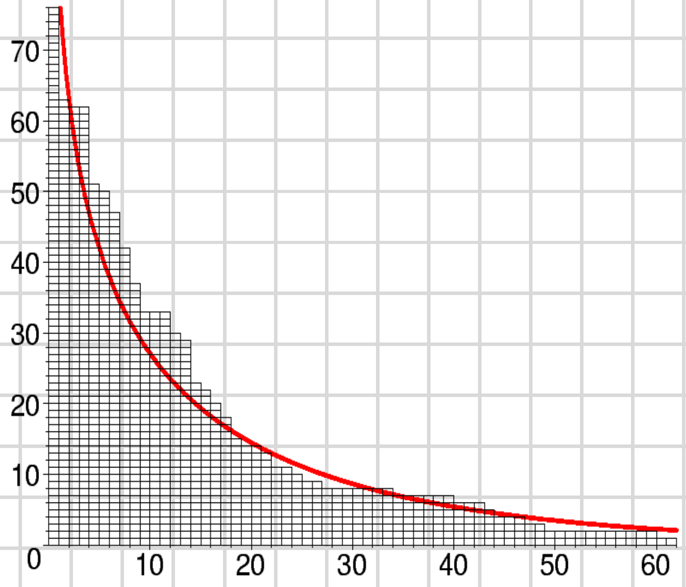
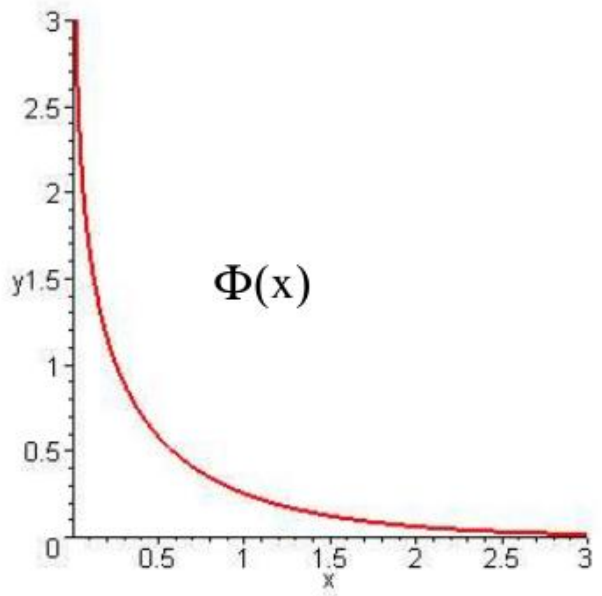
Igor Pak, UCLA

March 17, 2021

Limit shape of partitions via bijections





Stephen DeSalvo, CPC

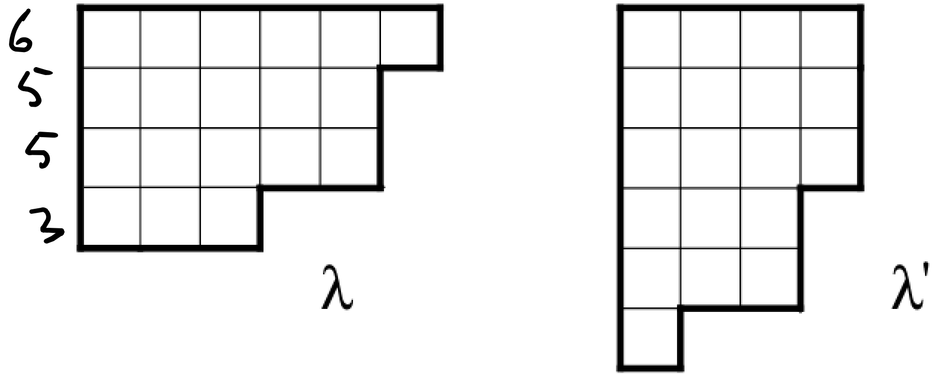
Partitions

$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is a *partition* of n if $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$ and $\lambda_1 + \lambda_2 + \dots + \lambda_\ell = n$.

λ_i - *parts of the partition* λ
$\ell = \ell(\lambda)$ - *the number of parts* in λ.

Example $n = 5$
(5), $(4,1)$, $(3,2)$, $(3,1,1)$, $(2,2,1)$, $(2,1,1,1)$, $(1,1,1,1,1)$
all partitions of 5.

Young diagrams



Young diagrams of partitions $\lambda = (6,5,5,3)$ and $\lambda' = (4,4,4,3,3,1)$.

Theorem (Euler)

$$\prod_{i=1}^{\infty} \frac{1}{1-t^i} = 1 + \sum_{n=1}^{\infty} p(n)\, t^n,$$

where $p(n)$ is the number of partitions of n.

Similarly:

$$\prod_{i=1}^{\infty} (1+t^i) = 1 + \sum_{n=1}^{\infty} q(n)\, t^n,$$

where $q(n)$ is the number of partitions of n into distinct parts.

Euler's Theorem

- *Number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts.*

$$\prod_{i=1}^{\infty}(1+t^i) = \prod_{i=1}^{\infty}\frac{1}{1-t^{2i-1}}$$

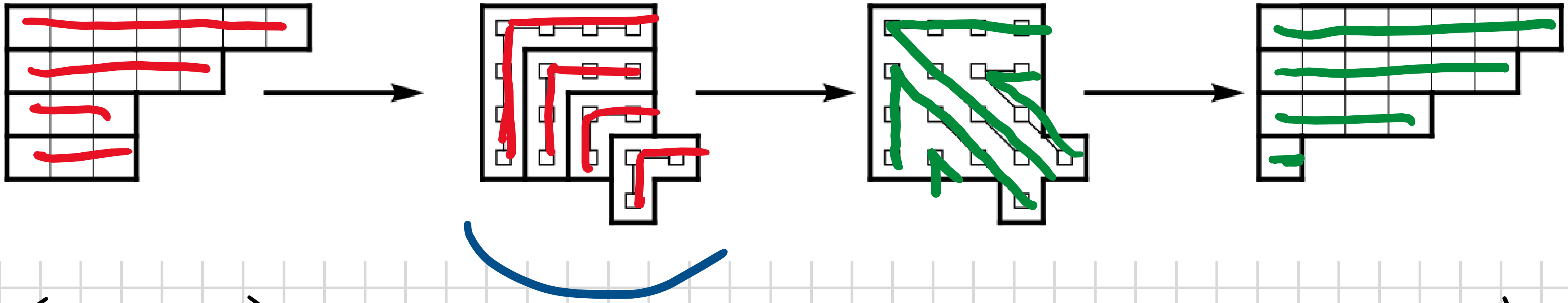
Proof:

$$\prod_{i=1}^{\infty}(1+t^i) = \prod_{i=1}^{\infty}\frac{(1-t^i)(1+t^i)}{(1-t^i)} = \prod_{i=1}^{\infty}\frac{1-t^{2i}}{(1-t^{2i-1})(1-t^{2i})} = \prod_{i=1}^{\infty}\frac{1}{1-t^{2i-1}}$$

Glaisher's bijection (*distinct* $\longrightarrow$ *odd*)

$$(7,6,4,1) \to (7,4,3,3,1) \to (7,3,3,2,2,1) \to (7,3,3,2,1,1,1) \to (7,3,3,1,1,1,1,1)$$

Sylvester's bijection



(7 5 3 3)
odd

(7 6 4 1) distinct

J. J. Sylvester:

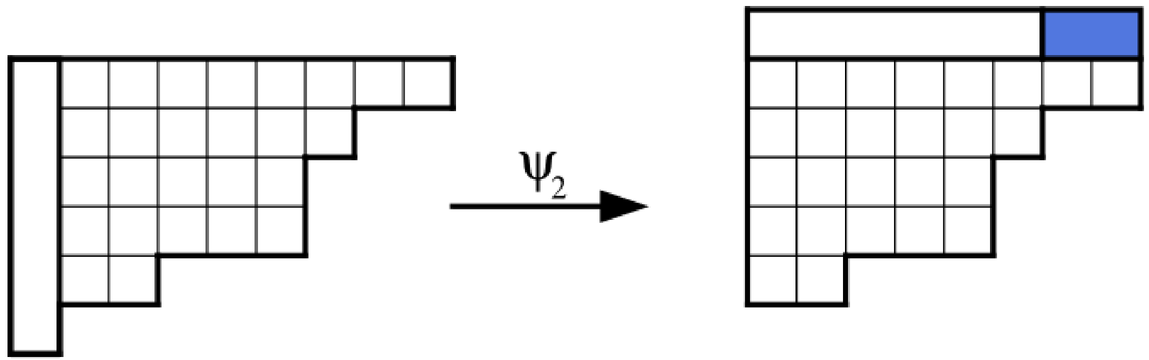
[Glaisher's] *correspondence is eminently arithmetic and transcendental in its nature, depending as it does on the forms of the numbers of repetitions of each integer with reference to the number 2.*

Very different is [Sylvester's correspondence] *which is essentially graphical, as in its operation, which is to bring into correspondence the two systems, not as wholes but separated each other of them into distinct classes; and it is a striking fact that the pairs arithmetically and graphically associated will be entirely different, thus evidencing that correspondence is rather a creation of the mind than a property inherent in the things associated.*

Iterated Dyson's map

О нескольких теоремах Файна,
об Эндрюсе, Дайсоне
и об упущенных возможностях[†]

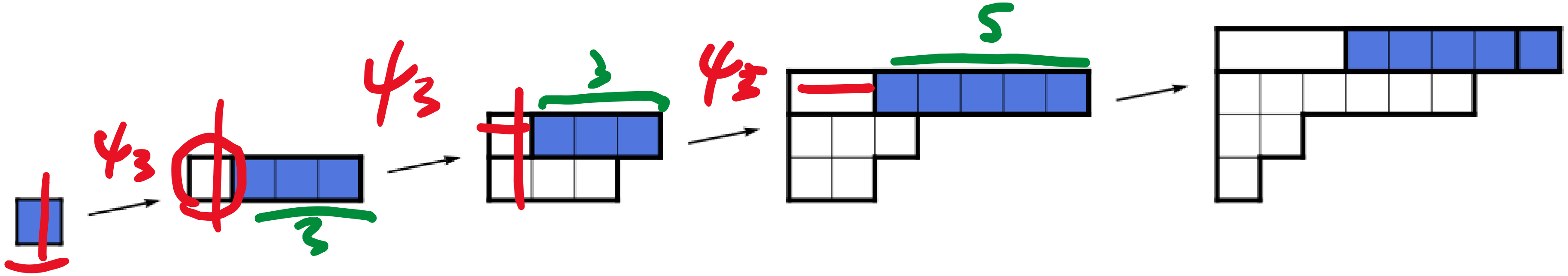
И. Пак*



Dyson's map ψ_2.

← Dyson: new proof of Euler's Pentag theorem

Andrews



$$O_n \to D_n$$

$(5, 5, 3, 3, 1) \to (8, 6, 2, 1)$

Th (P.) This works

Limit shapes

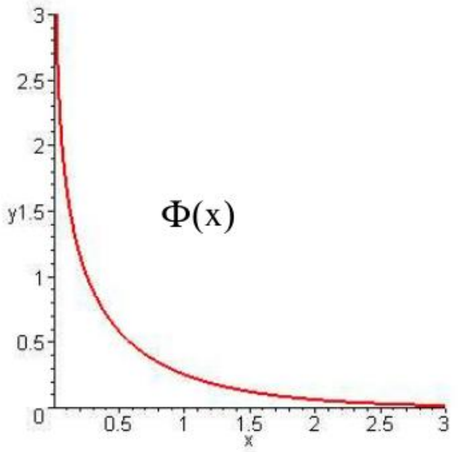
3.2. **Limiting shape.** Let $\mathcal{A} = \cup_n \mathcal{A}_n$ be a set of partitions. For every $x > 0$, think of the a scaled shape value $\alpha_x := \widetilde{\rho}_\lambda(x)$, $\alpha_x : \lambda \to \mathbb{R}$ as of a statistic on $\mathcal{A}$. We say that $\mathcal{A}$ is *asymptotically stable* if α_x are asymptotically stable statistics with uniform convergence: there exist a function $a = a(x) : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty a(x)\,dx = 1$, and for every $\varepsilon > 0$

$$\mathbf{P}\big(|\alpha_x(\lambda) - a(x)| < \varepsilon \ \text{ for all } x > 0 \mid \lambda \in \mathcal{A}_n\big) \to 1 \quad \text{as} \quad n \to \infty,$$

The function $a(x)$ is called the *limiting shape* of $\mathcal{A}$.

Example 3.5. It was shown in [23] that the set of all partitions $\mathcal{P}$ is asymptotically stable with the limiting shape $\Phi(x)$ defined as:

$$\Phi(x) = -\frac{1}{c}\log\left(1 - e^{-cx}\right), \quad \text{where } c = \frac{\pi}{\sqrt{6}}.$$



[23] A. M. Vershik, Statistical mechanics of combinatorial partitions, and their limit shapes, *Funct. Anal. Appl.* **30:2** (1996), 90–105.

[8] A. Dembo, A. Vershik, and O. Zeitouni, Large deviations for integer partitions, *Markov Process. Related Fields* **6** (2000), 147–179.

temperley

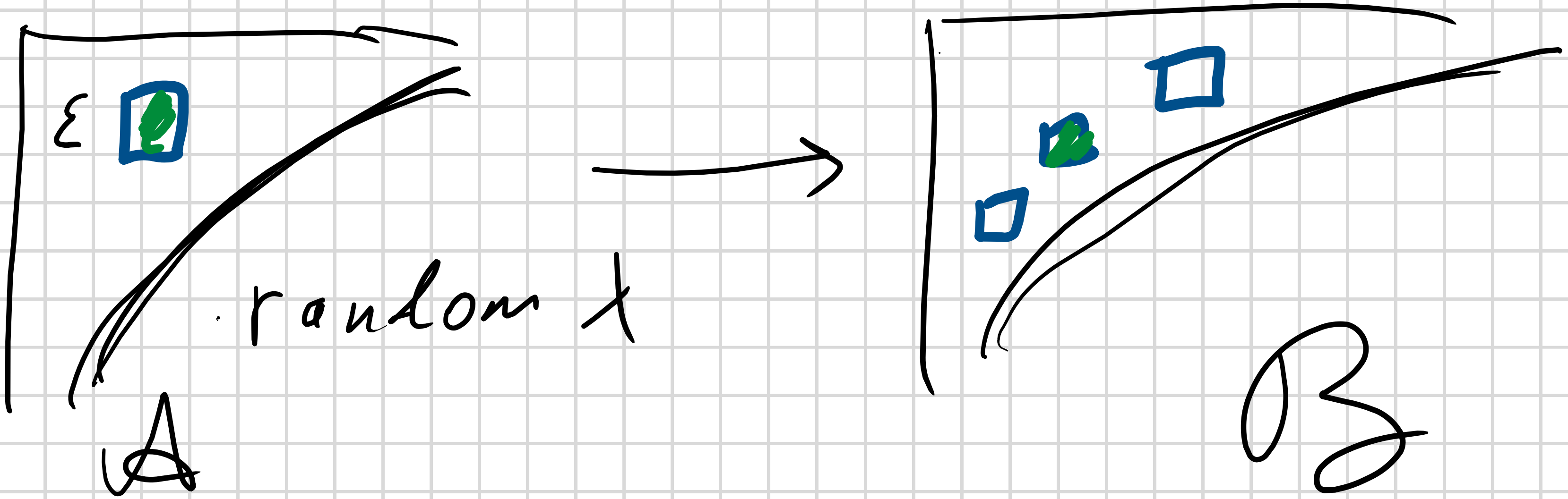
The nature of partition bijections II. Asymptotic stability (P., 2004)

$\mathcal{A} \xrightarrow{\varphi} \mathcal{B}$

Definition 3.9. Let $\mathcal{A}$, $\mathcal{B}$ be asymptotically stable sets of partitions. We say that bijection $\varphi : \mathcal{A} \to \mathcal{B}$ defined as above is *asymptotically stable* if $\Theta_\lambda(X, Y)$ are asymptotically stable statistics with uniform convergence, i.e. there exist a measurable function $F(X, Y) \geq 0$, such that for every $\varepsilon > 0$

$$\mathbf{P}\big(|\Theta_\lambda(X,Y) - F(X,Y)| < \varepsilon \ \text{ for all } X \subset V,\, Y \subset W \mid \lambda \in \mathcal{A}_n\big) \to 1 \ \text{ as } n \to \infty,$$

where the probability is over uniform $\lambda \in \mathcal{P}_n$.



unknown limit shape

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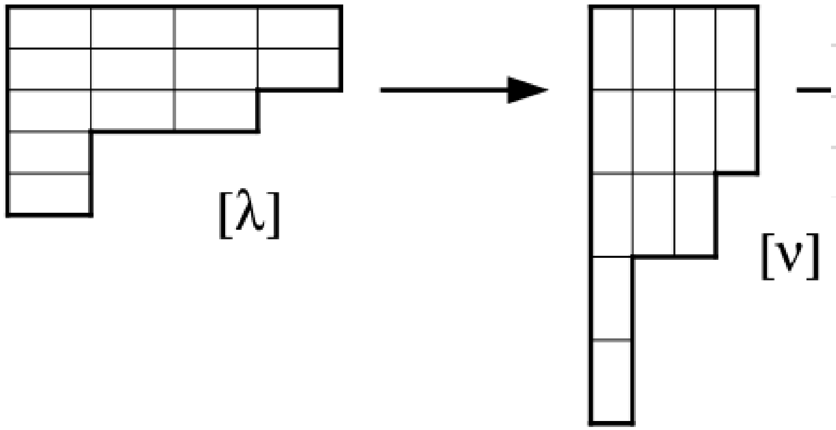
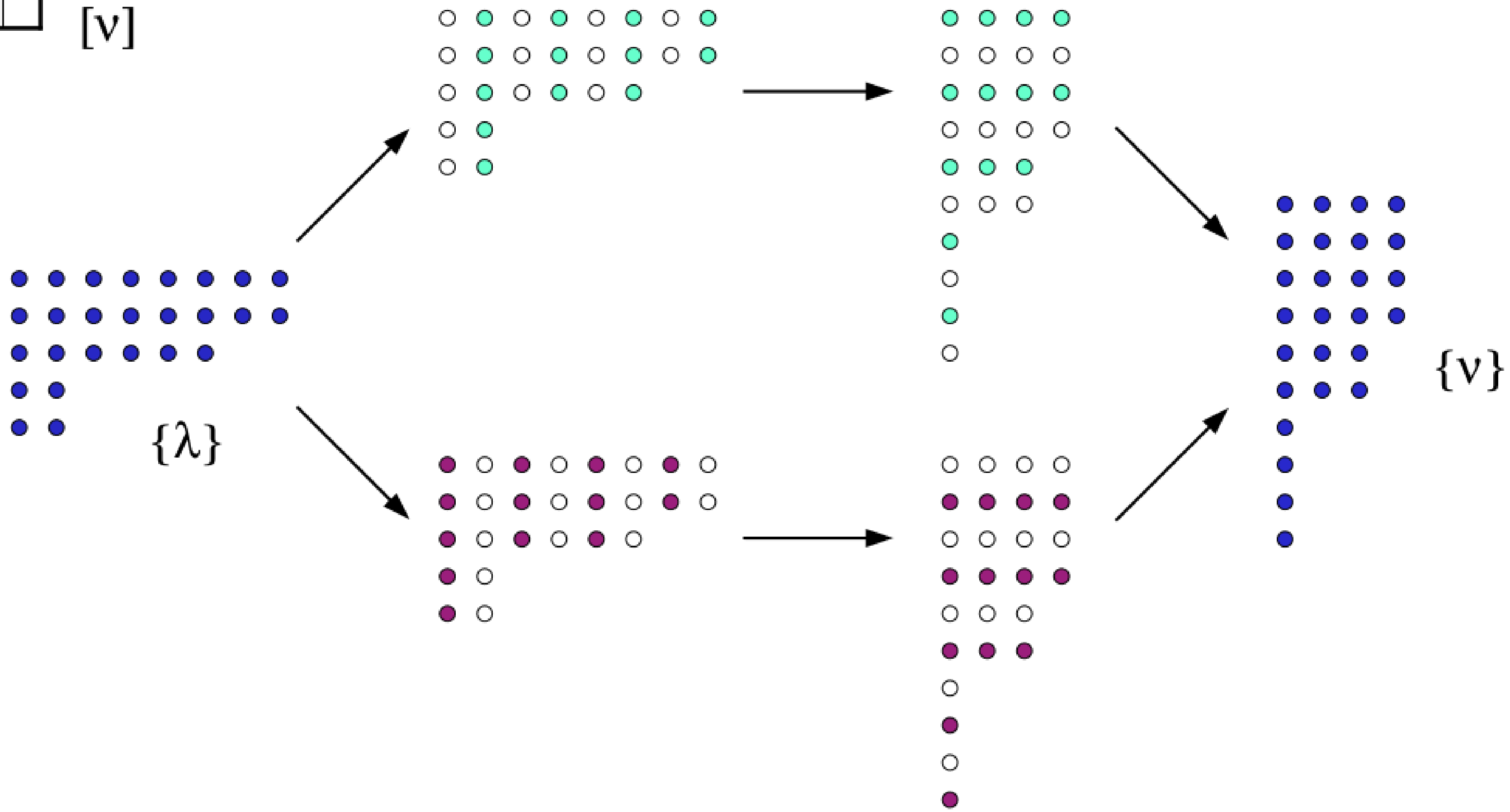
where the probability is over uniform $\lambda \in \mathcal{P}_n$.

Meta Theorem. *All known "good" partition bijections are asymptotically stable.*

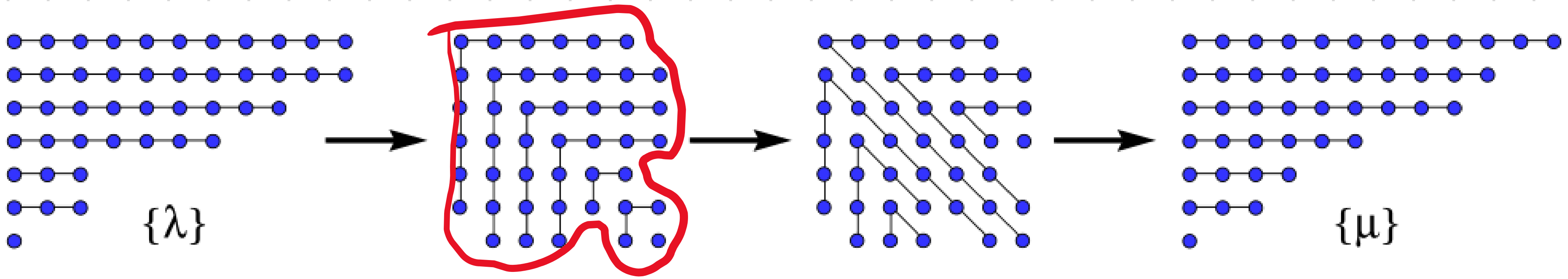
Theorem 4.1. *Let $\mathcal{A} \subset \mathcal{P}$ be an asymptotically stable set of partitions, let $\mathcal{B} \subset \mathcal{P}$, and let $\vartheta : \mathcal{A} \to \mathcal{B}$ be a size preserving one-to-one correspondence. Suppose a natural geometric bijection φ defines ϑ. Then φ is asymptotically stable. Moreover, this holds if φ is an AG-bijection with asymptotically stable parameters.*

AG-bijections. (P., 2004)

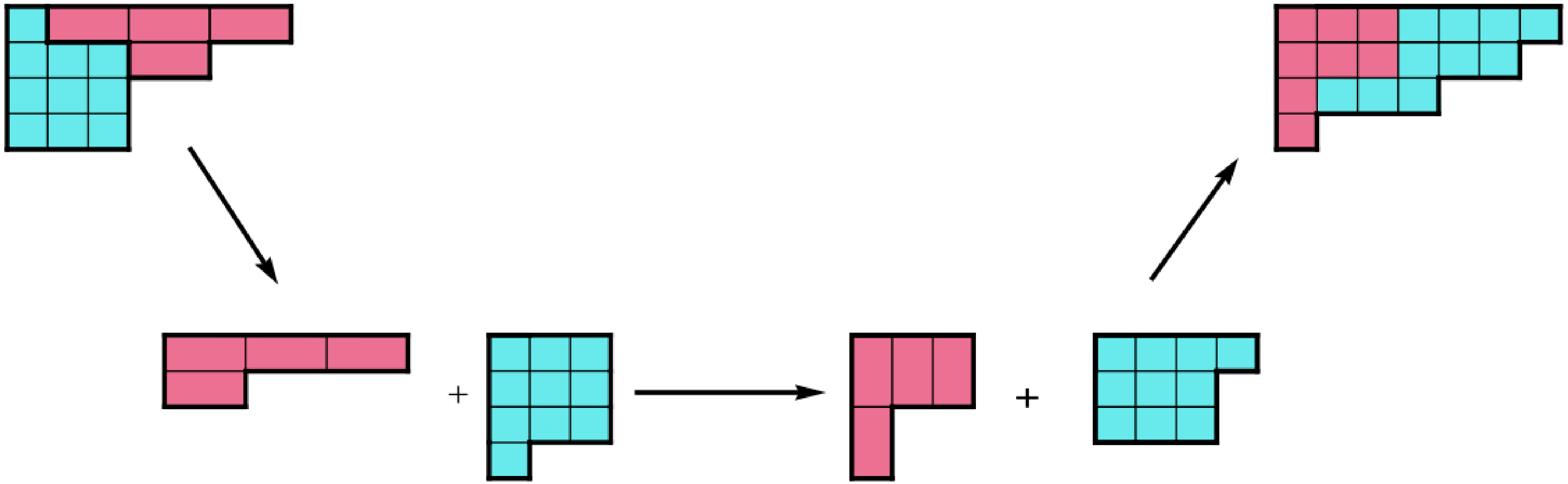


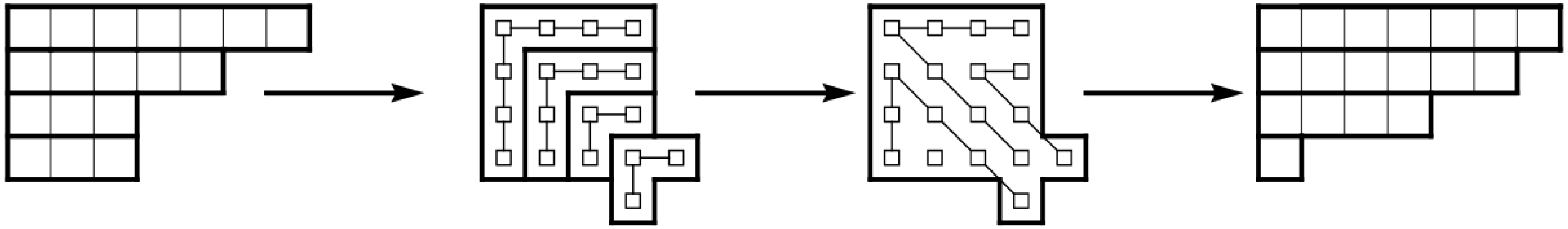


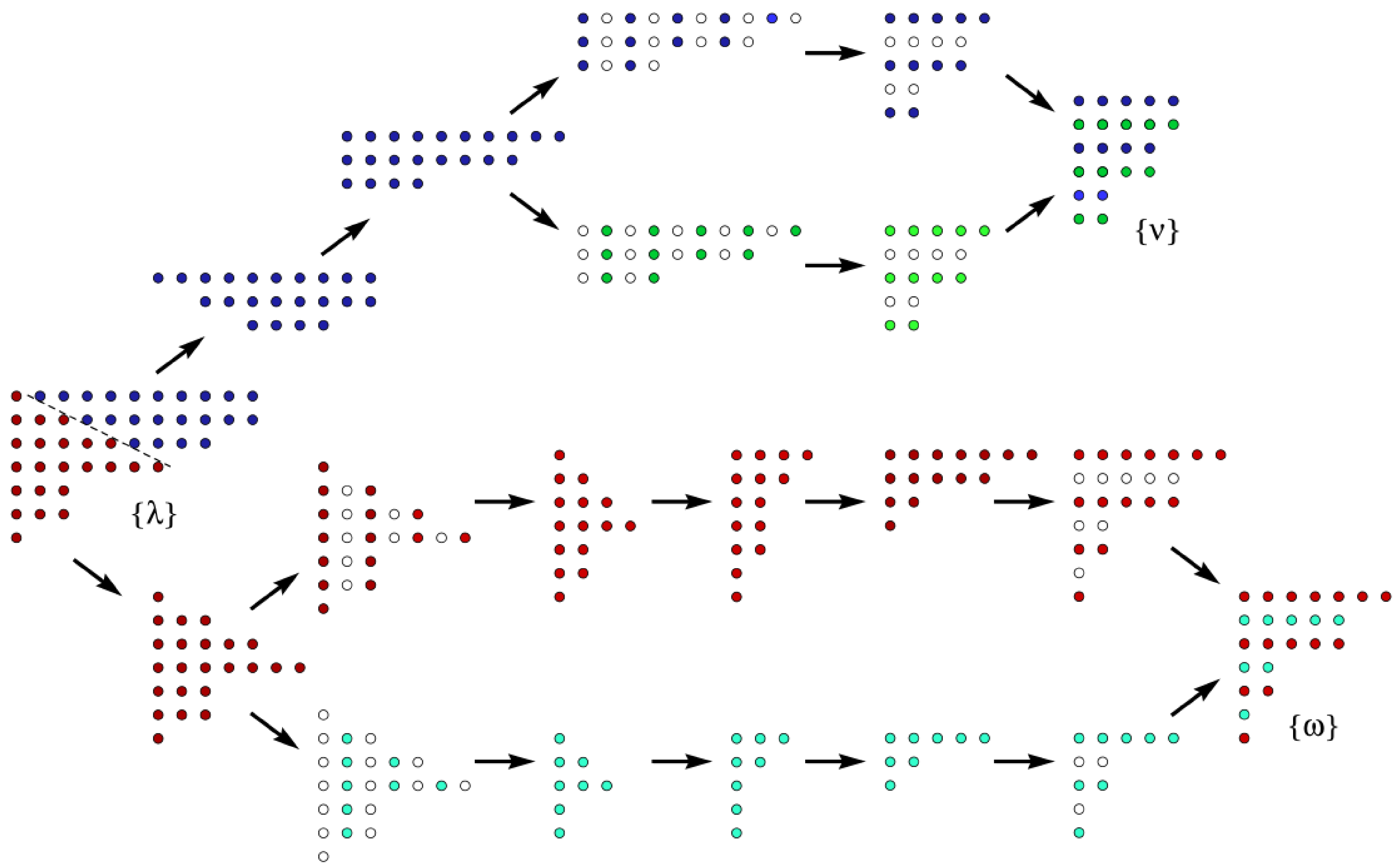
Is Sylvester's bijection a *good* bijection?



Pak-Postnikov version of Sylvester's bijection (1998)






{λ}
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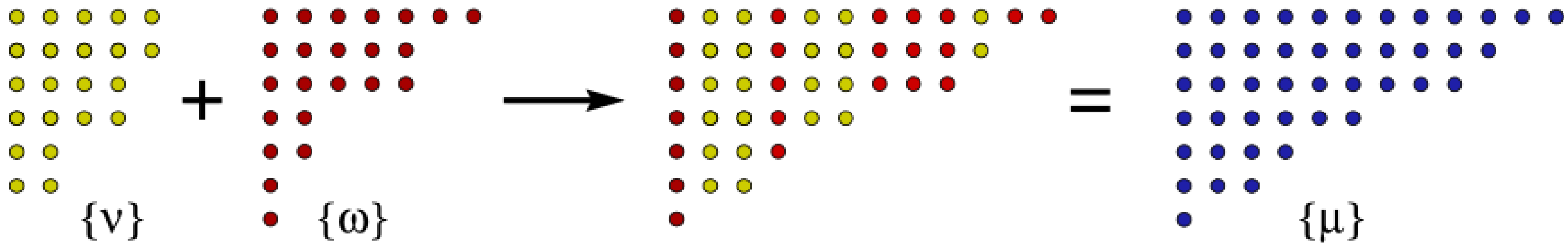
{ω}



Corollary 9.2. *One-to-one correspondence* $\psi : \mathcal{D}^r \to \mathcal{O}^r$ *defined above is given by an asymptotically stable AG-bijection.*

Theorem (Vershik, 1996) *Let* $y = \Psi(x)$ *denote the limit shape of integer partitions into distinct parts. Then:*

$$e^{dy} - e^{-dx} = 1, \qquad x > 0, \quad \text{where} \quad d = \frac{\pi}{\sqrt{12}}\,. \tag{1.2}$$

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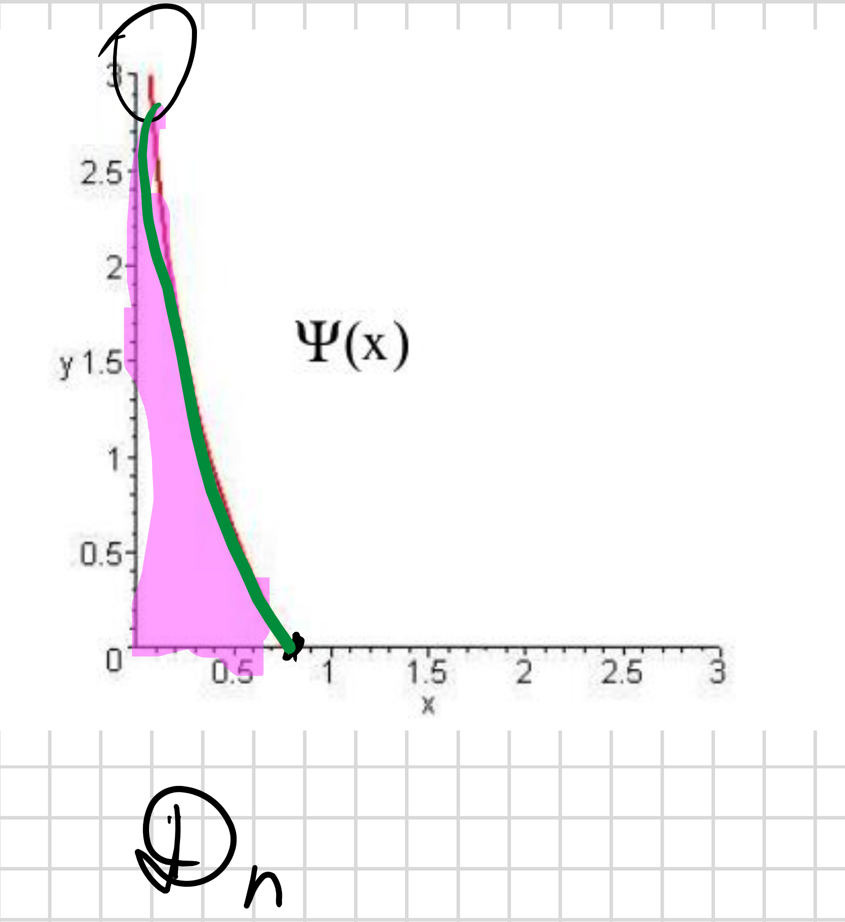


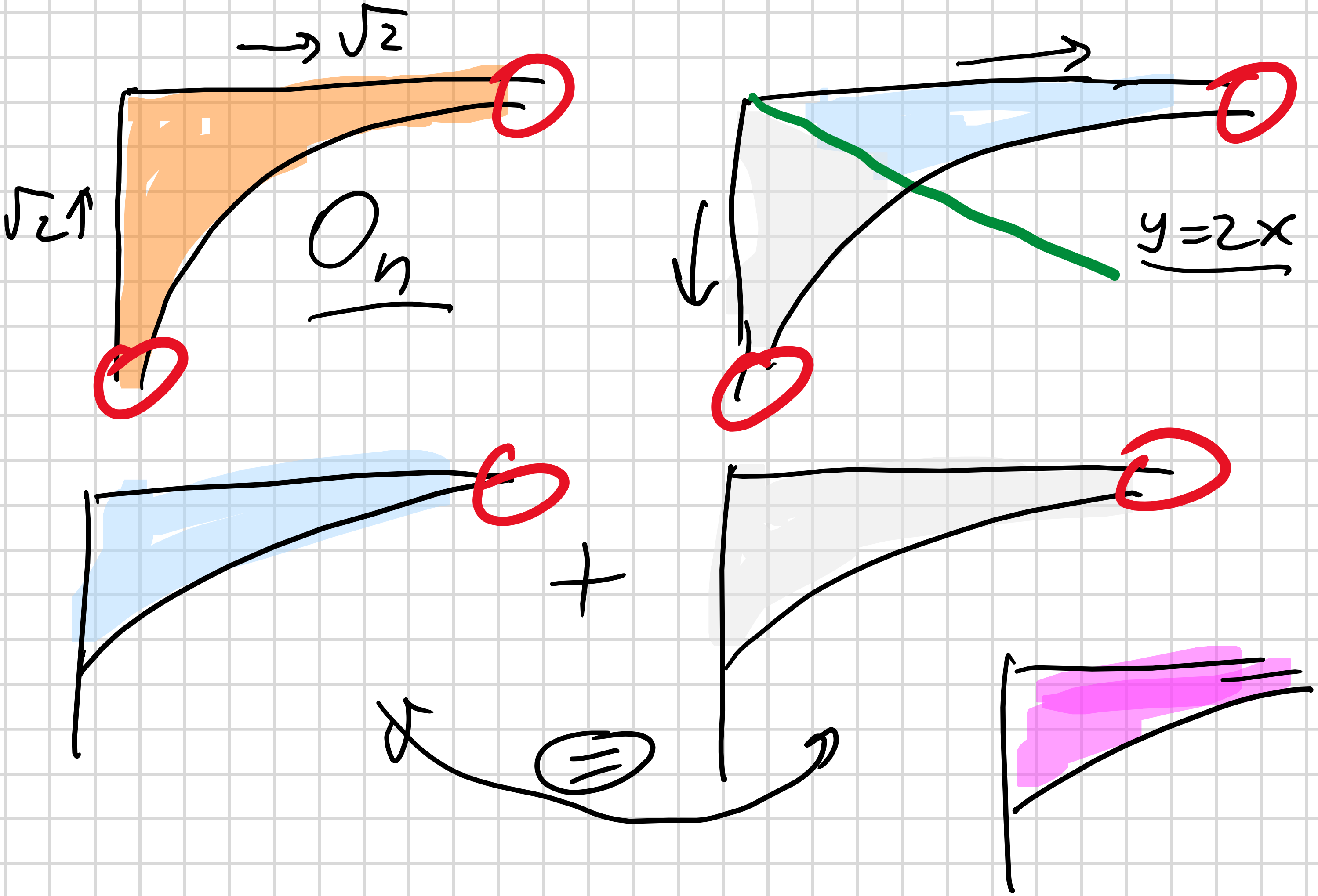
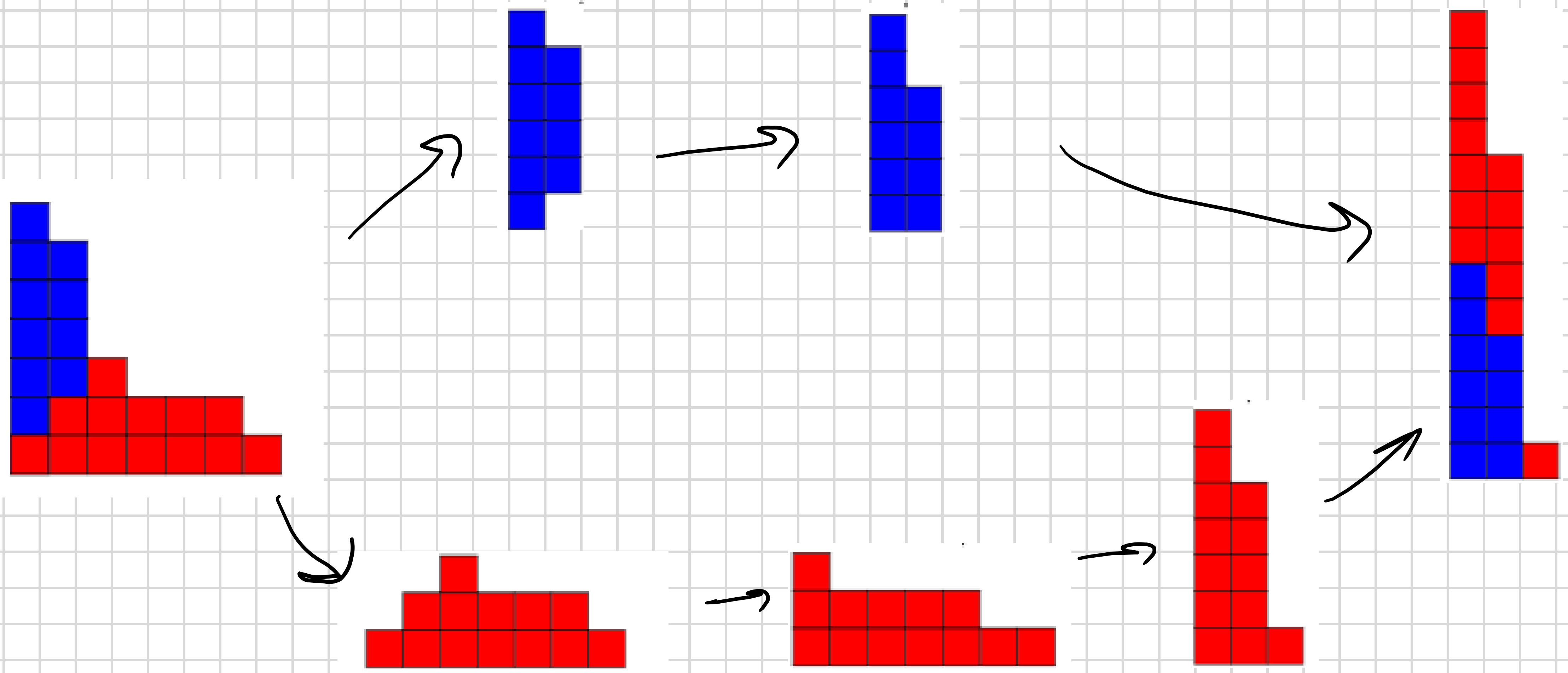


FIGURE 2. The bijection between self-conjugate partitions and partitions into distinct odd parts.



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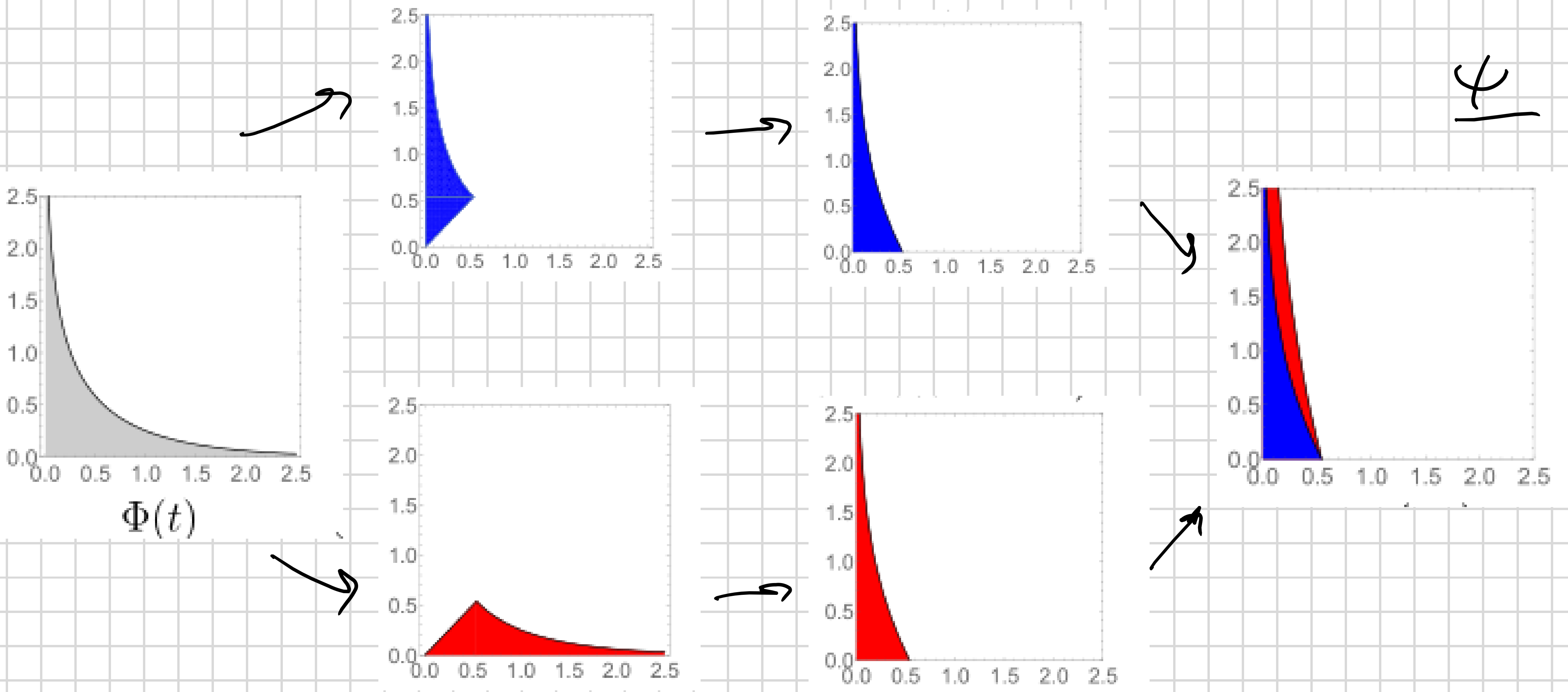

New results (some special cases)

Corollary 1.4. *Let $\mathcal{L}$ denote the set of partitions μ with consecutive parts $\mu_i - \mu_{i-1} \geq 2$ for even part sizes μ_i, and $\mu_i - \mu_{i-1} \geq 4$ for all odd part sizes μ_i, $i \geq 1$. Let $m(x)$ denote the limit shape of $\mathcal{L}$, and let $w = e^{\frac{\pi}{2}m(x)}$, and $u = e^{\frac{\pi}{4}x}$. Then the limit shape satisfies*

$$u = \frac{w+1}{w^2 - w}.$$

Corollary 1.5. *Let k_n denote the number of parts in a random partition of size n in $\mathcal{L}$. We have*

$$\frac{k_n}{\sqrt{n}} \longrightarrow_P \frac{2\log\left(1+\sqrt{2}\right)}{\pi} = 0.561099852\ldots, \tag{1.4}$$

where $\rightarrow_P$ denotes convergence in probability.

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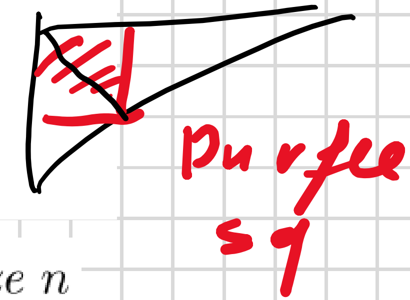

Corollary 1.6. *Let δ_n denote the size of the largest Durfee square in a random partition of size n in $\mathcal{L}$. Let $y_\circ = 4.171195932\ldots$ denote the real–valued solution to*

$$-1 + 2y - 9y^2 - 7y^3 - 2y^4 + y^5 = 0.$$

Then we have:

$$\frac{\delta_n}{\sqrt{n}} \longrightarrow_P \frac{4}{\pi} \log \frac{1}{14}\left(5 - 30y_\circ - 24y_\circ^2 - 9y_\circ^3 + 4y_\circ^4\right) = 0.454611067\ldots, \tag{1.5}$$

where $\rightarrow_P$ denotes convergence in probability.

Lebesgue's identity

$$\sum_{r=1}^{\infty} t^{\binom{r+1}{2}} \frac{(1+zt)(1+zt^2)\cdots(1+zt^r)}{(1-t)(1-t^2)\cdots(1-t^r)} = \prod_{i=1}^{\infty}(1+zt^{2i})(1+t^i)\,.$$

Let $\mathcal{A}_n$ denote the set of partitions of size n into distinct parts which are congruent to $0, 1$ or 2 modulo 4. Let $\mathcal{L}_n$ denote the set of partitions μ of size n with consecutive parts $\mu_i - \mu_{i-1} \geq 2$ for even part sizes μ_i and $\mu_i - \mu_{i-1} \geq 4$ for all odd part sizes μ_i, $i \geq 1$.

Bressoud's bijection $\varphi : \mathcal{A}_n \to \mathcal{L}_n$

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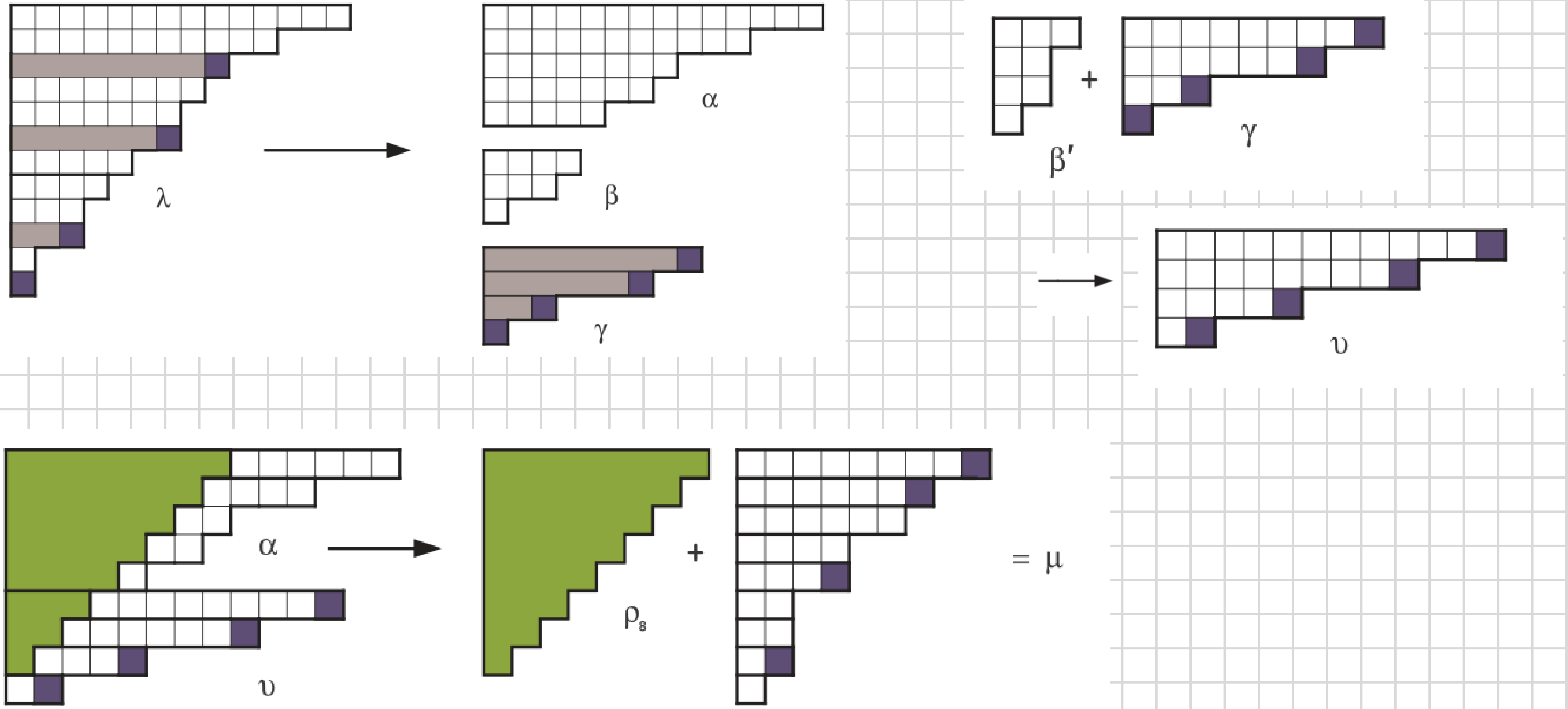
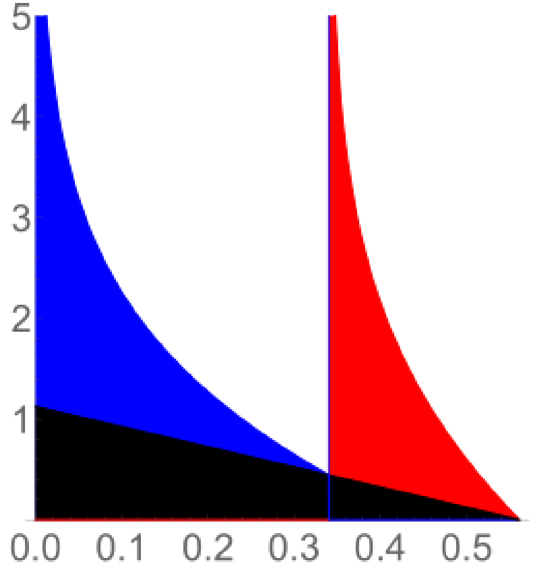


Limit shape of Lebesgue partitions



← Key to Bressoud's Bijection
asymptotic limit shape

Theorem 7.1. *Let $1 \leq \ell < k$. Let $\mathcal{L}^{\ell,k}$ denote the set of partitions μ into parts congruent to 0 or ℓ mod k such that parts differ by at least k, and parts congruent to ℓ mod k differ by at least $2k$. The limit shape of $\mathcal{L}^{\ell,k}$ is given by*

$$\frac{2\sqrt{2}}{\pi\sqrt{k}} \log\left[\frac{1}{2}\left(1+e^{-\frac{\pi t}{2\sqrt{2}\,k}}+\sqrt{1+e^{-\frac{\pi t}{\sqrt{2}\,k}}+6e^{-\frac{\pi t}{2\sqrt{2}\,k}}}\right)\right], \quad t \geq 0. \tag{7.1}$$

Thank you!