Domes over Curves

Igor Pak, UCLA

(joint work with Alexey Glazyrin, UTRGV)

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Integral curves

A PL-curve $\gamma \subset \mathbb{R}^3$ is called integral if comprised of unit length intervals.

A dome is a 2-dim PL-surface $S \subset \mathbb{R}^3$ comprised of unit equilateral triangles.

Integral curve $\gamma$ can be domed if there is a dome $S$ s.t. $\partial S = \gamma$.

Problem [Kenyon, c. 2005]: Can every closed integral curve be domed?

Examples: Domes over square, pentagon, regular 10-gon and 12-gon.
Other domes

Glass rooftop, Louvre pyramid, Buckminster Fuller’s real dome and his sketch of the *Dome over Manhattan* (1960).

**Bonus questions:** Are the second and third domes polyhedral?

Is the boundary curve \( \partial S \) a regular polygon?

Is it even planar?
Negative results:

**Theorem 1** [Glazyrin–P., 2020+]
Let \( \rho(a, b) \subset \mathbb{R}^3 \) be a unit rhombus with diagonals \( a, b > 0 \). Suppose \( \rho(a, b) \) can be domed. Then there is a nonzero polynomial \( P \in \mathbb{Q}[x, y] \), such that \( P(a^2, b^2) = 0 \).

**Theorem 2** [Glazyrin–P., 2020+]
Let \( \rho(a, b) \subset \mathbb{R}^3 \) be a unit rhombus with diagonals \( a, b > 0 \).
If \( a \notin \overline{\mathbb{Q}} \) and \( a/b \in \overline{\mathbb{Q}} \), then \( \rho(a, b) \) cannot be domed.

**Theorem 2′** [Glazyrin–P., 2020+]
Let \( a \notin \overline{\mathbb{Q}} \), and let \( a^2 \) and \( b^2 \) be algebraically dependent with the minimal polynomial \( P(a^2, b^2) = 0 \),
\[
P(x, y) = x^k y^{m-k} + \sum_{i+j<m} c_{ij} x^i y^j, \quad c_{ij} \in \overline{\mathbb{Q}}.
\]
Then the unit rhombus \( \rho(a, b) \) cannot be domed.

**Examples:** \( \rho(\frac{1}{\pi}, \frac{e^\pi}{\sqrt{97}}) \leftarrow \text{Thm 1}, \quad \rho(\frac{1}{\pi}, \frac{1}{\pi}) \) and \( \rho(\frac{e}{\sqrt{7}}, \frac{e}{\sqrt{8}}) \leftarrow \text{Thm 2}, \quad \rho(\frac{1}{\pi}, \frac{1}{\pi^2}) \) and \( \rho(\frac{1}{\sqrt{5e}}, \sqrt{e^2 + e - 7}) \leftarrow \text{Thm 2′} \)
Positive results:

**Theorem 3** [Glazyrin–P., 2020+]
For every integral curve $\gamma \subset \mathbb{R}^3$ and $\varepsilon > 0$, there is an integral curve $\gamma' \subset \mathbb{R}^3$, such that $|\gamma| = |\gamma'|$, $|\gamma, \gamma'|_F < \varepsilon$ and $\gamma'$ can be domed.

Here $|\gamma, \gamma'|_F$ is the *Fréchet distance* $|\gamma, \gamma'|_F = \max_{1 \leq i \leq n} |v_i, v'_i|$.

**Theorem 4** [Glazyrin–P., 2020+]
Every regular integral $n$-gon in the plane can be domed.

**Open**: Can all planar unit rhombi $\rho(a, b)$ be domed?
Can all integral triangles $\Delta = (p, q, r)$, $p, q, r \in \mathbb{N}$ be domed?

More conjectures and open problems later in the talk.
Prior work: polyhedra with regular faces

Star pyramid, small stellated dodecahedron, heptagrammic cuploid, and dodecahedral torus.

*Square surfaces:* Dolbilin–Shtanko–Shtogrin (1997)

*Pentagonal surfaces:* Alevy (2018+)
Steinhaus problem (*Scottish book*, 1957)

1) Does there exist a closed tetrahedral chain? ← Coxeter helix

2) Are the end-triangles dense in the space of all triangles?

Part 1) was resolved negatively by Świątkowski (1959)

Part 2) was partially resolved by Elgersma–Wagon (2015) and Stewart (2019)

*Idea:* The group of face reflections is isomorphic to $\mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2 \ast \mathbb{Z}_2$ which is dense in $O(3, \mathbb{R})$
Integral triangles (+if pigs can fly results)

**Conjecture 1.** An isosceles triangle $\Delta = (2,2,1)$ cannot be domed.

**Proposition:** Conjecture 1 false $\Rightarrow$ every isosceles triangle $\Delta = (p,q,r)$ can be domed.

**Conjecture 2.** Every closed dome is rigid.

**Proposition:** Conjecture 1 false $\Rightarrow$ Conjecture 2 false.
Space colorings

Γ ← unit distance graph of \( \mathbb{R}^3 \)

**Conjecture 3:** Let \( \rho = [uvwx] \subset \mathbb{R}^3 \) be a rhombus with edge lengths 2 and diagonal 1. Then \( \exists \) coloring \( \chi : \Gamma \rightarrow \{1, 2, 3\} \) with no rainbow (1-2-3) triangles, s.t. \( \chi(u) = \chi(v) = 1, \chi(w) = 2, \chi(x) = 3. \)

**Proposition:** Conjecture 3 \( \Rightarrow \) Conjecture 1.

**Proof:** Dome over \( \Delta = (2, 2, 1) \) \( \Rightarrow \) dome \( S \) over \( \rho \).

Sperner’s Lemma for (general) 2-manifolds applied to \( S \cup \rho \) \( \Rightarrow \) # of 1-2-3 \( \Delta \) is even \( \leftarrow \) [Musin, 2015]

Since \( \rho \) has one 1-2-3 \( \Delta \), dome \( S \) also has at least one 1-2-3 \( \Delta \), a contradiction. \( \Box \)
Domes over regular polygons

Rhombus Lemma
Fix \( a \notin \overline{Q} \). The set of \( b \) for which rhombus \( \rho(a, b) \) which can be domed is dense in \( (0, \sqrt{4 - a^2}) \).

Construction sketch:
Tilt blue triangles by \( \angle \theta \). Make near-planar rhombi until the center is overshot.
Use continuity to find \( \theta \) for which the tip of the slice is on the vertical axis.

Wayman AME Church in Minneapolis
Domes over generic integral curves

**Step 1:** Generic integral curves \(\rightarrow\) Generic near-planar integral curves

**Idea:** Use 2-flips to triangles \(v_{i-1}v_iv_{i+1} \rightarrow v'_iv_{i+1}\) until curve is near-planar.

**Step 2:** Generic near-planar integral curves \(\rightarrow\) Generic compact near-planar integral curves

**Idea:** Use 2-flips to obtain the desired permutation of unit vectors \(v_i\rightarrow v'_i\). Now apply

**Steinitz Lemma:** Let \(u_1, \ldots, u_n \in \mathbb{R}^2\) be unit vectors, \(u_1 + \ldots + u_n = 0\).

Then there exists \(\sigma \in S_n\), s.t. \(|u_{\sigma(1)} + \ldots + u_{\sigma(k)}| \leq \sqrt{\frac{5}{4}}\), for all \(1 \leq k \leq n\).

[Steinitz, 1913] → general dimensions, [Bergström, 1931] → optimal constant \(\sqrt{\frac{5}{4}}\)
Domes over generic integral curves (continued)

**Step 3:** Break the curve into unit rhombi and pentagons.

**Step 5:** Use an ad hoc construction for pentagons.

**Step 6:** Fix combinatorial data and undo the construction using the Rhombus Lemma. □
Doubly periodic surfaces

$K$ ← pure simplicial 2-dim complex homeomorphic to $\mathbb{R}^2$, with a free action of $\mathbb{Z} \oplus \mathbb{Z} = \langle a, b \rangle$

$\theta : K \rightarrow \mathbb{R}^3$ ← linear mapping of $K$, and equivariant w.r.t. $\mathbb{Z} \oplus \mathbb{Z}$, s.t. $a \equiv a \beta$, $b \equiv b \beta$

$(K, \theta)$ is called a doubly periodic triangular surface

$G(K)$ ← set of Gram matrices of $(\alpha, \beta)$, over all $(K, \theta)$

**Theorem** [A. Gaifullin – S. Gaifullin, 2014]

Then there is a one-dimensional real affine algebraic subvariety of $\mathbb{R}^3$ containing $G(K)$.

In particular, the entries of each Gram matrix $G$ from $G(K)$

\[
\begin{align*}
P(g_{11}, g_{12}, g_{22}) &= 0 \\
Q(g_{11}, g_{12}, g_{22}) &= 0
\end{align*}
\]

for some $P, Q \in \mathbb{Z}[x, y, z]$. 

![Diagram of doubly periodic surfaces](image-url)
Easy special case of Theorem 1

Proposition
Let $S$ be a dome over a rhombus $\gamma = \rho(a, b)$ homeomorphic to a disc.
Then there is a nonzero polynomial $F \in \mathbb{Q}[x, y]$, s.t. $F(a^2, b^2) = 0$.

Proof: Attach copies of $\gamma$ and $-\gamma$ as in Figure. Since $\alpha$ and $\beta$ are orthogonal, the Gram matrix is diagonal. By G–G Theorem, we have $F \leftarrow P$ or $F \leftarrow Q$. 
**G–G Theorem does not generalize**

**Theorem** [A. Gaifullin – S. Gaifullin, 2014] Every embedded doubly periodic triangular surface homeomorphic to a plane has at most one-dimensional doubly periodic flex.

**Theorem** [Glazyrin–P., 2020+, formerly G–G Open Problem] There is a doubly periodic triangular surface whose doubly periodic flex is three-dimensional.

**Moral:** Need a better technical result.
Ingredients of the proof of theorems 1 and 2

- heavy use of *theory of places*

- elementary but lengthy and tedious inductive topological argument

Cf. [Conelly–Sabitov–Walz, 1997], [Connelly, 2009], [Gaifullin–Gaifullin, 2014]

*Case 1 of the induction step.*
More conjectures and open problems

**Conjecture 4:** The set of $a$, s.t. planar rhombus $\rho(a, \sqrt{4-a^2})$ can be domed, is countable.

**Conjecture 5:** There are unit triangles $\Delta_1, \Delta_2 \subset \mathbb{R}^3$, such that $\Delta_1 \cup \Delta_2$ cannot be domed.

**Conjecture 6** [“cobordism for domes”]:
For every integral curve $\gamma \in \mathbb{R}^3$, there is a unit rhombus $\rho$, and a dome over $\gamma \cup \rho$.

\[
\gamma = [v_1 \ldots v_n] \leftarrow \text{integral curve, } n \geq 5
\]
\[
L_n = \mathbb{Q}[t_1, \ldots, t_{n-2}], \quad t_i \leftarrow |v_i v_{i+1}|^2 \text{ squared diagonals of } \gamma.
\]
\[
\text{CM}_n \subset L_n \leftarrow \text{ideal spanned by all Cayley–Menger determinants on } \{v_1, \ldots, v_n\}
\]

**Conjecture 7:** If $\gamma$ can be domed, then there is a nonzero $P \in L_n$, s.t. $P(t_1, \ldots, t_{n-2}) = 0$ and $P \notin \text{CM}_n$. 
Thank you!