Domes over Curves

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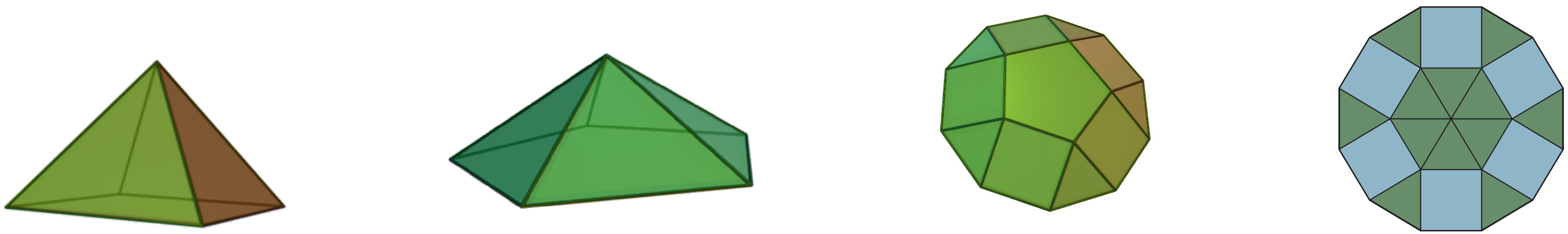
Integral curves

A PL-curve $\gamma \subset \mathbb{R}^3$ is called *integral* if comprised of unit length intervals.

A *dome* is a 2-dim PL-surface $S \subset \mathbb{R}^3$ comprised of unit equilateral triangles.

Integral curve γ *can be domed* if there is a dome S s.t. $\partial S = \gamma$.

Problem [Kenyon, c. 2005]: Can every closed integral curve be domed?



Examples: Domes over square, pentagon, regular 10-gon and 12-gon.

Other domes

Louvre pyramid, glass rooftop, Buckminster Fuller's real dome and his sketch of the *Dome over Manhattan* (1960).

Bonus questions:

Are the second and third domes polyhedral?

Is the boundary curve ∂S a regular polygon?

Is it even planar?

Positive results:

Theorem 1 [Glazyrin–P., 2020+]
For every integral curve $\gamma \subset \mathbb{R}^3$ and $\varepsilon > 0$, there is an integral curve $\gamma' \subset \mathbb{R}^3$, such that $|\gamma| = |\gamma'|$, $|\gamma, \gamma'|_F < \varepsilon$ and the curve γ' can be domed.

Here $|\gamma, \gamma'|_F$ is the *Fréchet distance* $|\gamma, \gamma'|_F = \max_{1 \le i \le n} |v_i, v_i'|$.

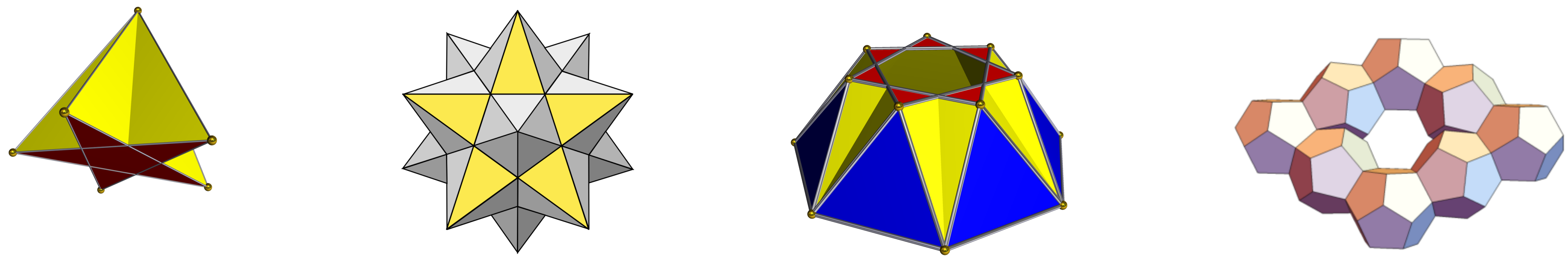
Theorem 2 [Glazyrin–P., 2020+]
Every regular integral n-gon in the plane can be domed.

Open: Can all planar unit rhombi $\rho(a, b)$ be domed?

Can all integral triangles $\Delta = (p, q, r)$, $p, q, r \in \mathbb{N}$ be domed?

More conjectures and open problems later in the talk.

Prior work: polyhedra with regular faces



Star pyramid, small stellated dodecahedron, heptagrammic cuploid, and dodecahedral torus.

***Johnson solids*:** Johnson (1966), Zalgaller (1969)

***Square surfaces*:** Dolbilin–Shtanko–Shtogrin (1997)

***Pentagonal surfaces*:** Alevy (2018+)

Steinhaus problem (*Scottish book*, 1957)

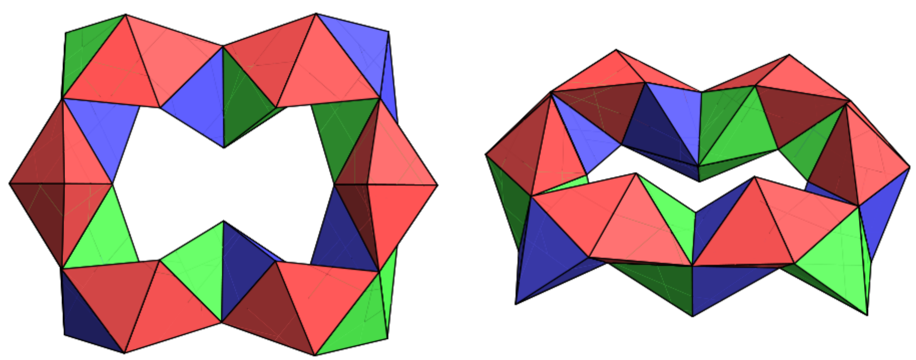
(1) Does there exist a closed tetrahedral chain? $\longleftarrow$ *Coxeter helix*

(2) Are the end-triangles dense in the space of all triangles?

Part (1) was resolved negatively by Świerczkowski (1959)

Part (2) was partially resolved by Elgersma–Wagon (2015) and Stewart (2019)

Idea: The group of face reflections is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ which is dense in $O(3, \mathbb{R})$



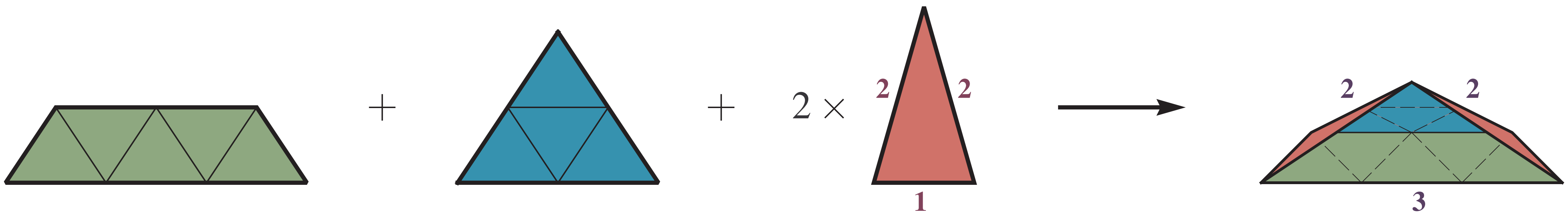
A length-36 fake tetratorus with a final gap of about 0.0005 cm.

Note: 0.0005 cm = 5000 nm.

Integral triangles

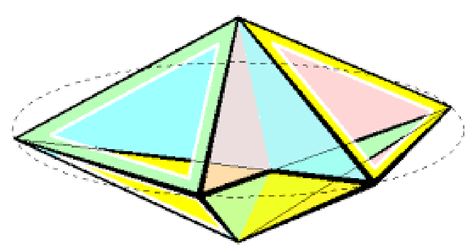
Conjecture 1. An isosceles triangle $\Delta = (2, 2, 1)$ cannot be domed.

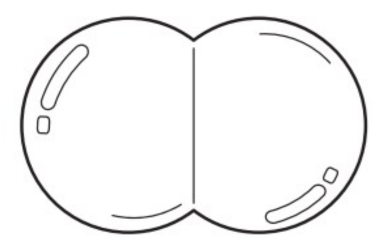
Proposition: Conjecture 1 false $\Rightarrow$ every triangle $\Delta = (p, q, r)$ can be domed.



Conjecture 2. Every non-degenerate closed dome is rigid.

Proposition: Conjecture 1 is false $\Rightarrow$ Conjecture 2 is false.





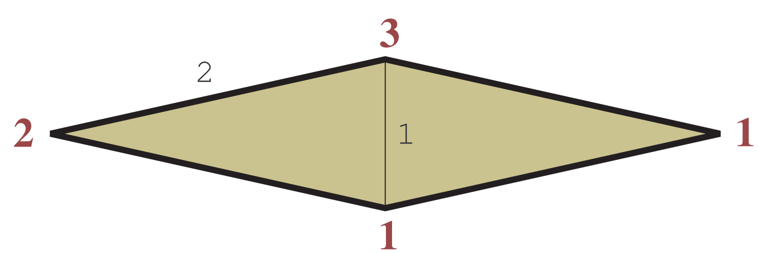
Space colorings

$\Gamma \leftarrow$ *unit distance graph* of $\mathbb{R}^3$

Conjecture 3: Let $\rho = [uvwx] \subset \mathbb{R}^3$ be a rhombus with edge lengths 2 and diagonal 1. Then $\exists$ coloring $\chi : \Gamma \to \{1, 2, 3\}$ with no *rainbow* (1-2-3) *triangles*, s.t. $\chi(u) = \chi(v) = 1,\ \chi(w) = 2,\ \chi(x) = 3.$

Proposition: Conjecture 3 $\Rightarrow$ Conjecture 1.

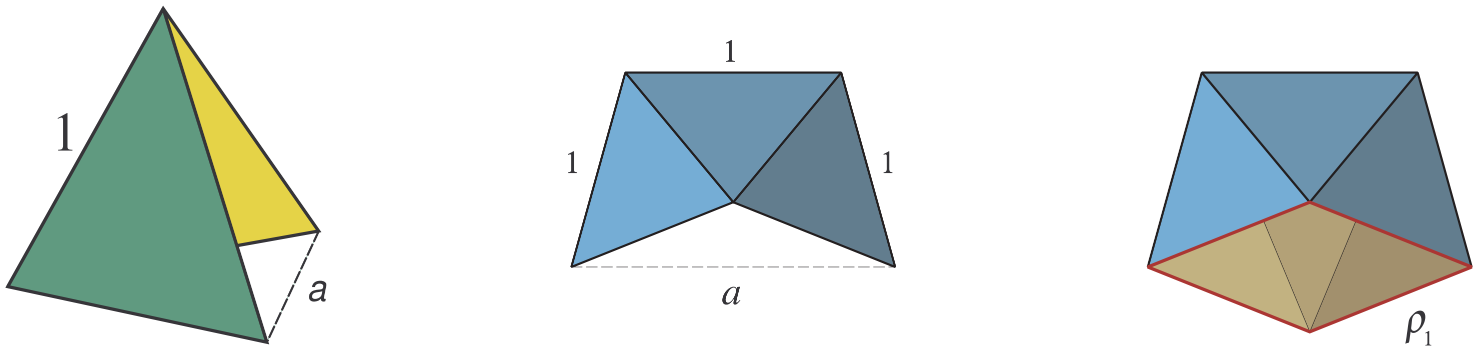
Proof: Dome over $\Delta = (2, 2, 1)$ $\Rightarrow$ dome S over ρ.
Sperner's Lemma for (general) 2-manifolds applied to $S \cup \rho$ $\Rightarrow$ # of 1-2-3 Δ is even $\leftarrow$ [Musin, 2015]
Since ρ has one 1-2-3 Δ, dome S also has at least one 1-2-3 Δ, a contradiction. $\square$



How to prove positive results?

Rhombus Lemma

Fix $a \notin \overline{Q}$. The set of b for which rhombus $\rho(a,b)$ which can be domed is dense in $\left(0, \sqrt{4-a^2}\right)$.

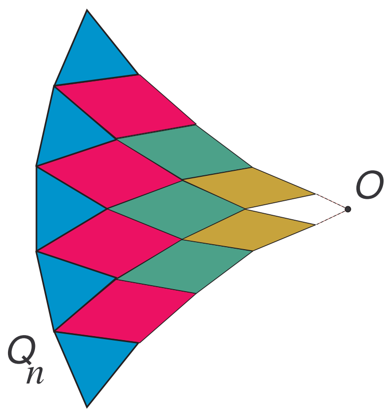


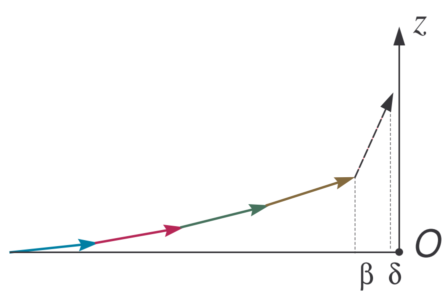
Domes over regular polygons

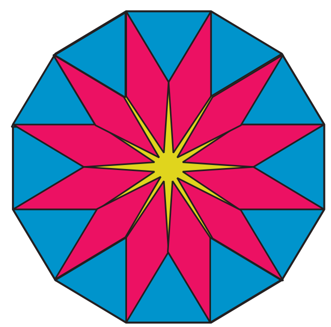
Construction sketch:

Tilt blue triangles by $\angle\theta$. Make near-planar rhombi until the center is overshot.

Use continuity to find θ for which the tip of the slice is on the vertical axis.





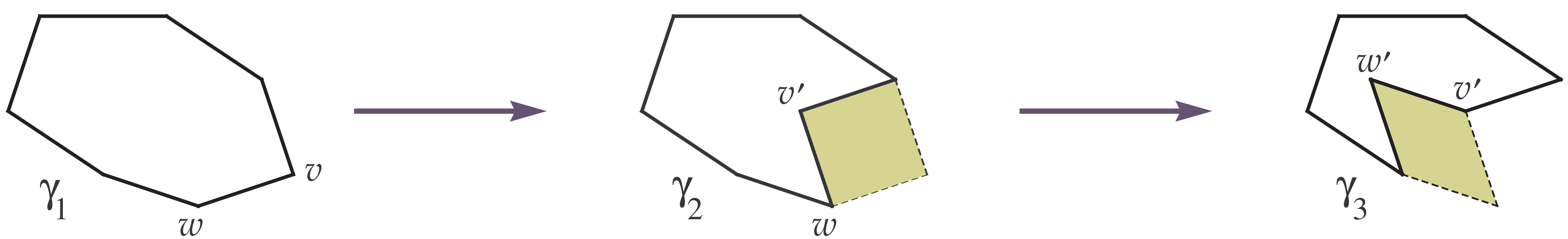


Domes over generic integral curves

Definition: Integral curve is $[v_1 \dots v_n]$ is *generic* if all *small diagonals* $|v_i v_{i+2}| \notin \overline{Q}$, and the same holds after all finite flips sequences

Step 1: Generic integral curves $\longrightarrow$ Generic near-planar integral curves

Idea: Use 2-flips to triangles $v_{i-1}v_iv_{i+1} \to v_{i-1}v_i'v_{i+1}$ until curve is near-planar.



Domes over generic integral curves (continued)

Step 2: Generic near-planar integral curves $\longrightarrow$ Generic compact near-planar integral curves

Idea: Use 2-flips to obtain the desired permutation of unit vectors $\overrightarrow{v_i v_{i+1}}$. Now apply

Steinitz Lemma: Let $u_1, \ldots, u_n \in \mathbb{R}^2$ be unit vectors, $u_1 + \ldots + u_n = 0$.

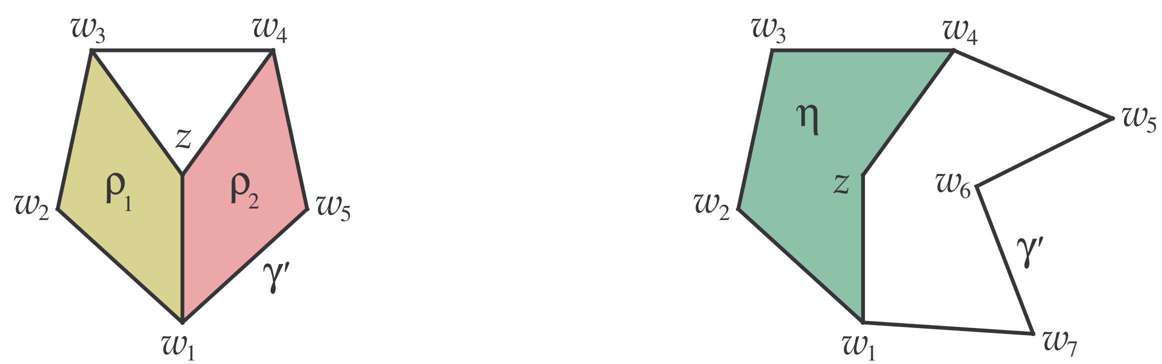
Then there exists $\sigma \in S_n$, s.t. $\left| u_{\sigma(1)} + \ldots + u_{\sigma(k)} \right| \leq \sqrt{\frac{5}{4}}$, for all $1 \leq k \leq n$.

[Steinitz, 1913] $\to$ general dimensions, [Bergström, 1931] $\to$ optimal constant $\sqrt{\frac{5}{4}}$

Step 3: Break the curve into unit rhombi and pentagons.

Domes over generic integral curves (continued)

Step 4: Use an ad hoc construction for pentagons.



Step 5: Fix combinatorial data and undo the construction using the Rhombus Lemma. ☐

Negative results:

Theorem 3 [Glazyrin–P., 2020+]
Let $\rho(a,b) \subset \mathbb{R}^3$ be a unit rhombus with diagonals $a, b > 0$. Suppose $\rho(a,b)$ can be domed.
Then there is a nonzero polynomial $P \in \mathbb{Q}[x,y]$, such that $P(a^2, b^2) = 0$.

Theorem 4 [Glazyrin–P., 2020+]
Let $\rho(a,b) \subset \mathbb{R}^3$ be a unit rhombus with diagonals $a, b > 0$.
If $a \notin \overline{\mathbb{Q}}$ and $a/b \in \overline{\mathbb{Q}}$, then $\rho(a,b)$ cannot be domed.

Examples:

$\rho\big(\frac{1}{\pi}, \frac{e^\pi}{\sqrt{97}}\big) \ \leftarrow$ Thm 3,

$\rho\big(\frac{1}{\pi}, \frac{1}{\pi}\big)$ and $\rho\big(\frac{e}{\sqrt{17}}, \frac{e}{\sqrt{19}}\big) \ \leftarrow$ Thm 4.

Doubly periodic surfaces

K ← pure simplicial 2-dim complex homeomorphic to $\mathbb{R}^2$, with a free action of $\mathbb{Z} \oplus \mathbb{Z} = \langle a, b \rangle$

$\theta : K \to \mathbb{R}^3$ ← linear mapping of K, and equivariant w.r.t. $\mathbb{Z} \oplus \mathbb{Z}$, s.t. $a \curvearrowright \alpha$, $b \curvearrowright \beta$

(K, θ) is called a *doubly periodic triangular surface*

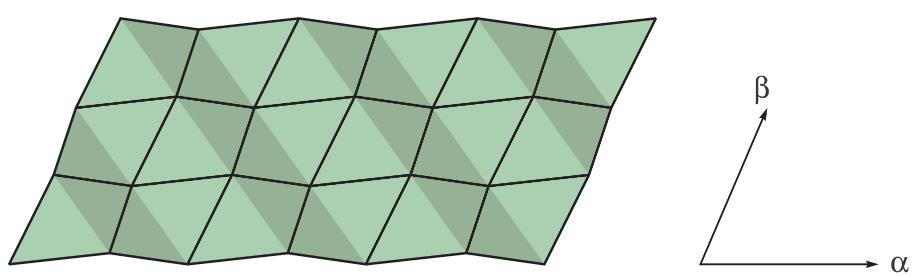
$\mathcal{G}(K)$ ← set of Gram matrices of (α, β), over all (K, θ)

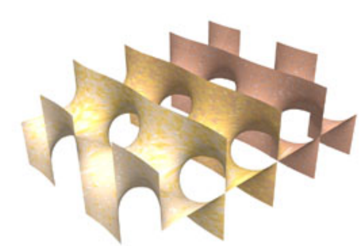
Theorem [A. Gaifullin – S. Gaifullin, 2014]

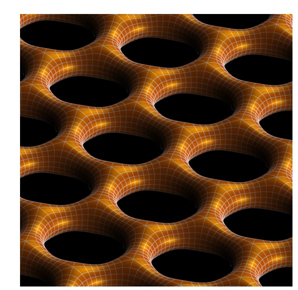
There is a one-dimensional real affine algebraic subvariety of $\mathbb{R}^3$ containing $\mathcal{G}(K)$.

In particular, the entries of each Gram matrix G from $\mathcal{G}(K)$

$$\begin{cases} P(g_{11}, g_{12}, g_{22}) = 0 \\ Q(g_{11}, g_{12}, g_{22}) = 0 \end{cases} \qquad \text{for some } \; P, Q \in \mathbb{Z}[x, y, z].$$







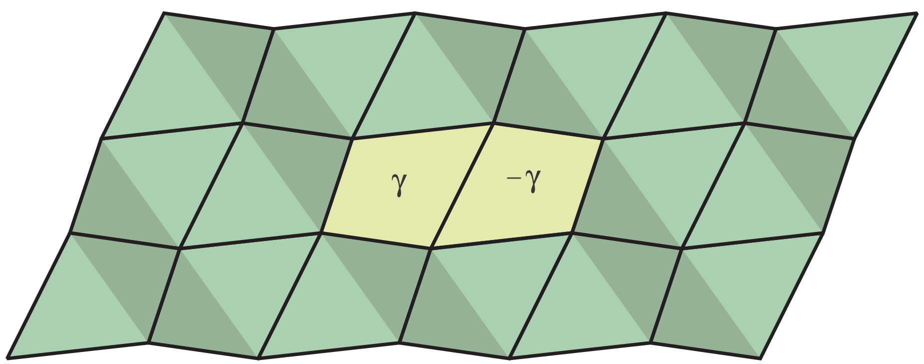
Special case of Theorem 3

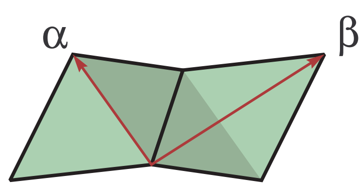
Proposition

Let S be a dome over a rhombus $\gamma = \rho(a, b)$ homeomorphic to a disc.

Then there is a nonzero polynomial $F \in \mathbb{Q}[x, y]$, s.t. $F(a^2, b^2) = 0$.

Proof: Attach copies of γ and $-\gamma$ as in Figure. Since α and β are orthogonal, the Gram matrix is diagonal. By G–G Theorem, we have $F \leftarrow P$ or $F \leftarrow Q$.

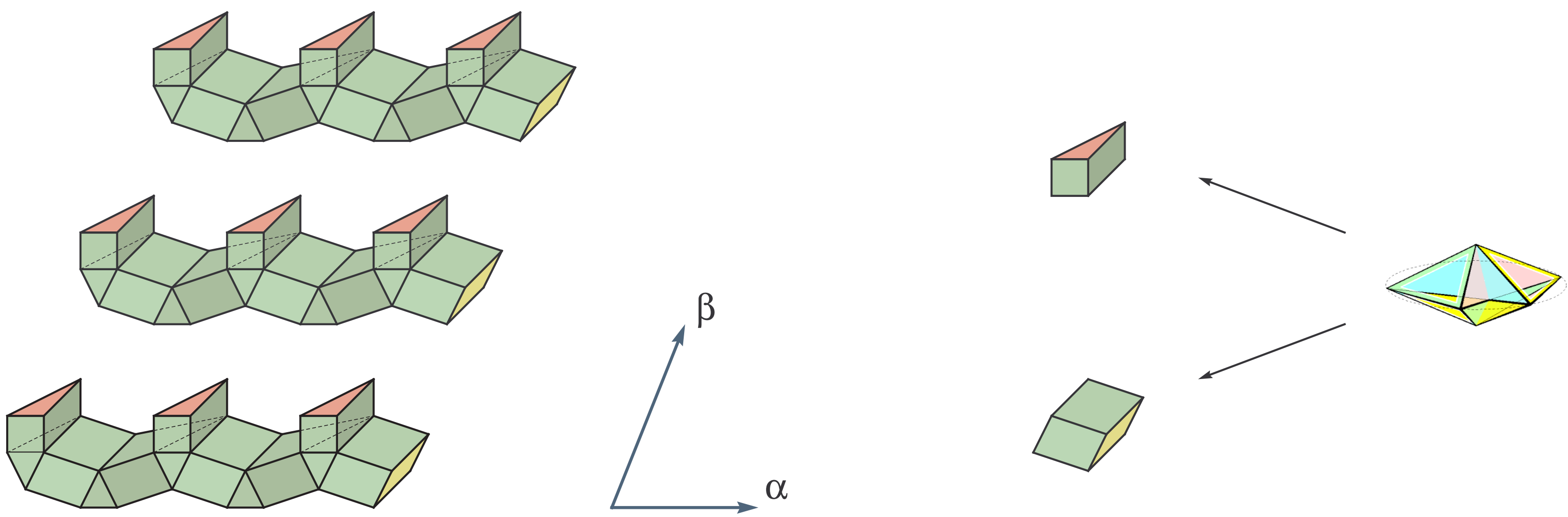




G–G Theorem does not generalize

Theorem [A. Gaifullin – S. Gaifullin, 2014] Every embedded doubly periodic triangular surface homeomorphic to a plane has at most one-dimensional doubly periodic flex.

Theorem [Glazyrin–P., 2020+, formerly G–G Open Problem] There is a doubly periodic triangular surface whose doubly periodic flex is three-dimensional.

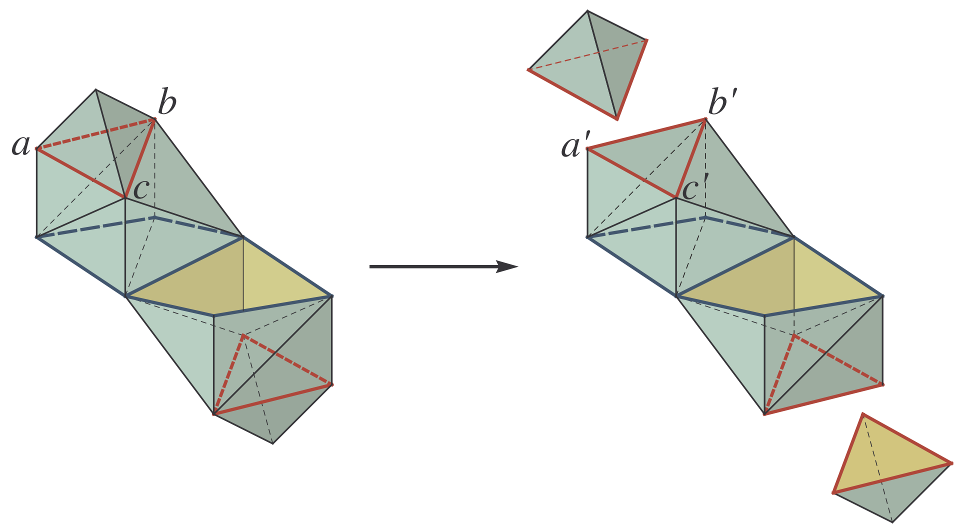


Moral: Need a better technical result.

Ingredients of the proof of theorems 3 and 4

- heavy use of *theory of places*
- elementary but lengthy and tedious inductive topological argument

Cf. [Conelly–Sabitov–Walz, 1997], [Connelly, 2009], [Gaifullin–Gaifullin, 2014]



Case 1 of the induction step.

More conjectures and open problems

Conjecture 4:

The set of a, s.t. planar rhombus $\rho\big(a, \sqrt{4-a^2}\big)$ can be domed, is countable.

Conjecture 5:

There are unit triangles $\Delta_1, \Delta_2 \subset \mathbb{R}^3$, such that $\Delta_1 \cup \Delta_2$ cannot be domed.

Conjecture 6 ["cobordism for domes"]:

For every integral curve $\gamma \in \mathbb{R}^3$, there is a unit rhombus ρ, and a dome over $\gamma \cup \rho$.

Thank you!