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Cogrowth sequences in groups and graphs

Def $\Gamma = (V, E)$ – connected countable graph, bounded degree

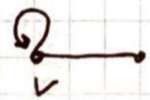
$v \in V$ – fixed vertex

$a_n = a_n(\Gamma, v)$ – number of closed walks

$v = v_0 \to v_1 \to v_2 \to \dots \to v_n = v$

/where $v_i \in V$, $(v_i, v_{i+1}) \in E$ /

Q: what can be said about $\{a_n\}$?

Ex: (1) $\Gamma =$ 

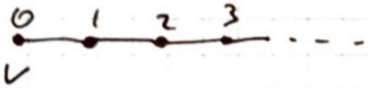
n	0	1	2	3	4	5
a_n	1	1	2	3	5	8

$a_n = Fib(n)$

Fibonacci number

$$A(t) = \frac{1}{1-t-t^2} = \sum_{n=0}^{\infty} a_n t^n$$

More examples

(2) $\Gamma =$ 

$a_{2n} = \frac{1}{n+1}\binom{2n}{n}$

$A(t) := \sum_{n=0}^{\infty} a_{2n}\, t^n = \dfrac{1 \text{ [redacted] } \sqrt{1-4t}}{2t}$ — algebraic

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(3) $\Gamma = \mathbb{Z}^2$ 

$a_{2n} = \binom{2n}{n}^2$

$A(t) := \sum_{n=0}^{\infty} a_{2n}\, t^n = {}_2F_1\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; 16t\right)$ — hypergeom function

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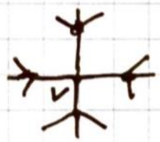
(4) $\Gamma = F_2$ 

$b_n := a_{2n}$

$n\, b_n + 2(9-14n)\, b_{n-1} + 96(2n-3)\, b_{n-2} = 0$

$B(t) := \sum_{n=0}^{\infty} b_n t^n = \dfrac{3}{1 + 2\sqrt{1-12t}}$ — algebraic

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Def $\quad A(t) = \dfrac{P(t)}{Q(t)}$, $P, Q \in \mathbb{Z}[t]$ — rational

$c_0 A^k + c_1 A^{k-1} + \dots + c_k = 0$, $c_i \in \mathbb{Z}[t]$ — algebraic

$c_0 A + c_1 A' + \dots + c_k A^{(k)} = 0$, $c_i \in \mathbb{Z}[t]$ — D-finite

...

Main Problem in EC (one of them)

classify sequences w.r.t. their analytic properties

Th 1 [folklore] $\Gamma = \mathrm{Cay}(\mathbb{Z}^d, S)$, $|S| < \infty$ $\Rightarrow$ $\mathcal{A}(t)$ is D-finite | **Ex:** $\mathbb{Z}^2 \to a_{2n} = \binom{2n}{n}^2$

Th 2 [Furstenberg, 1967] $\sum_{n=0}^{\infty} \binom{2n}{n}^2 t^n$ is NOT algebraic

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Th 3 [Haiman, 1993] $G = F_k$, S - standard, $\{a_{2n}(G, S)\}$

$\Rightarrow$ $\mathcal{A}(t) = \sum a_{2n} t^n$ is algebraic

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Key idea: asymptotics of $\{a_n\}$

Th [Birkhoff-Trjitzinsky + Katz + ...] $\boxed{a_n < c^n}$

$\mathcal{A}(t) = \sum a_n t^n$ - D-finite $\Rightarrow$ $a_n \propto K \lambda^n n^\alpha (\log n)^\beta$

where $K \in \mathbb{R}_+$, $\lambda \in \overline{\mathbb{Q}}$, $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{N}$

$\propto \leftrightarrow \Sigma$ over finite

Q [Kontsevich, 2014]

$\exists\ G \subset SL(n, \mathbb{Z})$, $G = \langle S \rangle$, $a_n = a_n(G, S)$

s.t. $\mathcal{A}(t) = \sum a_n t^n$ is NOT D-finite? $\circledast$

A: yes!

Th [Garrabrant - P., 2017]

(0) $\exists\ S$- nonstandard, $F_k^2 = \langle S \rangle$ $\boxed{\text{s.t. } \circledast}$

(1) amenable linear G of superpoly growth -

(2) G w/ weakly exp growth

(3) virtually solvable of exp growth

any S

(4) $BS(k,1)$, $k \geq 2$ Baumslag-Solitar

(5) $L(d, H) := H \wr \mathbb{Z}^d$ ← lamplighter

Th [Bell - Mishna, 2020] ← [GP Conj]

G - amenable, superpoly growth, $\langle S \rangle = G$

$\Rightarrow \mathcal{A}(t) = \sum a_n t^n$ is NOT D-finite

Th [Elder-Rechnitzer-van Rensburg-Wong, 2014]

$G = BS(k,k) = \langle x, y \mid x^k y = y x^k \rangle$, $S = \{x, x^{-1}, y, y^{-1}\}$ $\Rightarrow \mathcal{A}(t)$ is D-finite

Side Remark 1 Th [Revelle, 2003] $G = L(1, \mathbb{Z}_2)$

S - standard symmetric

$$\Rightarrow a_n \sim K 6^n n^{1/6} e^{-C\sqrt[3]{n}}, \quad K = \frac{2^{2/3}\pi^{5/6}}{\sqrt{3}\,(\log 2)^{2/3}}, \quad C = 3\sqrt[3]{2}\,(\pi \log 2)^{2/3}$$

Q: $\sum a_n t^n \in$ D-algebraic?

Side Remarks

Th [Dreyfus-Hardouin-Roques-Singer, 2018]

$\Gamma = (\mathbb{N}^2, S)$, $v = 0$, $a_n = \# \underbrace{v \to \dots \to v}_{n}$

$\Rightarrow \; A(t) = \sum_{n=0}^{\infty} a_n t^n$ is NOT D-algebraic

Galois Theory

Def $\mathcal{A}(t) = \sum a_n t^n$ – D-algebraic if $\exists Q \in \mathbb{Z}[t, \ldots]$

s.t. $Q(t, \mathcal{A}, \mathcal{A}', \mathcal{A}'', \ldots, \mathcal{A}^{(m)}) = 0$

ADE

Q: [Kontsevich, Stanley, Katzarkov, ...]

$\exists\, G \subset SL(n, \mathbb{Z})$, $G = \langle S \rangle$, $a_n = a_n(G, S)$

s.t. $\mathcal{A}(t) = \sum a_n t^n$ is NOT D-alg?

A: Yes! Th [G-P, much harder, non-asymp]

$\exists\, S$ - nonstandard / $S^{-1} \neq S$ /

s.t. $F_k^2 = \langle S \rangle$, $a_n = a_n(F_k, S)$

$\mathcal{A}(t) = \sum a_n t^n \in$ NOT D-alg.

{Proof uses explicit TM construction}

Q: Are there better constructions?

Def $\Gamma = (V, E)$, $\{a_n\}$, $\tilde{a}_n := a_0 + a_1 + \ldots + a_n$

cogrowth const $\rho := \lim_{n \to \infty} (\tilde{a}_n)^{1/n}$

Th [Cohen, Grigorchuk, ...]

$G = \langle S \rangle$, $S = S^{-1}$, $\rho := \rho(G, S)$

G - amenable $\iff$ $\rho = |S|$

Ex $G = \mathbb{Z}^2$, $|S| = 4$ $a_{2n} = \binom{2n}{n}^2 \sim C\, 4^{2n}\, n^{-1}$

$G = F_2$, $|S| = 4$ $a_{2n} \sim C\, (2\sqrt{3})^{2n}\, n^{-3/2}$

Main Question: Find $G = \langle S \rangle$ s.t $\rho(G, S) \notin \overline{\mathbb{Q}}$

$\Rightarrow$ $\mathcal{A}(t) = \sum a_n t^n \leftarrow$ NOT D-finite

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Th $\mathcal{R} := \{ \rho(G, S)$ over all $G = \langle S \rangle, |S| < \infty \}$

[Kassabov-P.] Then $\# \mathcal{R} = \# \mathbb{R} =$ continuum

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Note: our construction is inexplicit

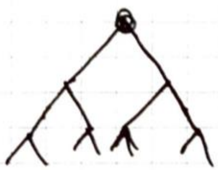
Open Problem $\exists\, G = \langle S \rangle$ s.t.

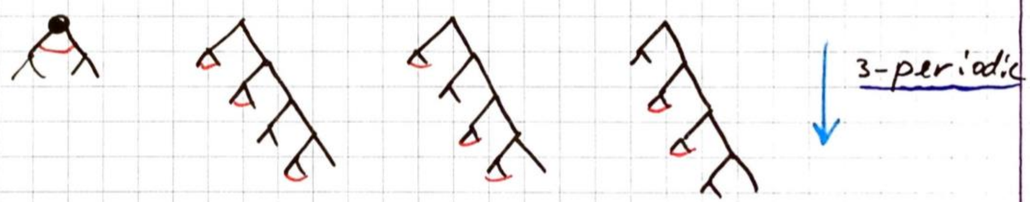
$\varrho(G,S) \notin \mathbb{Q}$ and G - finitely presented?

G - recursively presented?

Idea of the proof of K-P Thm

Recall 1) **Grigorchuk group** $G \subset \mathrm{Aut}(T_2)$

T_2 

S: 

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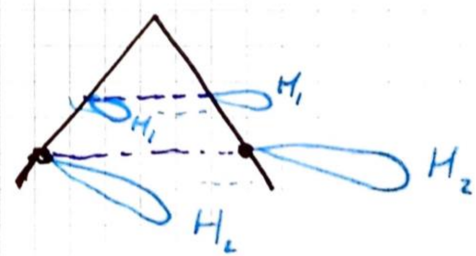
2) **Kassabov-P. construction** $G_{fg} \subset \mathrm{Aut}(\hat{T}_2)$

S: similar $\hat{T}_2 =$ 

$H_i \simeq SL(2, P_i)$

expanders

3) <u>New</u> <u>construction</u>

$\{H_k\}$ are replaced w/ $PSL(2, \mathbb{Z}[i, \tfrac{1}{2}])$, $i = \sqrt{-1}$

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varying levels of H_k allows some bounds to control $\rho(G,S)$

$\Rightarrow$ continuum family.

<u>Note</u>: Despite effort no explicit $\rho(G,S) \notin \mathbb{Q}$

<u>Please</u> <u>find</u> <u>one</u>!

<u>Thank you!</u>