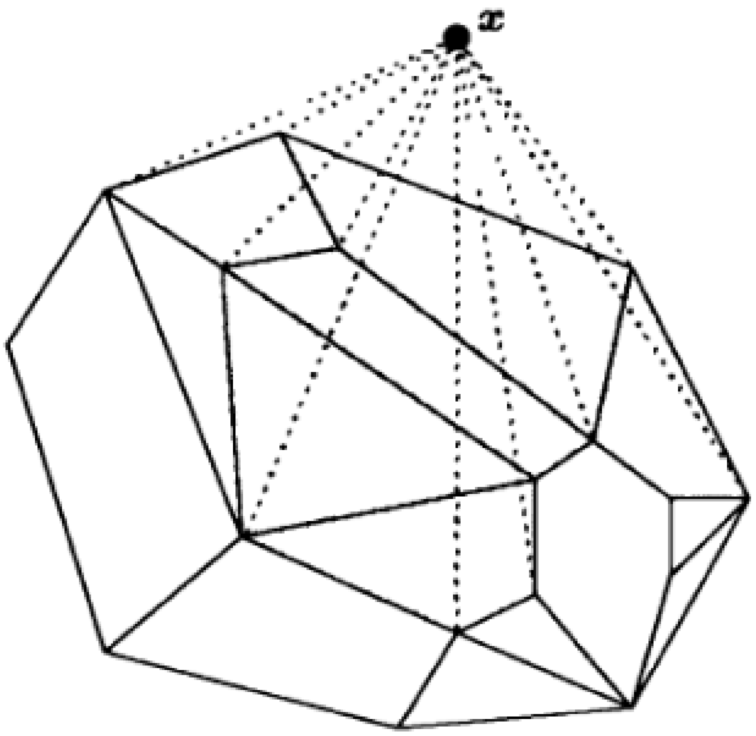

Brugesser–Mani'71

Topological result

Theorem [*former Alexander's conjecture*'30]
Every two PL homeomorphic simplicial complexes have combinatorially isomorphic stellar subdivisions.

Observation [Anderson-Mnëv'06]
Oda's conjecture implies Alexander's conjecture.

Thank you!