The computational complexity of integer programming with alternations

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What is this all about?

Let $P \subset \mathbb{R}^d$ be a convex polytope given by $A\mathbf{x} \leq \bar{b}$. Say, $d = 3$.

Can one compute $\#\mathrm{E}(P)$ – the *number of integer points* in P? (Yes!)

How about $\#\mathrm{E}(P \smallsetminus Q)$? Or $\#\big[\mathrm{E}(P)\downarrow_x\big]$? (Yes, yes!)

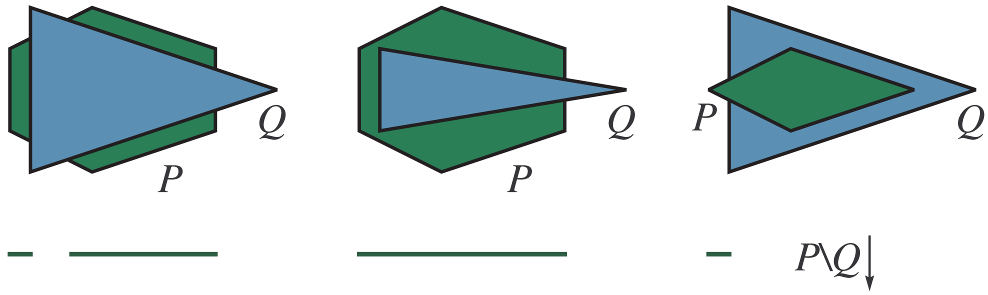
Theorem 1 (Nguyen–P.)
For $P, Q \in \mathbb{R}^3$, computing $\#\big[\mathrm{E}(P \smallsetminus Q)\downarrow_x\big]$ is $\#\mathsf{P}$-complete.

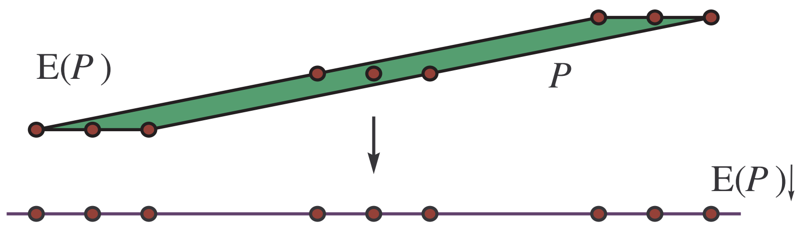
Theorem 2 (Nguyen–P.)
Given three polytopes $U_1, U_2, U_3 \subset \mathbb{R}^4$ and two boxes $I \subset \mathbb{Z}, K \subset \mathbb{Z}^3$, deciding the following sentence is NP-complete:

$$\exists x \in I \quad \forall \mathbf{z} \in K \quad : \quad (x, \mathbf{z}) \in U_1 \cup U_2 \cup U_3$$

Note: the abstract says $\mathbb{R}^4$ in Theorem 1. We improved this since then.

Examples by pictures:





Background: IP and #IP

Theorem (Lenstra, 1983) In $\mathbb{R}^d$, dimension d fixed, IP $\in \mathsf{P}$:

$$(\text{IP}) \qquad \exists \mathbf{x} \in \mathbb{Z}^d \,:\, A\mathbf{x} \,\leq\, \bar{b}.$$

Theorem (Barvinok, 1993) In $\mathbb{R}^d$, dimension d fixed, #IP $\in \mathsf{FP}$:

$$(\#\text{IP}) \qquad \#\big\{\mathbf{x} \,:\, A\mathbf{x} \,\leq\, \bar{b}\big\}.$$

Note: The system can be *long* here (i.e. has unbounded size)

Proof ideas: 1) Geometry of numbers (flatness theorem), lattice reduction (LLL).

2) Brion–Verge generating function approach, cone subdivisions, combinatorial tools.

From Long to Short

Theorem (Doignon–Bell–Scarf)

Let A be a $n \times d$ real matrix and $\bar{b} \in \mathbb{R}^d$. Suppose

$$\{\mathbf{x} \in \mathbb{Z}^d \,:\, A\mathbf{x} \le \bar{b}\} \;=\; \varnothing.$$

Then there is a subset S of rows of A, $|S| \le 2^d$, s.t.

$$\{\mathbf{x} \in \mathbb{Z}^d \,:\, A_s\mathbf{x} \le \bar{b}_S\} \;=\; \varnothing.$$

Corollary: It suffices to solve IP for short systems (of bounded size n).

Note: One should think of this as the *integral version* of the Helly Theorem.

Indeed, Helly's theorem says: $(d+1)$-intersections are nonempty $\Rightarrow$ all are nonempty.

More background: PIP and #PIP

Theorem (Kannan, 1990) For all dimensions d, k fixed, PIP $\in \mathsf{P}$:

$$(\text{PIP}) \qquad \forall \mathbf{y} \in Q \cap \mathbb{Z}^k \;\; \exists \mathbf{x} \in \mathbb{Z}^d \, : \, A\mathbf{x} + B\mathbf{y} \, \leq \, \bar{b}.$$

Theorem (Barvinok–Woods, 2003) For all dimensions d, k fixed, #PIP $\in \mathsf{FP}$:

$$(\#\text{PIP}) \qquad \#\big\{\mathbf{y} \in Q \cap \mathbb{Z}^k \;\; \exists \mathbf{x} \in \mathbb{Z}^d \, : \, A\mathbf{x} + B\mathbf{y} \, \leq \, \bar{b}\big\}.$$

Translation: These are $\mathrm{E}(Q) \subseteq_? \mathrm{E}(P){\downarrow}$ and $\#\big[\mathrm{E}(Q) \cap \mathrm{E}(P){\downarrow}\big]$.

Proof ideas: More of the same (geometry of numbers, GFs, + ad hoc arguments)

Note: DBS theorem applies, so PIP and #PIP hold for long systems.

What happens for three quantifiers?

Open Problem (Kannan, 1990) Is GIP $\in \mathsf{P}$ for all dimensions d, k, ℓ fixed?

$$\text{(GIP)} \qquad \exists \mathbf{z} \in R \cap \mathbb{Z}^\ell \;\; \forall \mathbf{y} \in Q \cap \mathbb{Z}^k \;\; \exists \mathbf{x} \in \mathbb{Z}^d \; : \; A\mathbf{x} + B\mathbf{y} + C\mathbf{z} \le \bar{b}.$$

Theorem 3 (Nguyen–P.) For dimensions $d \ge 3$, $k, \ell \ge 1$ fixed, GIP is NP-complete. The corresponding counting version #GIP is $\#\mathsf{P}$-complete.

Theorem (Nguyen–P., STOC'17) KPT implies that SHORT-GIP $\in \mathsf{P}$.

KPT = *Kannan's Partition Theorem* (1990) is the Main Lemma
in the proof of Kannan's PIP Theorem.

Note: DBS theorem no longer can be applied in this case (so no contradiction).

Many alternating quantifiers

Theorem (Schöning, 1997) Fix $k \geq 1$. Let $\Psi(\mathbf{x}, \mathbf{y})$ be a Boolean combination of linear inequalities with integer coefficients in the variables $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{Z}^k$ and $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{Z}^3$. Then deciding the sentence

$$(\star) \quad Q_1 \, x_1 \in \mathbb{Z} \quad \ldots \quad Q_k \, x_k \in \mathbb{Z} \quad Q_{k+1} \, \mathbf{y} \in \mathbb{Z}^3 \quad : \quad \Psi(\mathbf{x}, \mathbf{y})$$

is $\mathbf{\Sigma}_k^{\mathsf{P}}$-complete if $Q_1 = \exists$, and $\mathbf{\Pi}_k^{\mathsf{P}}$-complete if $Q_1 = \forall$. Here $Q_1, \ldots, Q_{k+1} \in \{\forall, \exists\}$ are $(k+1)$ alternating quantifiers.

Theorem (Nguyen–P.) Integer Programming $(\star)$ in a fixed number of variables with $(k+2)$ alternating quantifiers is $\mathbf{\Sigma}_k^{\mathsf{P}}/\mathbf{\Pi}_k^{\mathsf{P}}$-complete, depending on whether $Q_1 = \exists/\forall$. Here the problem is allowed to contain only a system of inequalities.

Note Tradeoff: *Boolean system* $\longleftrightarrow$ *extra quantifier.*

Proof idea: reduction to GSA

For a vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{Q}^d$ and an integer $k \in \mathbb{Z}$, let

$$\{\{k\boldsymbol{\alpha}\}\} = \max_{1 \le i \le d} \{\{k\alpha_i\}\},$$

where for each rational $\beta \in$ Q, the quantity $\{\beta\}$ is defined as:

$$\{\{\beta\}\} := \min_{n \in \mathbb{Z}} |\beta - n| = \min\{\beta - \lfloor\beta\rfloor, \lceil\beta\rceil - \beta\}.$$

GOOD SIMULTANEOUS APPROXIMATION (GSA)

Input: A rational vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{Q}^d$ and $N \in \mathbb{N}$, $\varepsilon \in \mathbb{Q}$.

Problem: Is an integer $x \in [1, N]$ such that $\{\{x\boldsymbol{\alpha}\}\} \le \varepsilon$?

Theorem (Lagarias, 1985) GSA is NP-complete.

Main ideas: Use continuing fraction for $\varepsilon = p/q$ to study integer points under $y \le \varepsilon x$ line. Note that for p, q Fibonacci numbers the resulting set is both large and has poly-size description. Generalize this observation. Convert the problem into a problem about polytopes by adding auxiliary variables. Proofs of all theorems 1,2 and 3 follow this pattern.

Coming attractions

Theorem (Nguyen–P., FOCS 2017)
Problem Short-GIP is NP–complete.

Note: This is a strong extension of our Theorem 3.
It should be compared to our STOC theorem: KPT $\Rightarrow$ Short-GIP $\in$ P.

Natural Questions: Did we prove P = NP? (No!)

Is STOC Theorem correct? (Yes!)

Is FOCS Theorem correct? (Yes!)

What gives? (We'll explain in Berkeley. See you then!)

Thank You!