Combinatorics and complexity of Stanley’s inequality

Igor Pak, UCLA

[Image of a building and a person]

Link to the paper.
“I actually don't remember this specific talk though I remember going to many of these seminars at Knuth’s house … when I was a postdoc at Berkeley 1971-73.” -- Richard Stanley

“I was watching a lecture by Knuth and he mentioned that apparently you were present at the very first lecture Richard Karp gave where he stated all those combinatorial problems which are NP-complete. I guess you didn't much like the talk … ” -- IP
Richard Stanley and P vs NP problem:

Richard’s email also included an interesting story how he was offered, accepted by mail and then declined over the phone Stanford’s joint Math/CS postdoc offer.

“I felt especially bad that I had in some sense betrayed Knuth, but things worked out okay in the end. But who knows, if I had gone to Stanford maybe now it would not be a conjecture that $P \neq NP$ ... (ha!).”
Linear extensions

Let $P = (X, \prec)$ be a poset on $n = |X|$ elements.

*Linear extension* of $P$ is a bijection $f : X \to \{1, \ldots, n\}$, s.t. $f(x) < f(y)$ for all $x \prec y$.

Denote $\mathcal{E}(P)$ the set of linear extensions of $P$, and $e(P) := |\mathcal{E}(P)|$.

**Example:** $X = \{a, b, c, d\}$, $n = 4$, $e(P) = 5$. 

![Diagram of linear extensions](image)
Stanley’s inequality

Let \( P = (X, <) \) be a poset on \( n := |X| \) elements.

A linear extension of \( P \) is a bijection \( f : X \rightarrow \{1, \ldots, n\} \), s.t. \( f(x) < f(y) \) for all \( x < y \).

Denote by \( \mathcal{E}(P) \) the set of linear extensions of \( P \).

Fix \( x \in X \), and let \( N(a) := \# \{ f \in \mathcal{E}(P) : f(x) = a \} \).

**Theorem** [Stanley, 1981]: \( N(a)^2 \geq N(a - 1)N(a + 1) \) for all \( 1 < a < n \).

Conjectured: Kislitsyn (1968), Rival, Chung-Fishburn-Graham (1980)

**Log-concavity**
Stanley’s inequality

Let $P = (X, \prec)$ be a poset on $n := |X|$ elements.

A **linear extension** of $P$ is a bijection $f : X \rightarrow \{1, \ldots, n\}$, s.t. $f(x) < f(y)$ for all $x \prec y$.

Denote by $\mathcal{E}(P)$ the set of linear extensions of $P$.

Fix $x \in X$, and let $N(a) := \# \{ f \in \mathcal{E}(P) : f(x) = a \}$.

**Theorem** [Stanley, 1981]: $N(a)^2 \geq N(a - 1)N(a + 1)$ for all $1 < a < n$.

\[
\begin{align*}
N(2) &= 1, \
N(3) &= 2, \
N(4) &= 2
\end{align*}
\]
Stanley’s inequality

Let \( P = (X, \prec) \) be a poset on \( n = |X| \) elements.

*Linear extension* of \( P \) is a bijection \( f : X \to \{1, \ldots, n\} \), s.t. \( f(x) < f(y) \) for all \( x \prec y \).

Denote \( \mathcal{E}(P) \) the set of linear extensions of \( P \).

Fix \( x, z_1, \ldots, z_k \in X, \ c_1, \ldots, c_k \in \{1, \ldots, n\} \).

Let \( N(a) := \# \{ f \in \mathcal{E}(P) : f(x) = a, f(z_1) = c_1, \ldots, f(z_k) = c_k \} \).

**Theorem** [Stanley, 1981]: \( N(a)^2 \geq N(a - 1)N(a + 1) \) for all \( 1 < a < n \).
Stanley's inequality

**Theorem [Stanley, 1981]:** \( N(a)^2 \geq N(a - 1)N(a + 1) \) for all \( 1 < a < n \).

*Sketch of proof:* Let \( P = \{v_1, \ldots, v_{n-1}, v\} \). Let \( K \) be the set of all points \((t_1, \ldots, t_{n-1}) \subseteq \mathbb{R}^{n-1}\) satisfying:

(a) \( 0 \leq t_i \leq 1 \),

(b) if \( v_i \leq v_j \) in \( P \), then \( t_i \leq t_j \),

(c) if \( v_i < v \), then \( t_i = 0 \).

Similarly define \( L \subseteq \mathbb{R}^{n-1} \) by (a), (b), and:

(c') if \( v_i > v \), then \( t_i = 1 \).

Then \( K \) and \( L \) are convex polytopes. By an explicit decomposition of \( xK + yL \) into products of simplices, it can be computed that \( V_i(K, L) = N_{i+1}/(n - 1)! \). The proof follows from Theorem 4. \( \square \)
Variations on Stanley’s inequality

Let $P = (X, \prec)$ be a poset on $n = |X|$ elements.

Order preserving maps: $h : X \to \{1, \ldots, t\}$, s.t. $f(x) \leq f(y)$ for all $x \prec y$.

Denote $\mathcal{O}(P, t)$ the set of OPMs, and $\Omega(P, t) = |\mathcal{O}(P, t)|$ the order polynomial.

Theorem 4.2 (log-concavity, Brenti [Bre89, Thm 7.6.5]). Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then, for all integer $t \geq 2$, we have:

(4.2) $\Omega(P, t)^2 \geq \Omega(P, t + 1) \Omega(P, t - 1)$. 

Injective proof!
Variations on Stanley’s inequality

Let $P = (X, \prec)$ be a poset on $n = |X|$ elements.

**Order preserving maps**: $h : X \to \{1, \ldots, t\}$, s.t. $f(x) \leq f(y)$ for all $x \prec y$.

Denote $\mathcal{O}(P, t)$ the set of OPMs, and $\Omega(P, t) = |\mathcal{O}(P, t)|$ the order polynomial.

---

**Theorem 9.9 (Daykin–Daykin–Paterson inequality [DDP84, Thm 4])**. Let $P = (X, \prec)$ be a finite poset, and let $x \in X$. Denote by $\Omega(P, t; x, a)$ the number of order preserving maps $h : X \to [t]$, such that $h(x) = a$. Then, for all integer $t > a > 1$, we have:

\begin{equation}
\Omega(P, t; x, a)^2 \geq \Omega(P, t; x, a + 1) \cdot \Omega(P, t; x, a - 1).
\end{equation}

---

**Log-concavity**

Injective proof!
Variations on Stanley’s inequality

Let $P = (X, \prec)$ be a poset on $n = |X|$ elements.

**Order preserving maps:** $h : X \rightarrow \{1, \ldots, t\}$, s.t. $f(x) \leq f(y)$ for all $x \prec y$.

Denote $\mathcal{O}(P,t)$ the set of OPMs, and $\Omega(P,t) = |\mathcal{O}(P,t)|$ the order polynomial.

**Theorem 9.10** (generalized DDP inequality [DDP84, Thm 4]). Let $P = (X, \prec)$ be a finite poset, let $x \in X$. Fix $k \in \mathbb{N}$ and let $z \in X^k$. Denote by $\Omega(P,t; z, c; x, a)$ the number of order preserving maps $h : X \rightarrow [t]$, such that $h(x) = a$, and $h(z_i) = c_i$ for all $1 \leq i \leq k$. Then, for all integer $t > a > 1$, we have:

\[
\Omega(P,t; z, c; x, a)^2 \geq \Omega(P,t; z, c; x, a+1) \cdot \Omega(P,t; z, c; x, a-1).
\]
Injective proof of Stanley’s inequality?

\[ \Rightarrow \quad |A| \leq |B| \]

“It appears unlikely that Stanley’s Theorem for linear extensions quoted earlier can be proved using the kind of injection presented here.” [DDP84, §4].
Injective proof of Stanley’s inequality?

**Main Theorem:** [Swee Hong Chan, IP, 2023] $\xrightarrow{} \{\text{level 0}\}$

*There is no nice injective proof of Stanley’s inequality.*

**Definition:** Injection $\varphi : A \rightarrow B$ is *nice* if both $\varphi$ and $\varphi^{-1}$ are poly-time computable.
Injective proof of Stanley’s inequality?

Main Theorem: [Swee Hong Chan, IP, 2023] ← \{level 1\}

For \( k \geq 2 \), there is no **nice** injective proof of Stanley’s inequality unless **something bad** happens in CS.

Note: Nice injection gives a combinatorial interpretation for

\[ |B| - |A| = \# \{ b \in B : b \notin \varphi(A) \} \]
Injective proof of Stanley’s inequality?

**Main Theorem:** [Swee Hong Chan, IP, 2023] ← \{level 2\}

For \( k \geq 2 \), the defect \( \delta(P, a) := N(a)^2 - N(a - 1)N(a + 1) \) of Stanley’s inequality does not have a combinatorial interpretation unless something bad happens in CS.
Injective proof of Stanley’s inequality?

**Main Theorem:** [Swee Hong Chan, IP, 2023] ← \{level 3\}

For \( k \geq 2 \), the defect \( \delta(P, a) := N(a)^2 - N(a - 1)N(a + 1) \) of Stanley’s inequality is not in \( \#P \) unless something bad happens in CS.
Injective proof of Stanley’s inequality?

Main Theorem: [Swee Hong Chan, IP, 2023]  \[
\text{for } k \geq 2, \text{ the defect } \delta(P, a) := N(a)^2 - N(a - 1)N(a + 1) \text{ of Stanley’s inequality is not in } \#P \text{ unless the polynomial hierarchy PH collapses.}
\]
Injective proof of Stanley’s inequality?

**Main Theorem:** [Swee Hong Chan, IP, 2023] ← \{level 4’\}

*For* \( k \geq 2 \), the defect \( \delta(P, a) := N(a)^2 - N(a - 1)N(a + 1) \) of Stanley’s inequality is not in \#P unless \( \text{PH} = \Sigma_2 \).
Injective proof of Stanley’s inequality?

Main Theorem: [Swee Hong Chan, IP, 2023]  \(\leftarrow\) \{level 5\}

For \(k \geq 2\), the equality \(\{N(a)^2 = N(a-1)N(a+1)\}\) of Stanley’s inequality is not in \text{coNP} unless \text{PH} collapses.

“We can ask about the conditions for equality…”

-- [Stanley’81]
Injective proof of Stanley’s inequality?

Main Theorem: [Swee Hong Chan, IP, 2023] ← {level 6, final}

For $k \geq 2$, the equality \( N(a)^2 = N(a-1)N(a+1) \) of Stanley’s inequality is not in PH unless PH collapses.
Injective proof of Stanley’s inequality?

**Main Theorem:** [Swee Hong Chan, IP, 2023] ← \{level 6, final\}

For \( k \geq 2 \), the equality \( \{ N(a)^2 =? N(a - 1)N(a + 1) \} \) of Stanley’s inequality is not in PH unless PH collapses.

**Theorem:** [Shenfeld – van Handel’23, Chan–P’23]

For \( k \in \{0, 1\} \), the equality \( \{ N(a)^2 =? N(a - 1)N(a + 1) \} \) of Stanley’s inequality is in P.
Proof ingredients:

Main Lemma [Swee Hong Chan, IP’23]  
For $k \geq 2$,
Given $P = (X, \prec)$, deciding \{\(N(a) = ? N(a + 1)\)\} is not in PH unless PH collapses.

$|A| = |B|$
Proof ingredients:

Main Lemma [Swee Hong Chan, IP’23] \(\text{For } k \geq 2,\)
Given \(P = (X, \prec),\) deciding \(\{N(a) =? N(a + 1)\}\) is not in PH unless PH collapses.

Theorem [Brightwell–Winkler’91, formerly Linial Conjecture’84]
The number of linear extensions \(e(P)\) is \(#P\)-complete.
Proof ingredients:

**Theorem** [Kravitz–Sah’21]

\[ \mathcal{T}_e(n) \supseteq \{1, \ldots, c^{n/(\log n)}\} \text{ for some } c > 1, \text{ where} \]

\[ \mathcal{T}_e(n) := \{ e(P) : P = (X, \prec), |X| = n, \text{width}(P) = 2 \}. \]

\[ S_n(m) := \sum_{i=0}^{s} a_i(m) \text{ where } \frac{m}{n} = a_0 + \frac{1}{a_1 + \frac{1}{\cdots + \frac{1}{a_s}}} \]

**Lemma 8.6** (Yao–Knuth [YK75]). We have:

\[ \frac{1}{n} \sum_{m \in [n]} S_n(m) = \frac{6}{\pi^2} (\log n)^2 + O((\log n)(\log \log n)^2) \text{ as } n \to \infty. \]
Happy Birthday, Richard!