

IHP (Dec 4 , 2023)

# What is beyond D-Finite?

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**Workshop:** *Computer Algebra for Functional Equations in Combinatorics and Physics*



# Deep Questions:



**First Question:** What is a *formula*?

**Second Question:** Does  $\mathcal{A}(t) = \sum_n a_n t^n$  have a formula?

**Third Question:** What is a *combinatorial object*?

**Fourth Question:** Does  $a_n = [t^n]\mathcal{A}(t)$  count any combinatorial objects?

# Classes of combinatorial sequences:

(1) *rational* GF  $\mathcal{A}(t) = P(t)/Q(t)$ ,  $P, Q \in \mathbb{Z}[t]$ .

E.g.  $a_n := \text{Fib}(n)$ ,  $\mathcal{A}(t) = 1/(1 - t - t^2)$ .

(2) *algebraic* GF  $c_0\mathcal{A}^k + c_1\mathcal{A}^{k-1} + \dots + c_k = 0$ ,  $c_i \in \mathbb{Z}[t]$ .

E.g.  $a_n := \text{Cat}(n)$ ,  $\mathcal{A}(t) = (1 - \sqrt{1 - 4t})/2t$ .

# Classes of combinatorial sequences:

(3) *diagonals*  $\mathcal{A}(t) = \text{diag } P/Q, \quad P, Q \in \mathbb{Z}[x_1, \dots, x_k].$

$$\mathcal{B} = \sum_{(i_1, \dots, i_k)} b(i_1, \dots, i_k) x_1^{i_1} \cdots x_k^{i_k} \implies \text{diag } \mathcal{B} := \sum_{n=0}^{\infty} b(n, \dots, n) t^n$$

E.g. *Delannoy numbers*  $\{D_n\}$  and *Apéry numbers*  $\{A_n\}$

$$D_n := \sum_{k=0}^n \binom{n+k}{n-k} \binom{2k}{k} \quad \text{and} \quad A_n := \sum_{k=0}^n \sum_{j=0}^k \binom{n}{k} \binom{n+k}{k} \binom{k}{j}^3$$

# Classes of combinatorial sequences:

(4) *D-finite* GF  $c_0\mathcal{A} + c_1\mathcal{A}' + \dots + c_k\mathcal{A}^{(k)} = 0$ ,  $c_i \in \mathbb{Z}[t]$ .

E.g.  $a_n := \#$  involutions in  $S_n$ ,  $a_n = a_{n-1} + (n-1)a_{n-2}$ .

The sequences  $\{a_n\}$  are called *P-recursive*

(5) *D-algebraic* GF  $Q(t, \mathcal{A}, \mathcal{A}', \dots, \mathcal{A}^{(k)}) = 0$ ,  $Q \in \mathbb{Z}[t, x_0, x_1, \dots, x_k]$

E.g.  $a_n = \#\{\sigma(1) < \sigma(2) > \sigma(3) < \dots \in S_n\}$ ,  $\mathcal{A}'' = \mathcal{A} \cdot \mathcal{A}'$ .

Also  $p(n) = \#$  integer partitions of  $n$ . Then  $F(t) = \sum_n p(n)t^n$  satisfies:

$$4F^3F'' + 5tF^3F''' + t^2F^3F^{(4)} - 16F^2(F')^2 - 15tF^2F'F'' - 39t^2F^2(F'')^2 \\ + 20t^2F^2F'F''' + 10tF(F')^3 + 12t^2F(F')^2F'' + 6t^2(F')^4 = 0.$$

(Jacobi, Ramanujan)

# Classes of combinatorial sequences:

$$\textit{Rational} \subsetneq \textit{Algebraic} \subsetneq \textit{Diagonal} \subsetneq \textit{D-finite} \subsetneq \textit{D-algebraic}$$

**Question:** What are the *positive analogues* of these classes? What classes are we missing?

# N-Rational

**Definition 1.** Let  $\mathcal{R}_1$  be the smallest set of functions  $F(t)$  which satisfies

- (1)  $0, t \in \mathcal{R}_1$ ,
- (2)  $F, G \in \mathcal{R}_1 \implies F + G, F \cdot G \in \mathcal{R}_1$ ,
- (3)  $F \in \mathcal{R}_1, F(0) = 0 \implies 1/(1 - F) \in \mathcal{R}_1$ .

Note that all  $F \in \mathcal{R}_1$  satisfy:  $F \in \mathbb{N}[[x]]$ , and  $F = P/Q$ , for some  $P, Q \in \mathbb{Z}[t]$ .

**Definition 1'.** Let  $\mathcal{R}_1$  be the class of  $\mathcal{A} = \sum_n a_n t^n$  where  $a_n$  is the number of  $s - t$  paths of length  $n$  in graph  $G$ .

For example,

$$\frac{1}{1 - x - x^2} \quad \text{and} \quad \frac{x^3}{(1 - x)^4} \in \mathcal{R}_1$$

**Theorem** [Schützenberger, 1962]

Def. 1  $\Leftrightarrow$  Def. 1'

# N-Rational (complete success)

**Theorem** [Berstel'71, Soittola'76]

Complete analytic characterization.

**Corollary:**  $Rational = \mathcal{R}_1 - \mathcal{R}_1$

**Note:** Koutschan (2008) has a code.

$$\frac{t + 5t^2}{1 + t - 5t^2 - 125t^3} \notin \mathcal{R}_1 \quad , \quad \frac{1 + t}{1 + t - 2t^2 - 3t^3} \in \mathcal{R}_1$$

[Gessel, 2003]

**Theorem** [Berstel–Reutenauer, 2008]

Every  $A(t) \in \mathcal{R}_1$  has *star height* at most two.



# N-Algebraic

**Definition 2.** Class of  $A_0 = \sum_n a_n t^n$ , where

$$\begin{cases} A_0(t) = f_0(A_0, \dots, A_k, t) \\ \vdots \\ A_k(t) = f_k(A_0, \dots, A_k, t) \end{cases} \quad \text{and} \quad f_i \in \mathbb{N}[x_0, \dots, x_k, t] \quad \text{are well-posed}$$

**Definition 2'.** Class of GFs for the number of accepting paths of PDA

**Example:**  $A(t) = 1 + tA(t)^2$

$A(t) = \text{Cat}(t) \in \mathbb{N}\text{-Algebraic}$

**Example:** The following  $A(t) = \sum_n a_n t^n$  counts number of certain bilabeled trees:

$$\begin{cases} A(t) = 1 + tA^2B + t^2B^3 \\ B(t) = 1 + tAB + t^2A^2 \end{cases} \quad \text{and} \quad A(0) = B(0) = 1$$

# $\mathbb{N}$ -Algebraic

**Theorem** [Banderier–Drmota, 2015]

*Asymptotic characterization:*  $A(t) \in \mathbb{N}$ -Algebraic,  $A(t) = \sum_n a_n t^n$

$\implies a_n \sim C n^\alpha \lambda^n$ , where  $\alpha \in \left\{ \frac{m}{2^k} \right\}$  and  $\alpha > -1$  or  $\alpha = -1 - \frac{1}{2^k}$  for some  $k \geq 1$

**Corollary:** *Tutte's GFs for the number of rooted triangulations is not  $\mathbb{N}$ -Algebraic:*

$$T_n = \frac{2(4n+1)!}{(n+1)!(3n+2)!} \sim C n^{-5/2} \left(\frac{64}{27}\right)^n$$

**Proposition** [Banderier–Drmota, 2015]: Algebraic =  $\mathbb{N}$ -Algebraic –  $\mathbb{N}$ -Algebraic

**Question:** Can we give a better description of this class?

# $\mathbb{N}$ -Diagonals = Positive Binomial Sums

**Definition 3.** Let  $\mathcal{R}_k$  be the smallest set of functions  $F(x_1, \dots, x_k)$  which satisfies

- (1)  $0, x_i \in \mathcal{R}_k$ ,
- (2)  $F, G \in \mathcal{R}_k \implies F + G, F \cdot G \in \mathcal{R}_k$ ,
- (3)  $F \in \mathcal{R}_1, F(0) = 0 \implies 1/(1 - F) \in \mathcal{R}_k$ .

Let  $\mathcal{F}$  be the set of diagonals of  $\mathcal{R}_k$

**Definition 3'.** Let  $\mathcal{F}$  be the class of  $\mathcal{A} = \sum_n a_n t^n$  where  $a_n$  is the number of  $s - t$  paths of length  $kn$  in  $k$ -edge colored graph  $G$  with equal numbers of each color.

**Definition 3''.** Let  $\mathcal{F}$  be the class of GFs of *binomial sums* with coefficients in  $\mathbb{N}$

$$D_n = \sum_{k=0}^n \binom{n+k}{n-k} \binom{2k}{k}, \quad A_n = \sum_{k=0}^n \sum_{j=0}^k \binom{n}{k} \binom{n+k}{k} \binom{k}{j}^3$$

**Theorem** [Garrabrant-P., 2014]

Def. 3  $\Leftrightarrow$  Def. 3'  $\Leftrightarrow$  Def. 3''

# $\mathbb{N}$ -Diagonals = Positive Binomial Sums

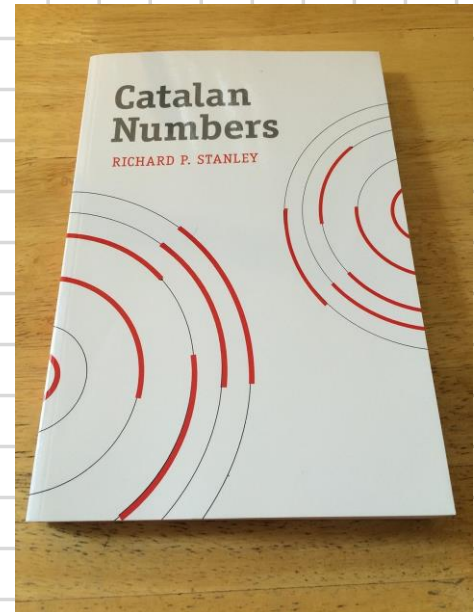
**Corollary** [Bostan–Lairez–Salvy’16]:  $\text{Diagonals} = \mathbb{N}\text{-Diagonals} - \mathbb{N}\text{-Diagonals}$

**Bad news:** Open whether  $\mathbb{N}\text{-Algebraic} \subseteq \mathbb{N}\text{-Diagonals}$

**Conjecture** [Garrabrant-P., 2014]:  $\text{Cat}(t) \notin \mathbb{N}\text{-Diagonal}$

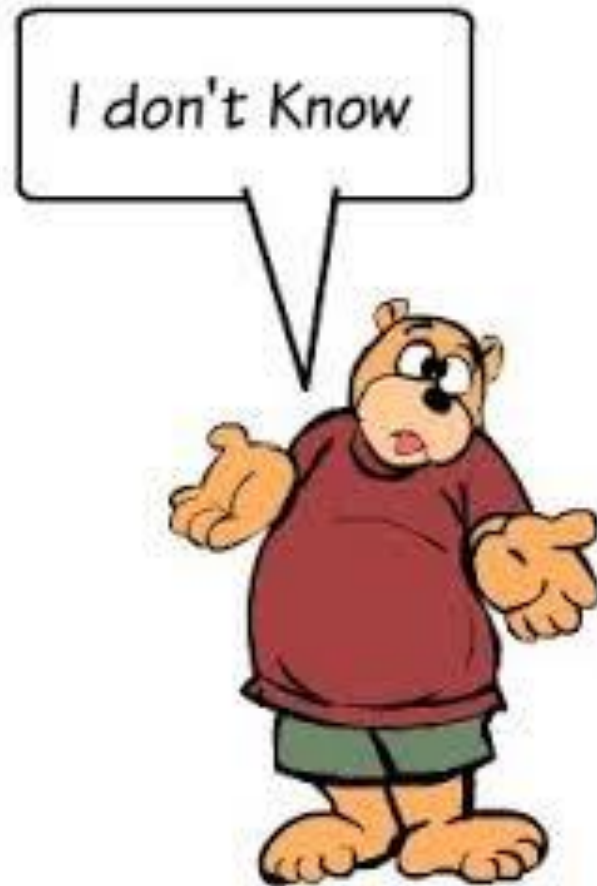
**Note:** One of only two open problems in Stanley’s *Catalan Numbers* book.

**Note:**  $\frac{1 - (1 - 2z)^{1/4} \sqrt{2z\sqrt{1 + 2z} + \sqrt{1 - 2z}}}{2z} \notin \mathbb{N}\text{-Diagonal}$  is potentially easier.



**Question:** Can we give an asymptotic characterization of this class?

N-D-Finite = ???

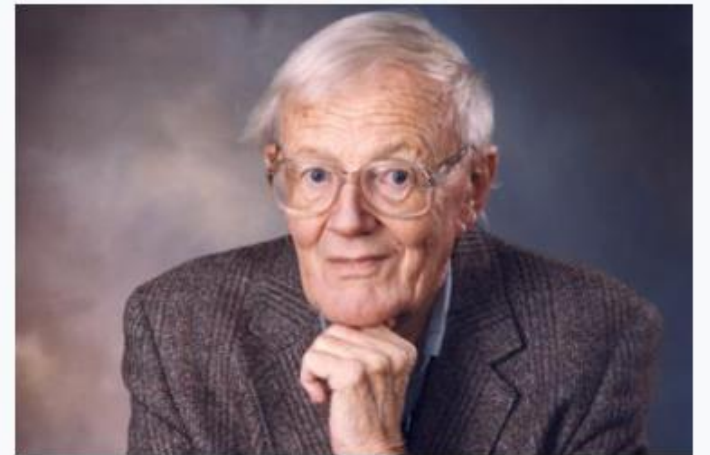


# $\mathbb{N}$ -D-Algebraic = $\mathbb{T}$ -Algebraic

**Definition 4** [Drmota-P, 2023+] Class of  $A_0 = \sum_n a_n t^n$ , where

$$\begin{cases} A'_0 = f_0(A_0, \dots, A_k, t) \\ \vdots \\ A'_k = f_k(A_0, \dots, A_k, t) \end{cases} \quad \text{and} \quad f_i \in \mathcal{R}_{k+2}$$

**W. T. Tutte**



# $\mathbb{N}$ -D-Algebraic = $\mathbb{T}$ -Algebraic

**Definition 4** [Drmota-P, 2023+] Class of  $A_0 = \sum_n a_n t^n$ , where

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**Example:**  $A(t) = \tan(t) + \sec(t)$  is  $\mathbb{T}$ -algebraic since it can be written as

$$\begin{cases} A' = B \\ B' = A \cdot B \end{cases}$$

**Motivating Example:** Tutte's GFs for the numbers  $T_n$  of rooted plane triangulations and numbers  $M_n$  of rooted plane maps are  $\mathbb{T}$ -Algebraic. Here

$$M_n = \frac{2(2n!)3^n}{n!(n+2)!} \quad \text{and} \quad \sum_{n=0}^{\infty} M_n t^n = -\frac{1 - 18t - (1 - 12t)^{3/2}}{54t^2} \notin \mathbb{N}\text{-Algebraic}$$

# $\mathbb{N}$ -D-Algebraic = $\mathbb{T}$ -Algebraic

**Theorem** [Drmota-P, 2023+] *Let  $A(t)$  be a  $\mathbb{T}$ -Algebraic function.*

*Then  $A(t)$  is either an entire function or has finitely many singularities on the circle of convergence  $|t| = \rho$ , where  $0 < \rho < \infty$ .*

**Corollary:** *Partition and theta functions are D-Algebraic, but not  $\mathbb{T}$ -algebraic*

$$P(t) = \prod_{k=1}^{\infty} \frac{1}{1 - t^k} = \sum_{n=0}^{\infty} p(n)t^n \quad \text{and} \quad \Theta(t) = \sum_{n=-\infty}^{\infty} t^{n^2}$$

**Corollary:** *D-Algebraic  $\neq$   $\mathbb{T}$ -Algebraic  $\rightarrow$   $\mathbb{T}$ -Algebraic*

**Open Problem:** Are there any other necessary conditions?



# $\mathbb{N}$ -D-Algebraic = $\mathbb{T}$ -Algebraic

**Theorem** [Kauers–Koutschan–Zeilberger’09] and [Banderier–Drmota’15]

*The GF  $\mathcal{G}(t) = \sum_{n=0}^{\infty} G_n t^n$  for Gessel numbers is Algebraic, but not  $\mathbb{N}$ -Algebraic*

$$\text{where } G_n = 16^n \frac{(5/6)_n (1/2)_n}{(5/3)_n (2)_n}, \quad (z)_m := z(z+1) \cdots (z+m-1).$$

**Open Problem:** *Decide if  $\mathcal{G}(t)$  is  $\mathbb{T}$ -Algebraic.*

# A-Recursive

## Definition:

$\{a_n\}$  is *P-recursive* if  $c_0a_n + c_1a_{n-1} + \dots + c_ka_{n-k} = 0$ ,  $c_i \in \mathbb{Z}[n]$ , for all  $n \in \mathbb{N}$ .

$\{a_n\}$  is *A-recursive* if  $Q(a_n, a_{n-1}, \dots, a_{n-k}, n) = 0$ ,  $Q \in \mathbb{Z}[x_0, \dots, x_n, n]$ , for all  $n \in \mathbb{N}$ .

## Two Open Problems:

- Build a theory of such sequences
- Find a nice positive integer subclass of such sequences

# A-Recursive

A006720 Somos-4 sequence:  $a(0)=a(1)=a(2)=a(3)=1$ ; for  $n \geq 4$ ,  $a(n) = (a(n-1) * a(n-3) + a(n-2)^2) / a(n-4)$ .<sup>88</sup>  
(Formerly M0857)

1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, 620297, 7869898, 126742987, 1687054711,  
47301104551, 1123424582771, 32606721084786, 1662315215971057, 61958046554226593,  
4257998884448335457, 334806306946199122193, 23385756731869683322514, 3416372868727801226636179 ([list](#);  
[graph](#); [refs](#); [listen](#); [history](#); [text](#); [internal format](#))

**Theorem** [Gale'91]  $a_n \in \mathbb{N}$

**Theorem** [Fomin–Zelevinsky'02]

Advanced generalization via *Laurent phenomenon*

**Theorem** [Speyer'06]

$\{a_n\}$  has a combinatorial interpretation

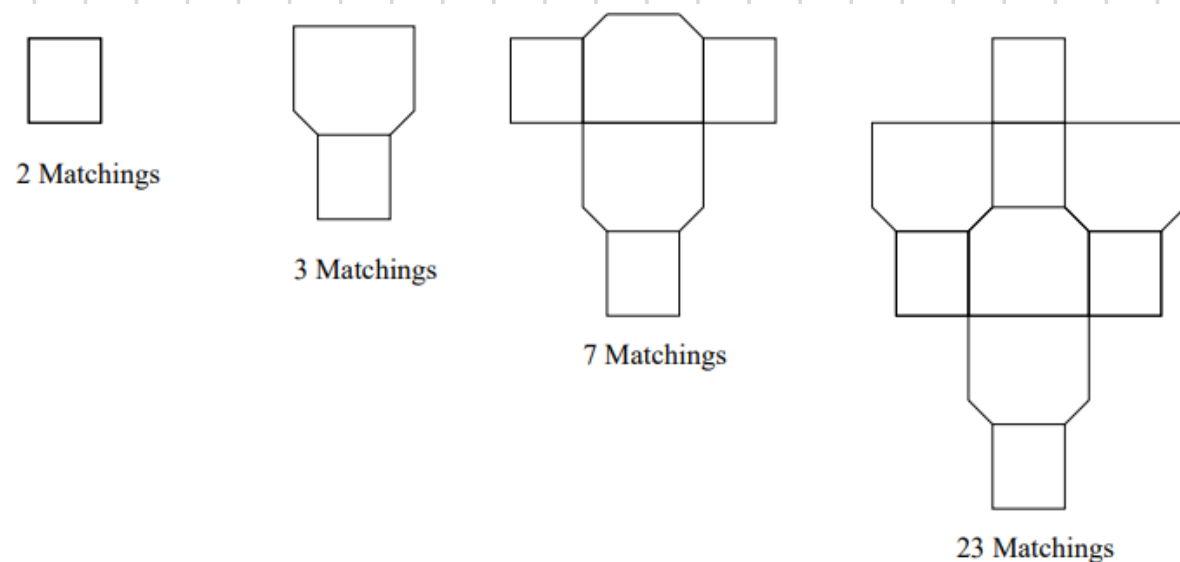
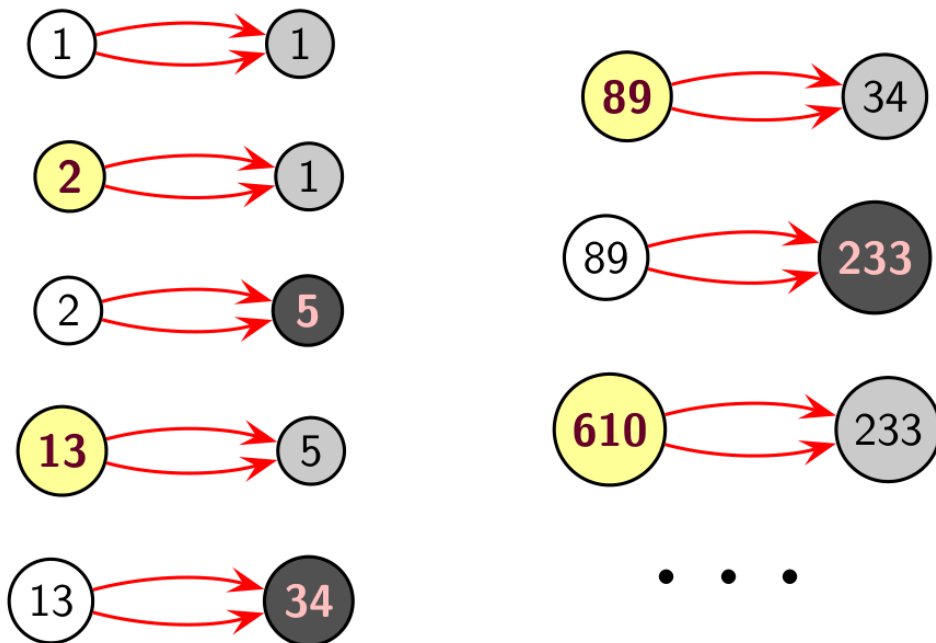
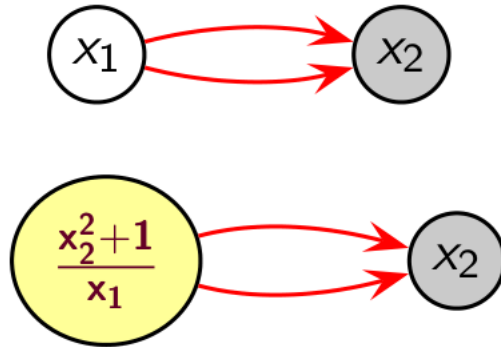


Figure 11: The First Four Nontrivial Somos-4 Graphs

# A-Recursive

Example:



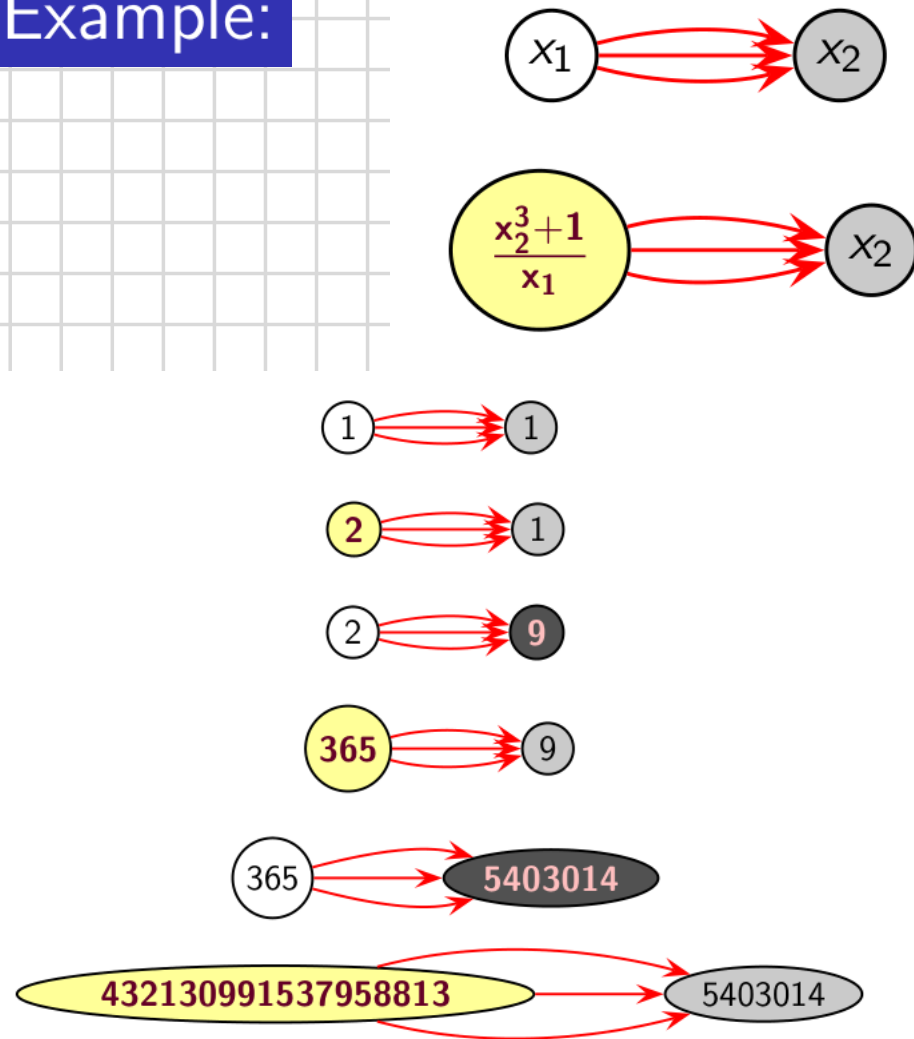
## Zamolodchikov periodicity and integrability

Pavel Galashin

*Odd Fibonacci numbers*  $\text{Fib}(2n - 1)$

# A-Recursive

Example:



## Zamolodchikov periodicity and integrability

Pavel Galashin

A003818  $a(1)=a(2)=1, a(n+1) = (a(n)^3 + 1)/a(n-1).$

1, 1, 2, 9, 365, 5403014, 432130991537958813,

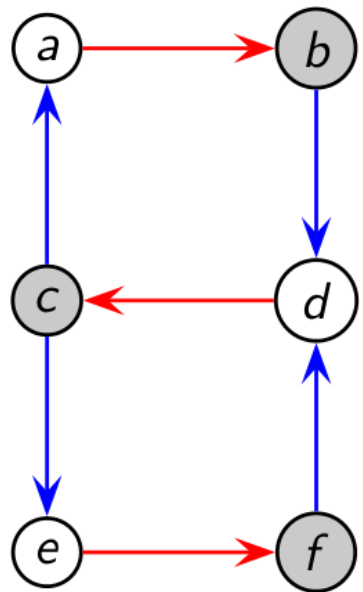
[graph](#); [refs](#); [listen](#); [history](#); [text](#); [internal format](#))

OFFSET 1,3

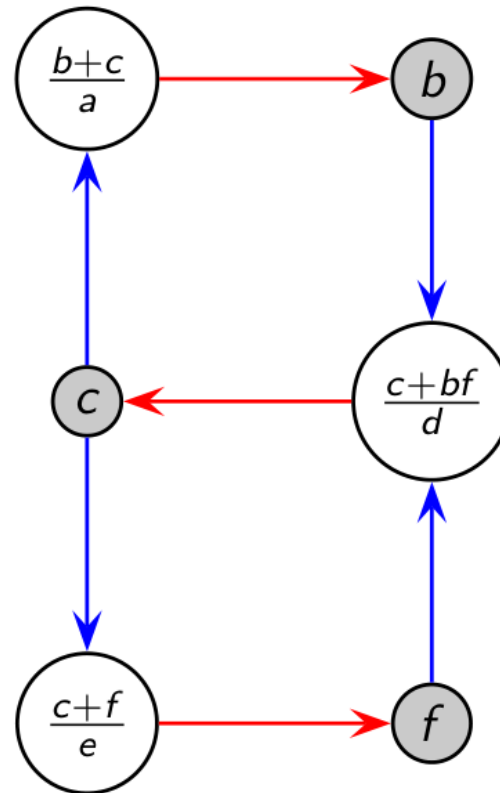
COMMENTS The term  $a(9)$  has 121 digits. - [Harvey P. Dale](#),

# A-Recursive

## Bipartite $T$ -system



$\longrightarrow$



Zamolodchikov periodicity and integrability

Pavel Galashin

# A-Recursive

## Zamolodchikov periodicity and integrability

### Bipartite $T$ -system

Pavel Galashin

**Theorem** [Galashin–Pylyavskyy’19]

Except for finite number of (known) examples and series,  $\{a_n\}$  grow doubly exponentially.

**Conjecture** [Galashin–Pylyavskyy’19]

For the examples and series in the theorem,  $\{a_n\}$  is periodic, grows as  $\exp \Theta(n)$  or  $\exp \Theta(n^2)$ .

**Note:** Theorem implies no combinatorial interpretation in  $\#P$ . But maybe  $\#EXP$ ?

**Open Problem:** Build a theory to show that  $\{a_n\}$  *do not* have a combinatorial interpretation for general A-Recursive functions.

*Thank you!*

