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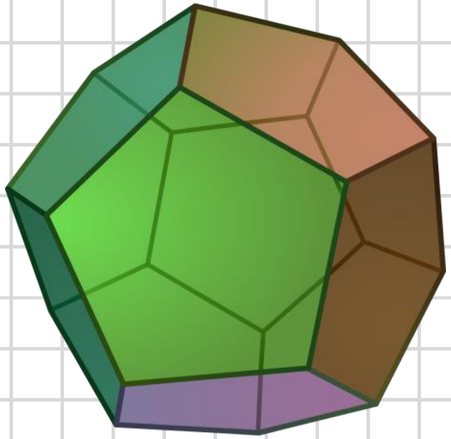
May 18, 2022

What is a combinatorial interpretation?

(joint work with **Christian Ikenmeyer**)

Open Problems in Algebraic Combinatorics (Minneapolis, 2022)



Plan of the talk: *four deep questions*

1) Why do we care about combinatorial interpretations?
2) Why are people so *positive* about their existence against both evidence, reason and experience?
3) What *are* combinatorial interpretations?
4) How can we prove that they don't exist?

arXiv:2204.13149 [pdf, ps, other] cs.CC

What is in #P and what is not?

Authors: Christian Ikenmeyer, Igor Pak

[Submitted on 27 Apr 2022] 82 pp.

Most wanted combinatorial interpretations

- ***Kronecker coefficients*** $g(\lambda,\mu,\nu) \in \mathbb{N}$

$$\chi^\mu \cdot \chi^\nu \;=\; \sum_{\lambda \vdash n} g(\lambda,\mu,\nu)\, \chi^\lambda \quad \text{where} \quad \mu,\nu \vdash n$$

describe *tensor products* of irreducible S_n-reps

generalize *Littlewood–Richardson coefficients*

Positivity Problems and Conjectures in Algebraic Combinatorics

Richard P. Stanley[1] (2000)

- ***plethysm coefficients*** $a_\lambda(\mu,\nu) \in \mathbb{N}$

$$s_\mu[s_\nu] \;=\; \sum_{\lambda} a_\lambda(\mu,\nu)\, s_\lambda$$

describe *Schur functors* of irreducible S_n-reps

crucial in *Geometric Complexity Theory*

- ***Schubert coefficients*** $c(u,v,w) \in \mathbb{N}$

$$\mathfrak{S}_u \cdot \mathfrak{S}_v \;=\; \sum_{w} c(u,v,w)\, \mathfrak{S}_w$$

describe *cohomology of the Grassmannian*

Why work on combinatorial interpretations?

When you ask the experts, they tell you:

1) Intellectual curiosity
2) Need to publish
3) Blind belief in the mission
4) Getting estimates
5) Saturation-type problems (after Knutson-Tao)
6) Vanishing problems

Estimate struggles:

$$1 \le g(\delta_k, \delta_k, \delta_k) \le f^{\delta_k} = \sqrt{n!}\, e^{-O(n)}$$

where $\delta_k = (k-1, \ldots, 2, 1)$, $n = \binom{k}{2}$, $f^{\delta_k} := \mathrm{SYT}(\delta_k)$

[Bessenrodt-Behns'04], [P.-Panova-Vallejo'16], [P.-Panova'20]

Saturation struggles:

Saturation easily fails for Kronecker coefficients, e.g.

$$g(2^2, 2^2, 2^2) = 1 \quad \text{but} \quad g(1^2, 1^2, 1^2) = 0.$$

Moreover, saturation fails for the *reduced Kronecker coefficients* [P.-Panova'20]

Vanishing struggles:

Deciding if $g(\lambda, \mu, \nu) > 0$ is *strongly* NP*-hard*

[Ikenmeyer-Mulmuley-Walter'17]

Why would they exist?

Positive experience

(1) *Young's rule*: $f^\lambda = \big|\mathrm{SYT}(\lambda)\big|$, where $f^\lambda := \chi^\lambda(1)$ [Young, 1900]

(2) *Littlewood–Richardson's rule*: $c^\lambda_{\mu\nu} = \big|\mathrm{LR}(\lambda/\mu,\nu)\big|$ [L–R, 1934]

(3) *Pipe dreams rule*: $\mathfrak{S}_w = \sum_D \boldsymbol{x}^D$ [Fomin-Kirillov'96], [Bergeron-Billey'93], [Knutson-Miller'05]

(4) (few/several/many) more extensions/generalizations/variations on the theme (many papers)

Perseverance & Optimism (as in *"why be discouraged by failures?"*)

Combinatorics Seminar

Thursday February 07, 2013

Sami Assaf (USC)

Stable Schur functions

Kroneckers real soon!

Combinatorics Seminar

Thursday May 11, 2017

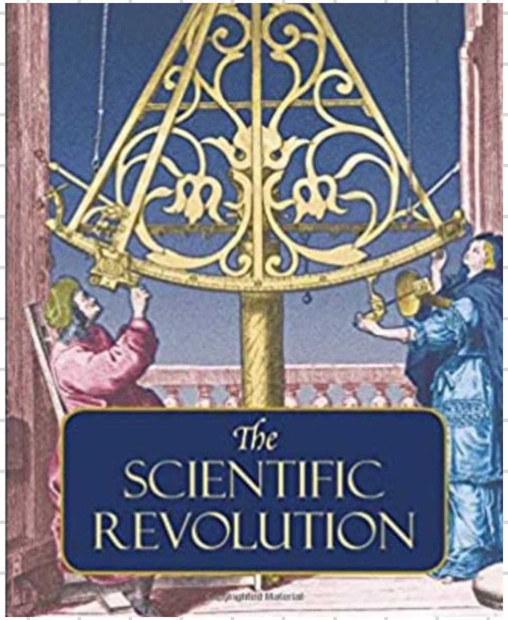
Sami Assaf (USC)

Schubert polynomials and slide polynomials

Schuberts real soon!

Why would they NOT exist?

Very brief history of negative results:



(1) *rational numbers* $<$ *geometric numbers*

Pythagoras (6th century BC): $\sqrt{2} \notin \mathbb{Q}$

(2) *ruler/compass numbers* $<$ *geometric numbers*

Gauss (1796): regular 7-gon cannot be constructed

(3) *radical numbers* $<$ *algebraic numbers*

Ruffini (1799), *Abel* (1824): quintic equations cannot be solved in the radicals

(4) *elementary functions* $<$ *their integrals*

Liouville (1833-41): $\int \frac{\sin x}{x}\, dx$, $\int e^{-x^2}\, dx$ are not elementary

(5) *algebraic numbers* $<$ *reals*

Liouville (1844), *Cantor* (1874): $\exists$ (many) transcendental numbers

Seriously, why *would* Kronecker coefficients have a combinatorial interpretation?

Imaginary conversation of 15 y.o. Gauss and his friend:

Friend: Why do you believe that the heptagon cannot be constructed?

Gauss: IDK. Because many smart people tried and failed.
Why do you believe that it can?

Friend: Isn't it obvious? We can construct so much: triangle, square, pentagon, hexagon, even octagon. I am very optimistic!

(Braunschweig, Germany, 1793)

(young Gauss)

What is a combinatorial interpretation?

Wrong Answer: *anything that we can count!*

What is Combinatorics?

Cherednik (2002):
Combinatorics is the science of counting the possible arrangements and ways of organizing collections of anything (e.g., atoms, pebbles, star clusters).

$g(\lambda, \mu, \nu)$

$g(\lambda, \mu, \nu)$

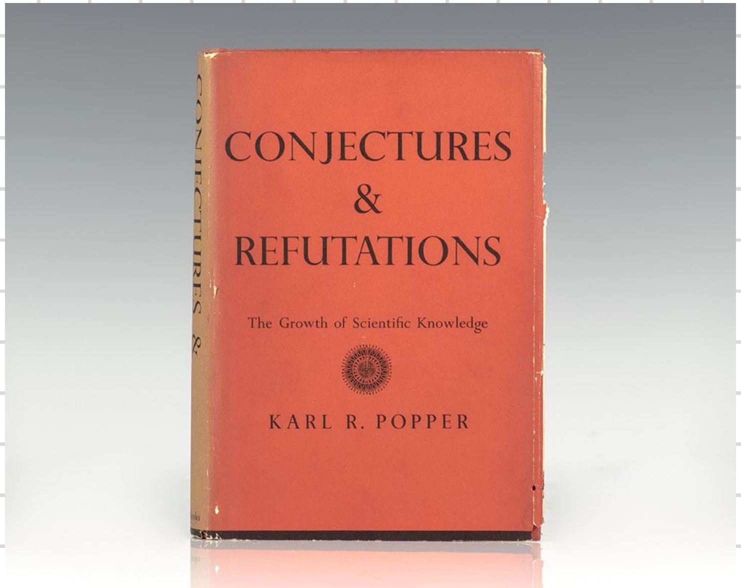
Billey (Feb. 2021): *I never say "It is an open problem to find a combinatorial interpretation for the Schubert coeff." They already count something!*

Billey to P. (Apr. 2022): *They count the number of points in a generic intersection of 3 Schubert varieties.*

What is a combinatorial interpretation?

Wrong Answer:

Popper: *A belief needs to be* disprovable *in order to be* scientific!



We need a formal definition!

What is a combinatorial interpretation?

Correct Answer: #P (a notion in *computational complexity*)

What is #P?

Quick and easy guide with examples:

(0) P – *class of poly-time decision problems*

FP – *class of poly-time counting problems*

Examples: GraphConnectivity, PerfectMatching, $c^{\lambda}_{\mu\nu} >^? 0 \quad \in \mathrm{P}$

#SpaningTrees, #PerfectMatching in planar graphs, f^{λ}, $s_{\lambda}(1,\ldots,1) \ \in \mathrm{FP}$

(1) NP – *class of decision problems where objects can be verified in poly-time*

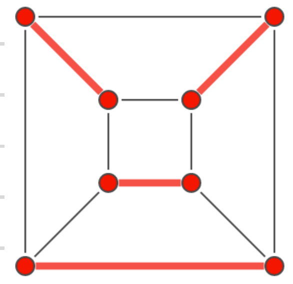
NP-complete – *class of hardest problems in NP*

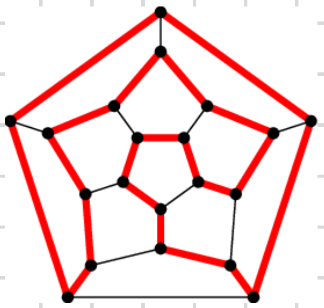
Examples: 3Coloring, HC (Hamiltonian cycle), Knapsack $\in$ NP-c

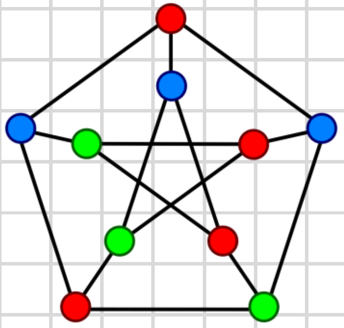
(2) #P – *class of counting problems where objects can be verified in poly-time*

#P-complete – *class of hardest problems in #P*

Examples: #3Coloring, #HC, #Knapsack, #PerfectMatching, $c^{\lambda}_{\mu\nu} \quad \in$ #P-c







Where are our favorite problems?

$$(3) \quad \textsc{GapP} := \#\mathrm{P} - \#\mathrm{P} \quad \text{and} \quad \textsc{GapP}_{\geq 0} := \textsc{GapP} \cap \mathbb{N}$$

$g(\lambda, \mu, \nu) \in \textsc{GapP}_{\geq 0}$ [Christandl-Doran-Walter'12], [P.-Panova'17]

$a_\lambda(\mu, \nu) \in \textsc{GapP}_{\geq 0}$ [Fischer-Ikenmeyer'20]

$c(u, v, w) \in \textsc{GapP}_{\geq 0}$ follows from [Postnikov-Stanley'09]

Translation to Algebraic Combinatorics lingo:

$\textsc{GapP}_{\geq 0}$ = "Combinatorial Interpretation"

$\#\mathrm{P}$ = "Manifestly Positive Combinatorial Interpretation"

Note: Billey's "combinatorial interpretation" is not in $\#\mathrm{P}$ because of Vakil's *Murphy's law* (2006)

Other problems in $\textsc{GapP}_{\geq 0}$?

(1) $e(P) - 1 \in \textsc{GapP}_{\geq 0}$

$P = (X, \prec)$ is a poset, $e(P) = \#$ linear extensions of P

(1) and (3) $\in \#\text{P}$ (easy)

(2) $m_k(G)^2 - m_{k+1}(G)\, m_{k-1}(G) \in \textsc{GapP}_{\geq 0}$

$m_k(G) := \#$ k-matchings in G [Heilmann-Lieb'72]

(2) $\in \#\text{P}$ by [Krattenthaler'96]

(3) $(2m)^{n-1} - n(n-1)^{n-1}\tau(G) \in \textsc{GapP}_{\geq 0}$

$G = (V, E)$, $|V| = n$, $|E| = m$, $\tau(G) = \#$ spanning trees [Grimmett'76]

(4) $f_k(G)^2 - f_{k+1}(G)\, f_{k-1}(G) \in \textsc{GapP}_{\geq 0}$

$f_k(G) := \#$ k-forests in G [Adiprasito-Huh-Katz'18]

(4) $\in^{?}$ $\#\text{P}$ is a major open problem. No combinatorial proof is known.

see also [Anari-Liu-Oveis_Gharan-Vinzant'18], [Brändén-Huh'20], [Chan-P.'21]

Can we prove anything at all?

Yes, now we can!

Proposition [Ikenmeyer-P.'22]

If $\mathrm{GAPP}^2 \subseteq \#\mathrm{P}$, then $\mathrm{PH} = \Sigma_2^{\mathsf{p}}$.

Note: $\mathrm{GAPP}^2 = (\#\mathrm{P} - \#\mathrm{P})^2 \subseteq \mathrm{GAPP}_{\geq 0}$

Explanation of Proposition: Let G, H be two graphs. Define

$$f(G,H) := \big(\#\text{3-colorings in } G - \#\text{3-colorings in } H\big)^2$$

Then $f(G,H) \notin \#\mathrm{P}$, i.e. *does not* have a combinatorial interpretation unless $(\circledast)$

Explanation of $(\circledast)$: You can decide $\exists\forall\exists\ldots\forall\,\Phi$ just as fast as $\exists\forall\Phi$.

This is universally conjectured to be *false* ($\mathrm{PH} \neq \Sigma_2^{\mathsf{p}}$ is stronger than $\mathrm{P} \neq \mathrm{NP}$).

Proof idea: Suppose $f(G,H) \in \#\mathrm{P}$. Then there is a poly-time certificate for $\big(\#\text{3-colorings of } G\big) \neq \big(\#\text{3-colorings of } H\big)$. But that would be too powerful, akin to poly-time certificate for $\big(\#\text{3-colorings of } G\big) = 0$.

Our results (a sampler):

The following are $\notin \#\mathrm{P}$ *(under some complexity assumptions)*

Cauchy inequality

$$(x_1y_1 + \ldots + x_ny_n)^2 \le (x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2)$$

Minkowski inequality

$$\prod_{i=1}^{n}(x_i^n + y_i^n) \ge \left[\prod_{i=1}^{n} x_i + \prod_{i=1}^{n} y_i\right]^n$$

Karamata inequality

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, such that $\boldsymbol{x} \trianglerighteq \boldsymbol{y}$.

Then, for every convex $f : \mathbb{R}^n \to \mathbb{R}$,

we have $f(x) \ge f(y)$.

Hadamard inequality

$$\det \begin{pmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{pmatrix}^2 \le \prod_{i=1}^{d}(a_{i1}^2 + \ldots + a_{id}^2)$$

Definition: Let $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ be nonincreasing sequences.

We say: $\boldsymbol{x}$ *majorizes* $\boldsymbol{y}$, write $\boldsymbol{x} \trianglerighteq \boldsymbol{y}$, if

$$x_1 + \ldots + x_i \ge y_1 + \ldots + y_i \quad \text{for all } 1 \le i < n, \quad \text{and}$$
$$x_1 + \ldots + x_n = y_1 + \ldots + y_n\,.$$

Case study: *Gessel sequence*

$$b_n := 2 \cdot 5^n - (3+4i)^n - (3-4i)^n, \quad \text{where } i = \sqrt{-1}$$

Note that $b_n \in \mathbb{Z}$ since

$$b_n = 2 \cdot 5^n - 2 \sum_r (-1)^r \binom{n}{2r} 3^{n-2r} 4^{2r}$$

and that $b_i \geq 0$ since $|3 \pm 4i| = 5$.

$$b_n = \left[2\,\mathrm{Im}(1+2i)^n\right]^2$$

$$b_n = -b_{n-1} + 5b_{n-2} + 125b_{n-3} \quad \text{for } n > 2.$$

$$B(t) := \sum_{n=0}^{\infty} b_n t^n = \frac{16t(1+5t)}{(1-5t)(1+6t+25t^2)}$$ not $\mathbb{N}$-rational

A250102 as a simple table

n	b_n
0	0
1	16
2	64
3	16
4	2304
5	5776
6	7744
7	309136
8	451584
9	2062096
10	38837824
11	27920656
12	424030464
13	4570300816
14	1039933504
15	74815378576
16	501671890944
17	2396689936

Open Problem: *Does* $\{b_n\}$ *have a combinatorial interpretation?*

DZ story

Case study: *generalized Gessel sequences*

Consider $\{a_n\} = \{a_n(f,g)\}$ defined as

$$a_n := 2\left(f^2 + g^2\right)^n - \left(f + gi\right)^{2n} - \left(f - gi\right)^{2n}.$$

$$a_n(1,2) = b_n$$

$$A(t) := \sum_{n=0}^{\infty} a_n t^n \in \mathbb{Z}(t) \cap \mathbb{N}[[t]].$$

Suppose now that f, g are #P functions.

$$a_1 = 4g^2 \in \#\mathrm{P}$$
$$a_2 = 16f^2g^2 \in \#\mathrm{P}$$
$$a_3 = 4g^2(3f^2 - g^2)^2 \notin \#\mathrm{P} \quad \text{unless } \mathrm{UP} = \mathrm{coUP}$$
$$a_4 = \left[8fg(f+g)(f-g)\right]^2 \notin \#\mathrm{P} \quad \text{unless } \mathrm{PH} = \Sigma_2^{\mathsf{p}}$$

Conjecture: $a_n(f,g) \notin \#\mathrm{P}$ for all $n \geq 3$.

Thank you!