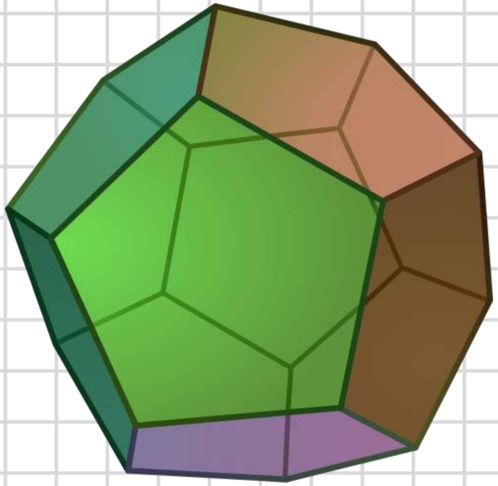
June 4, 2024

StanleyFest **2024**

Combinatorics and complexity of Stanley's inequality

Igor Pak, UCLA



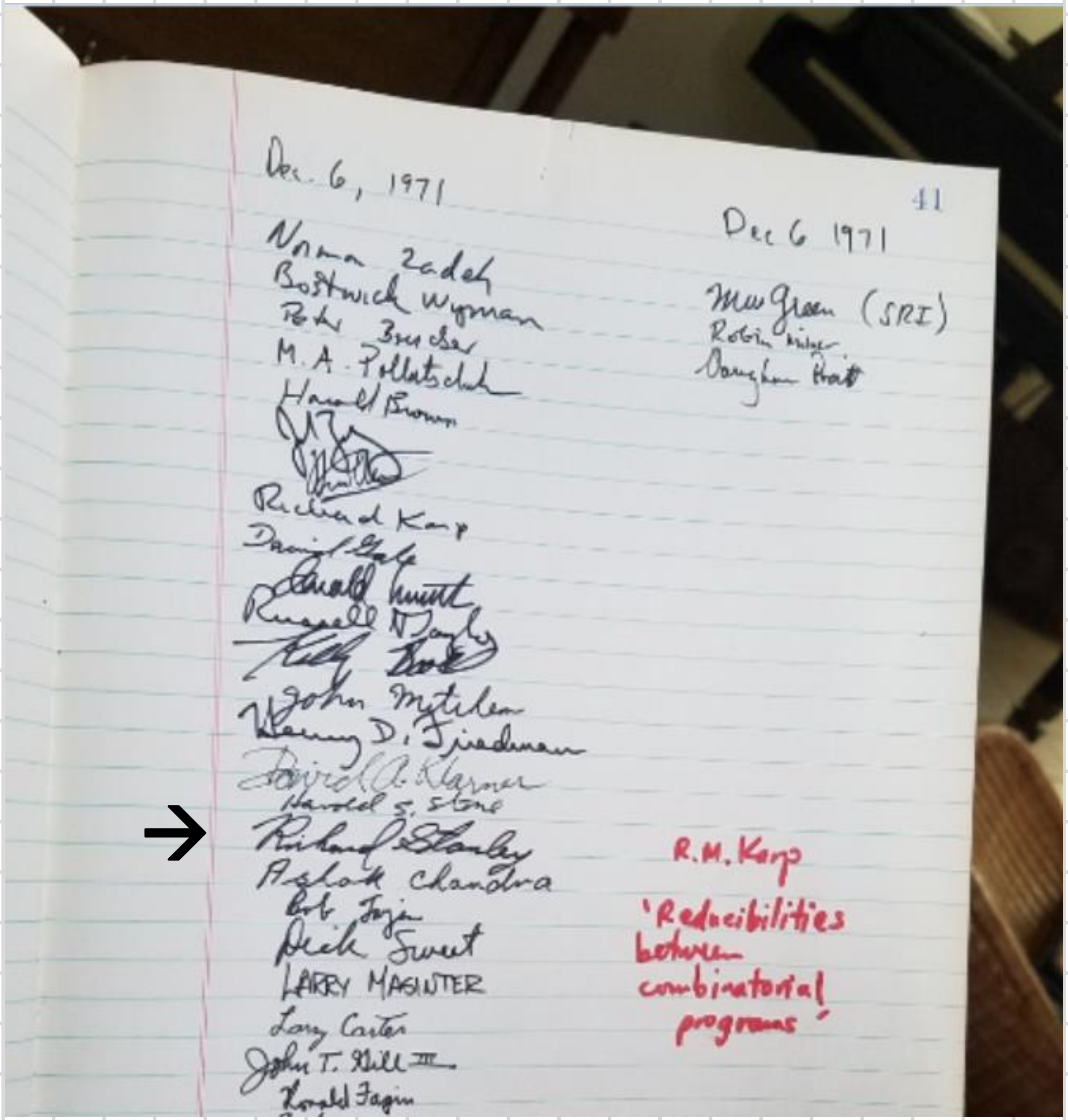
Link to the paper.

Richard Stanley and P vs NP problem:

"I was watching a lecture by Knuth and he mentioned that apparently you were present at the very first lecture Richard Karp gave where he stated all those combinatorial problems which are NP-complete. I guess you didn't much like the talk … " -- IP

"I actually don't remember this specific talk though I remember going to many of these seminars at Knuth's house … when I was a postdoc at Berkeley 1971-73." -- Richard Stanley

Richard Stanley and P vs NP problem:



Richard’s email also included an interesting story how he was offered, accepted by mail and then declined over the phone Stanford’s joint Math/CS postdoc offer.

“I felt especially bad that I had in some sense betrayed Knuth, but things worked out okay in the end. But who knows, if I had gone to Stanford maybe now it would not be a conjecture that P\neq NP ... (ha!).”

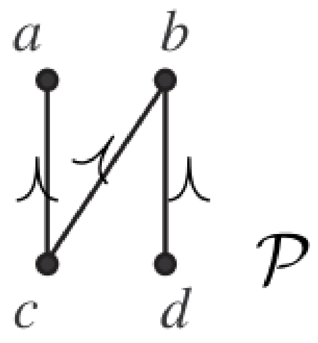
Linear extensions

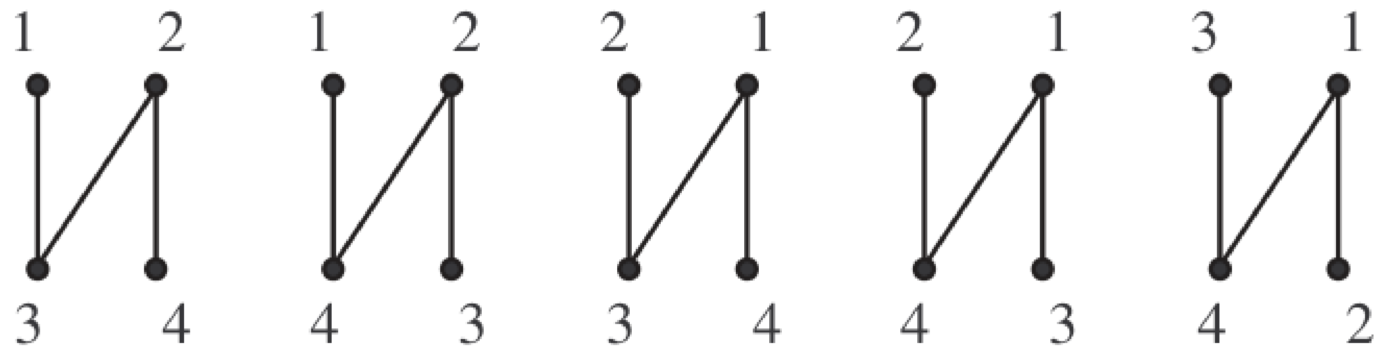
Let $P = (X, \prec)$ be a poset on $n = |X|$ elements.

Linear extension of P is a bijection $f : X \to \{1, \ldots, n\}$, s.t. $f(x) < f(y)$ for all $x \prec y$.

Denote $\mathcal{E}(P)$ the set of linear extensions of P, and $e(P) := |\mathcal{E}(P)|$.

Example: $X = \{a, b, c, d\}$, $n = 4$, $e(P) = 5$.





Stanley's inequality

Let $P = (X, \prec)$ be a poset on $n := |X|$ elements.

A *linear extension* of P is a bijection $f : X \to \{1, \ldots, n\}$, s.t. $f(x) < f(y)$ for all $x \prec y$.

Denote by $\mathcal{E}(P)$ the set of linear extensions of P.

Fix $x \in X$, and let $N(a) := \#\{f \in \mathcal{E}(P) : f(x) = a\}$.

Theorem [Stanley, 1981]: $N(a)^2 \geq N(a-1)\,N(a+1)$ for all $1 < a < n$.

Conjectured: Kislitsyn (1968), Rival, Chung-Fishburn-Graham (1980)

Log-concavity

Stanley's inequality

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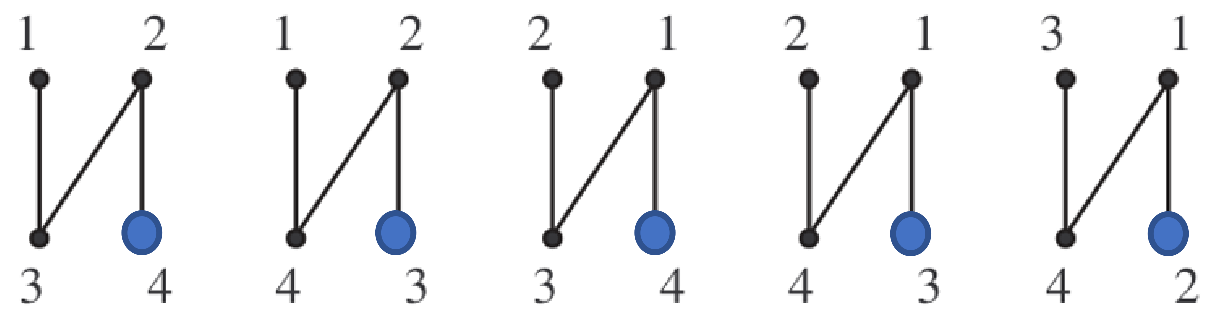


$N(2) = 1$, $N(3) = 2$, $N(4) = 2$

Stanley's inequality

Let $P = (X, \prec)$ be a poset on $n = |X|$ elements.

Linear extension of P is a bijection $f : X \to \{1, \ldots, n\}$, s.t. $f(x) < f(y)$ for all $x \prec y$.

Denote $\mathcal{E}(P)$ the set of linear extensions of P.

Fix $x, z_1, \ldots, z_k \in X$, $c_1, \ldots, c_k \in \{1, \ldots, n\}$.

Let $N(a) := \#\{f \in \mathcal{E}(P) : f(x) = a,\, f(z_1) = c_1, \ldots, f(z_k) = c_k\}$.

Theorem [Stanley, 1981]: $N(a)^2 \geq N(a-1)\,N(a+1)$ for all $1 < a < n$.

Log-concavity

JOURNAL OF COMBINATORIAL THEORY, Series A 31, 56–65 (1981)

Two Combinatorial Applications of the Aleksandrov–Fenchel Inequalities*

RICHARD P. STANLEY

Stanley's inequality

Theorem [Stanley, 1981]: $N(a)^2 \geq N(a-1)\,N(a+1)$ for all $1 < a < n$.

Log-Concave and Unimodal Sequences in Algebra, Combinatorics, and Geometry[a]

RICHARD P. STANLEY

Sketch of proof: Let $P = \{v_1, \ldots, v_{n-1}, v\}$. Let K be the set of all points $(t_1, \ldots, t_{n-1}) \in \mathbb{R}^{n-1}$ satisfying:

(a) $0 \leq t_i \leq 1$,
(b) if $v_i \leq v_j$ in P, then $t_i \leq t_j$,
(c) if $v_i < v$, then $t_i = 0$.

Similarly define $L \subset \mathbb{R}^{n-1}$ by (a), (b), and:

(c′) if $v_i > v$, then $t_i = 1$.

Then K and L are convex polytopes. By an explicit decomposition of $xK + yL$ into products of simplices, it can be computed that $V_i(K, L) = N_{i+1}/(n-1)!$. The proof follows from Theorem 4. □

Variations on Stanley's inequality

Let $P = (X, \prec)$ be a poset on $n = |X|$ elements.

Order preserving maps: $h : X \to \{1, \ldots, t\}$, s.t. $f(x) \leq f(y)$ for all $x \prec y$.

Denote $\mathcal{O}(P, t)$ the set of OPMs, and $\Omega(P, t) = |\mathcal{O}(P, t)|$ the *order polynomial*.

Theorem 4.2 (*log-concavity*, Brenti [Bre89, Thm 7.6.5]). *Let $P = (X, \prec)$ be a poset with $|X| = n$ elements. Then, for all integer $t \geq 2$, we have:*

$$\Omega(P, t)^2 \;\geq\; \Omega(P, t+1)\; \Omega(P, t-1). \tag{4.2}$$

Injective proof!

Log-concavity

Variations on Stanley's inequality

Let $P = (X, \prec)$ be a poset on $n = |X|$ elements.

Order preserving maps: $h : X \to \{1, \ldots, t\}$, s.t. $f(x) \leq f(y)$ for all $x \prec y$.

Denote $\mathcal{O}(P, t)$ the set of OPMs, and $\Omega(P, t) = |\mathcal{O}(P, t)|$ the *order polynomial*.

Theorem 9.9 (*Daykin–Daykin–Paterson inequality* [DDP84, Thm 4]). *Let $P = (X, \prec)$ be a finite poset, and let $x \in X$. Denote by $\Omega(P, t; x, a)$ the number of order preserving maps $h : X \to [t]$, such that $h(x) = a$. Then, for all integer $t > a > 1$, we have:*

$$\Omega(P, t; x, a)^2 \geq \Omega(P, t; x, a+1) \cdot \Omega(P, t; x, a-1). \tag{9.7}$$

Injective proof!

Log-concavity

Variations on Stanley's inequality

Let $P = (X, \prec)$ be a poset on $n = |X|$ elements.

Order preserving maps: $h : X \to \{1, \ldots, t\}$, s.t. $f(x) \le f(y)$ for all $x \prec y$.

Denote $\mathcal{O}(P, t)$ the set of OPMs, and $\Omega(P, t) = |\mathcal{O}(P, t)|$ the *order polynomial*.

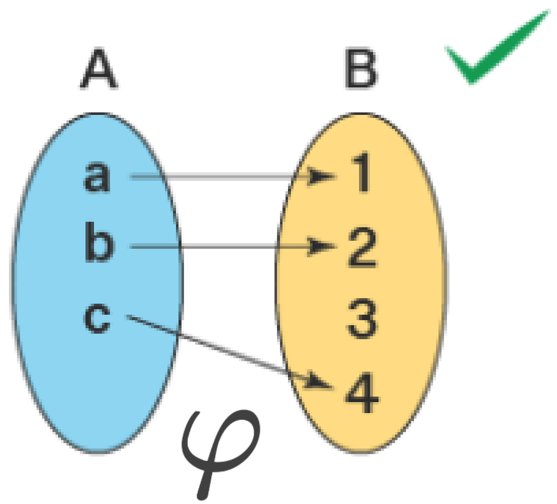
Theorem 9.10 (*generalized DDP inequality* [DDP84, Thm 4]). *Let $P = (X, \prec)$ be a finite poset, let $x \in X$. Fix $k \in \mathbb{N}$ and let $\boldsymbol{z} \in X^k$. Denote by $\Omega(P, t; \boldsymbol{z}, \mathbf{c}; x, a)$ the number of order preserving maps $h : X \to [t]$, such that $h(x) = a$, and $h(z_i) = c_i$ for all $1 \le i \le k$. Then, for all integer $t > a > 1$, we have:*

$$\Omega(P, t; \boldsymbol{z}, \mathbf{c}; x, a)^2 \ge \Omega(P, t; \boldsymbol{z}, \mathbf{c}; x, a + 1) \cdot \Omega(P, t; \boldsymbol{z}, \mathbf{c}; x, a - 1). \quad (9.8)$$

Injective proof!

Log-concavity

Injective proof of Stanley's inequality?



$$\Longrightarrow \quad |A| \leq |B|$$

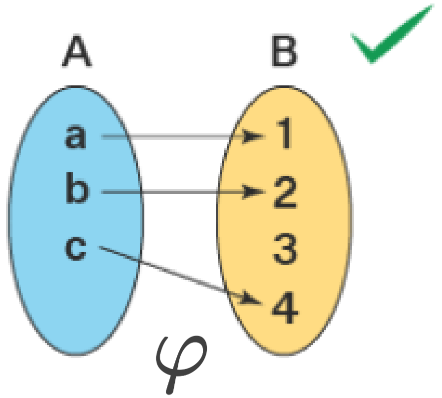
"It appears unlikely that Stanley's Theorem for linear extensions quoted earlier can be proved using the kind of injection presented here." [DDP84, §4].

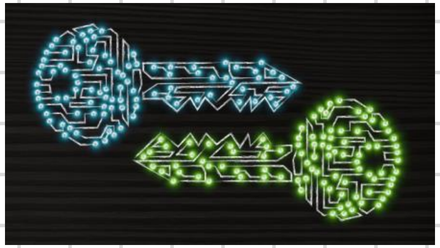
Injective proof of Stanley's inequality?

Main Theorem: [Swee Hong Chan, IP, 2023] $\leftarrow$ {level 0}

There is no nice injective proof of Stanley's inequality.

Definition: Injection $\varphi : A \to B$ is *nice*
if both φ and φ^{-1} are poly-time computable.

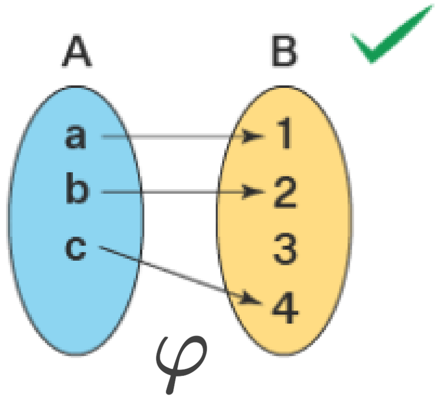




Injective proof of Stanley's inequality?

Main Theorem: [Swee Hong Chan, IP, 2023] $\leftarrow$ {level 1}

For $k \geq 2$, there is no nice injective proof of Stanley's inequality unless something bad happens in CS.

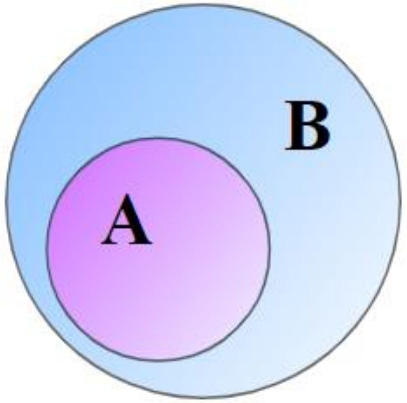


Note: Nice injection gives a combinatorial interpretation for $|B| - |A| = \#\{b \in B : b \notin \varphi(A)\}$

Injective proof of Stanley's inequality?

Main Theorem: [Swee Hong Chan, IP, 2023] $\leftarrow$ {level 2}

For $k \geq 2$, *the* defect $\delta(P, a) := N(a)^2 - N(a-1)N(a+1)$ *of Stanley's inequality does not have a combinatorial interpretation unless something bad happens in CS.*



arXiv > math > arXiv:2209.06142

Mathematics > Combinatorics

[Submitted on 13 Sep 2022]

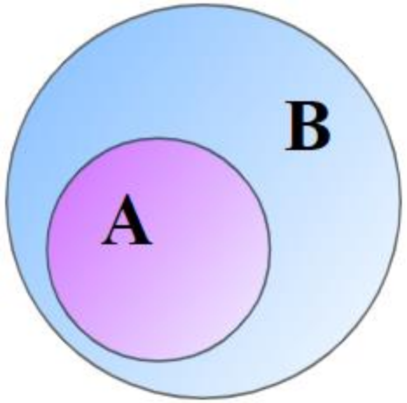
What is a combinatorial interpretation?

Igor Pak

Injective proof of Stanley's inequality?

Main Theorem: [Swee Hong Chan, IP, 2023] $\leftarrow$ {level 3}

For $k \geq 2$, *the* defect $\delta(P, a) := N(a)^2 - N(a-1)N(a+1)$ *of Stanley's inequality is not in* $\#\mathrm{P}$ *unless something bad happens in CS.*

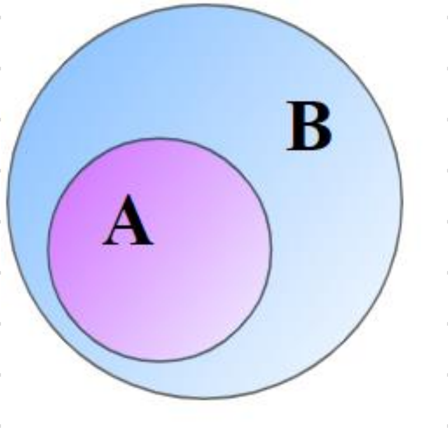


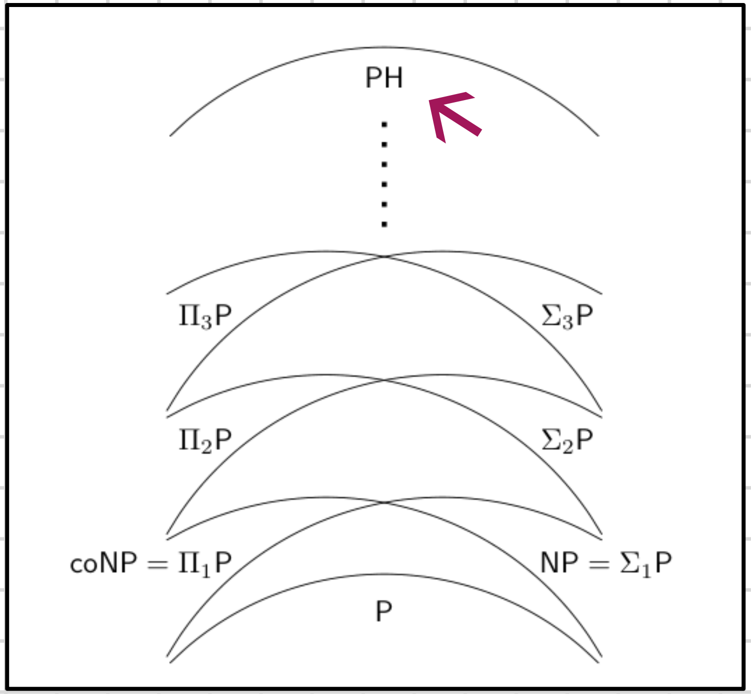
#P

Injective proof of Stanley's inequality?

Main Theorem: [Swee Hong Chan, IP, 2023] $\leftarrow$ {level 4}

For $k \geq 2$, *the* defect $\delta(P, a) := N(a)^2 - N(a-1)N(a+1)$ *of Stanley's inequality is not in* #P *unless the polynomial hierarchy* PH *collapses.*

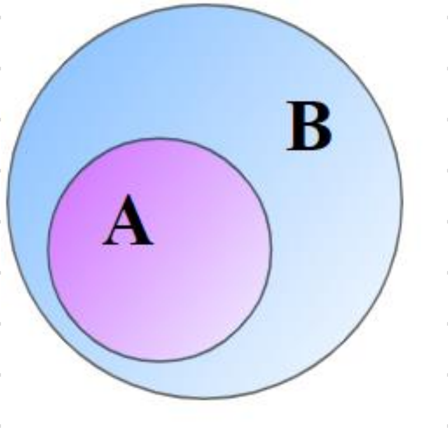


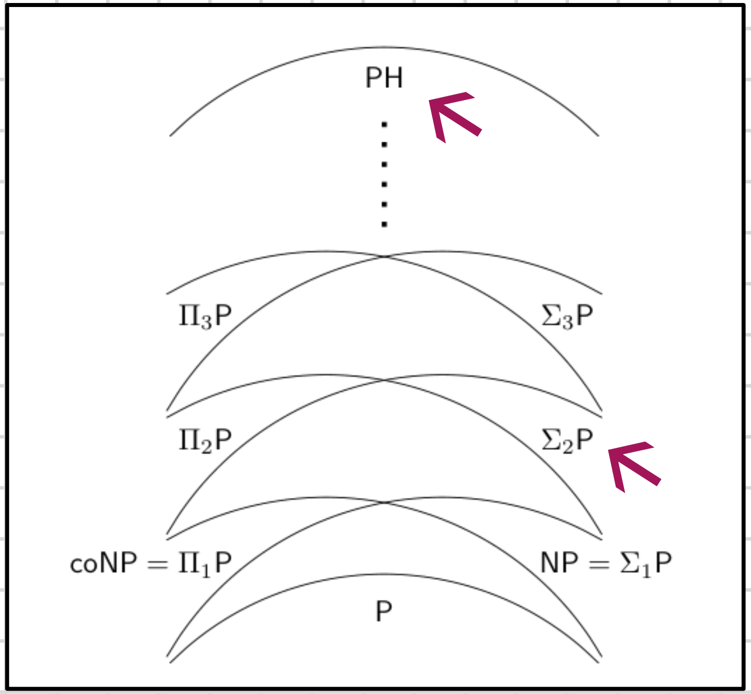


Injective proof of Stanley's inequality?

Main Theorem: [Swee Hong Chan, IP, 2023] $\leftarrow$ {level 4′}

For $k \geq 2$, *the* defect $\delta(P, a) := N(a)^2 - N(a-1)N(a+1)$ *of Stanley's inequality is not in* #P *unless* $\text{PH} = \Sigma_2$.



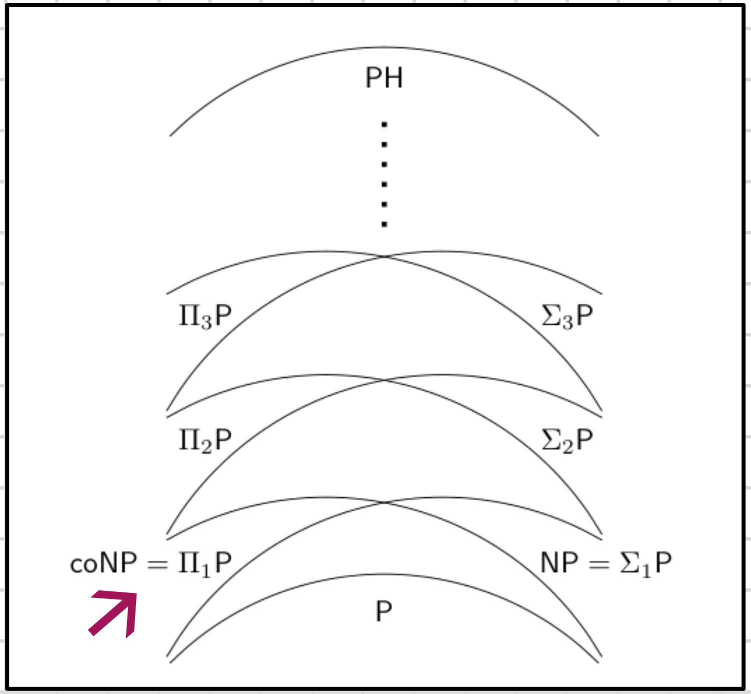


Injective proof of Stanley's inequality?

Main Theorem: [Swee Hong Chan, IP, 2023] $\leftarrow$ {level 5}

For $k \geq 2$, *the* equality $\{N(a)^2 =^? N(a-1)N(a+1)\}$ *of Stanley's inequality is not in* coNP *unless* PH *collapses.*

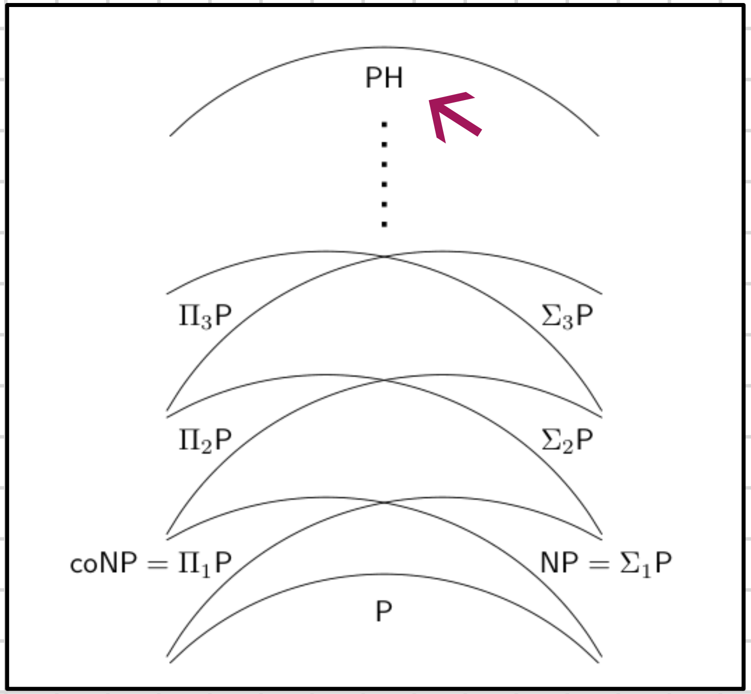
"We can ask about the conditions for equality..."
-- [Stanley'81]



Injective proof of Stanley's inequality?

Main Theorem: [Swee Hong Chan, IP, 2023] $\leftarrow$ {level 6, final}

For $k \geq 2$, *the* equality $\{N(a)^2 =^? N(a-1)N(a+1)\}$ *of Stanley's inequality is not in* PH *unless* PH *collapses.*



Injective proof of Stanley's inequality?

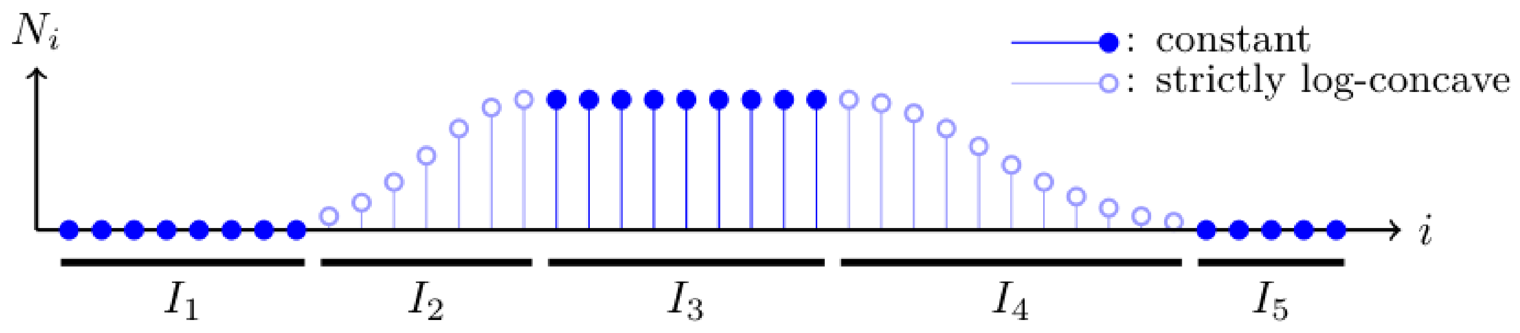
Main Theorem: [Swee Hong Chan, IP, 2023] $\leftarrow$ {level 6, final}
For $k \geq 2$, the equality $\{N(a)^2 =^? N(a-1)N(a+1)\}$ *of Stanley's inequality is not in* PH *unless* PH *collapses.*

Theorem: [Shenfeld – van Handel'23, Chan–P'23]
For $k \in \{0, 1\}$, the equality $\{N(a)^2 =^? N(a-1)N(a+1)\}$ *of Stanley's inequality is in* P.

Proof ingredients:

Main Lemma [Swee Hong Chan, IP'23] *For* $k \geq 2$,

Given $P = (X, \prec)$, deciding $\{N(a) =^? N(a+1)\}$ is not in PH unless PH collapses.



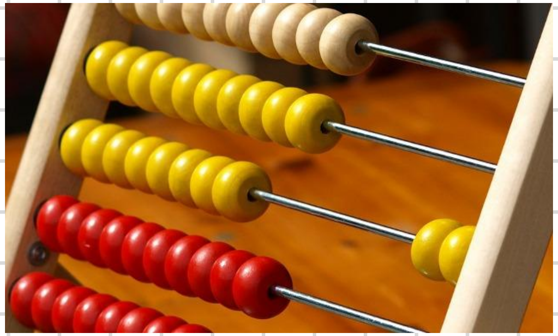
[Shenfeld – van Handel'22]

|A| = |B|

Proof ingredients:

Main Lemma [Swee Hong Chan, IP'23] *For* $k \geq 2$,
Given $P = (X, \prec)$, deciding $\{N(a) =^? N(a+1)\}$ is not in PH unless PH collapses.

Theorem [Brightwell–Winkler'91, formerly *Linial Conjecture*'84]
The number of linear extensions $e(P)$ is #P-complete.



Proof ingredients:

Theorem [Kravitz–Sah'21]

$$\mathcal{T}_e(n) \supseteq \{1, \ldots, c^{n/(\log n)}\} \quad \text{for some} \quad c > 1, \quad \text{where}$$

$$\mathcal{T}_e(n) := \{e(P) : P = (X, \prec), \ |X| = n, \ \text{width}(P) = 2\}.$$

$$S_n(m) := \sum_{i=0}^{s} a_i(m) \qquad \text{where} \qquad \frac{m}{n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \frac{1}{a_s}}}$$

Lemma 8.6 **(**Yao–Knuth [YK75]**).** *We have:*

$$\frac{1}{n} \sum_{m \in [n]} S_n(m) = \frac{6}{\pi^2} (\log n)^2 + O\big((\log n)(\log \log n)^2\big) \quad \textit{as} \quad n \to \infty.$$

Happy Birthday, Richard!