Oda's strong factorization conjecture on stellar subdivision of triangulations

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Outline

Projective space

We work over the field of complex numbers $\mathbb{C}$. The n-dimensional *complex projective space*, denoted by $\mathbb{P}^n_{\mathbb{C}}$, is the quotient:

$$(\mathbb{C}^{n+1} - \{(0, \dots, 0)\})/\mathbb{G}_m$$

where $\mathbb{G}_m := \mathbb{C}^*$ is acting diagonally on $\mathbb{C}^{n+1}$.

We write $[x_0 : \cdots : x_n]$ for a point in $\mathbb{P}^n_{\mathbb{C}}$. Note that

$$[x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n]$$

for $\lambda \neq 0$.

Projective space as union of tori

If we delete the hyperplane $H_0 := \{x_0 = 0\}$ from $\mathbb{P}^n_{\mathbb{C}}$, we simply obtain the n-dimensional affine space $\mathbb{C}^n$. We can write

$$\mathbb{P}^n_{\mathbb{C}} = \mathbb{C}^n \sqcup \mathbb{P}^{n-1}_{\mathbb{C}}.$$

Similarly, we can write

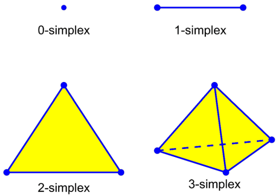
$$\mathbb{C}^n = \mathbb{G}^n_m \cup \underbrace{\mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C}^{n-1}}_{\text{n times}}.$$

This gives a description of the projective space as disjoint union of tori:

$$\mathbb{P}^n_{\mathbb{C}} := \mathbb{G}^n_m \sqcup \underbrace{\mathbb{G}^{n-1}_m \sqcup \cdots \sqcup \mathbb{G}^{n-1}_m}_{\text{n times}} \sqcup \mathbb{G}^{n-2}_m \sqcup \cdots \sqcup \mathbb{G}^0_m.$$

Combinatorial description

The n-dimensional projective space $\mathbb{P}^n_{\mathbb{C}}$ can be described by a n-dimensional simplex:



The k-dimensional faces correspond to copies of $\mathbb{G}_m^k$ and the picture respects containment of closures.

Smooth projective varieties

A *projective variety* $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ is the zero set of a finite set of homogeneous polynomial equations in $\mathbb{C}[x_0, \ldots, x_n]$. A projective variety $X \subseteq \mathbb{P}^n_{\mathbb{C}}$ is said to be *smooth* if it admits a tangent plane at every point $x \in X$. The *dimension* d of X is the dimension of any of the tangent spaces.

Example

The Fermat hypersuface

$$X_k := \{[x_0 : \cdots : x_n] \mid x_0^k + \cdots + x_n^k = 0\} \subset \mathbb{P}^n_{\mathbb{C}}$$

is a smooth projective variety of dimension $n - 1$.

A smooth projective variety of dimension 1 (resp. 2 and n) is called a *curve* (resp. *surface* and n-*fold*).

Rational maps

How to understand the geometry of a smooth projective variety X ?

A successful approach to understand a geometric object X is to understand all the functions $f\colon X \to Y$ that we can define on X.

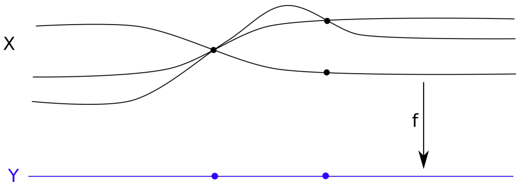
In the setting of smooth projective varieties, we are mostly interested on *rational maps*, i.e., functions $f\colon X \dashrightarrow Y$ that are defined by a ratio of polynomial functions (they may not be defined everywhere).

Rational maps from curves

A *smooth projective curve* is simply a Riemann surface.

If $f\colon X \dashrightarrow Y$ is a rational map between smooth projective curves, then it can be extended to a function on X and is either constant or a finite cover (possibly ramified).

Thus, rational maps between curves are either constant or they look as follows:

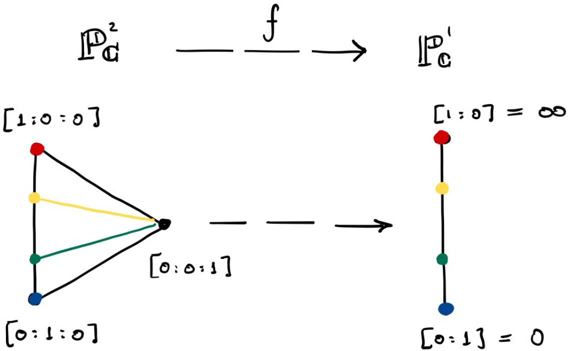


Rational maps from surfaces

Rational maps from surfaces are much more complicated. Lets consider the example:

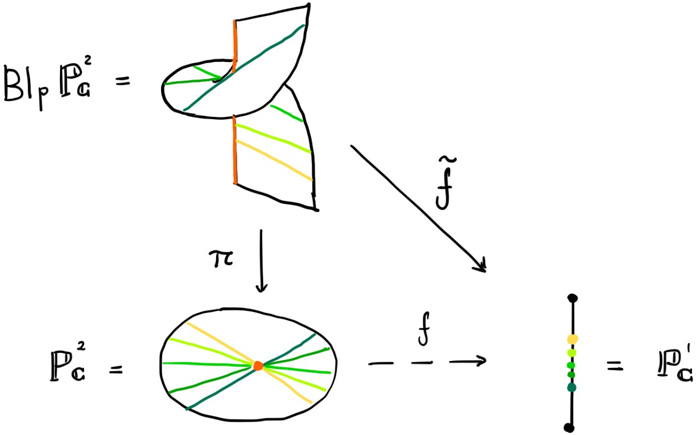
$$f \colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^1_{\mathbb{C}} \qquad [x_0 : x_1 : x_2] \mapsto [x_0 : x_1]$$

Using the previous combinatorial description, the map looks like this:



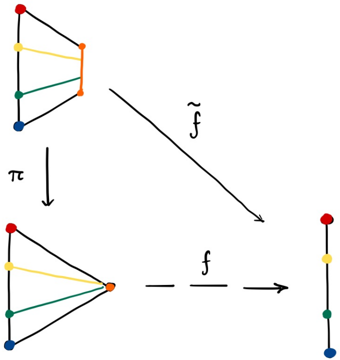
Resolving the indeterminancy locus

The approach to understand the map f is to replace it with the closure of its graph in the product $\mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$. This geometric process leads to the following picture:



Resolving the indeterminancy locus

The previous picture can be described combinatorially as follows:



By taking dual fans, the previous picture shows that stellar subdivisions allow us to understand resolution of indeterminancy.

Blow-ups

Let X be a smooth projective variety and $Z \subset X$ be a smooth subvariety. The *blow-up* of X at Z is a new variety obtained by deleting Z from X and gluing a variety that represents all the tangent directions of X from Z.

For instance, when blowing up a point p in a smooth projective surface X, we replace the point $p \in X$ with $\mathbb{P}^1_{\mathbb{C}}$. The outcome surface $\mathrm{Bl}_p(X)$ is called the *blow-up of X at p* and the induced projection function

$$\pi \colon \mathrm{Bl}_p(X) \to X$$

is called a *blow-down*.

Birational maps between surfaces

A *birational map* is a rational function $f\colon X \dashrightarrow Y$ between smooth projective varieties that induces an isomorphism between open dense subsets.

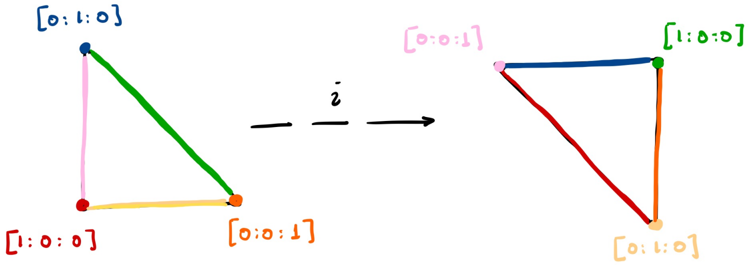
Example

The *plane Cremona transformation* is the map

$$i\colon \mathbb{P}^2_{\mathbb{C}} \dashrightarrow \mathbb{P}^2_{\mathbb{C}} \qquad [x_0 : x_1 : x_2] \mapsto [x_0^{-1} : x_1^{-1} : x_2^{-1}] = [x_1x_2 : x_0x_2 : x_0x_1]$$

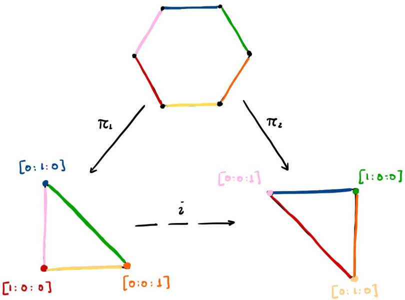
Picture of the plane Cremona

Here there is a picture that combinatorially describes the plane Cremona transformation:



Resolving the plane Cremona

The following picture shows how to resolve the indeterminancy of the plane Cremona:



Taking dual fans, the previous picture shows two triangulations of $\mathbb{Q}^2$. The hexagon on top gives a common refinement of both triangulations.

Factorization of birational maps between surfaces

The following theorem was proved by Castelnuovo in the 1920's:

Theorem (Castelnuovo, 1920)

Let $f\colon X \dashrightarrow Y$ be a birational map between smooth projective surfaces. Then, there is a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$

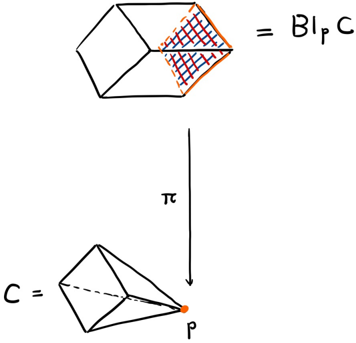
where both π_1 and π_2 are compositions of blow-ups of points.

An example in dimension three

Consider an ordinary double point of dimension three

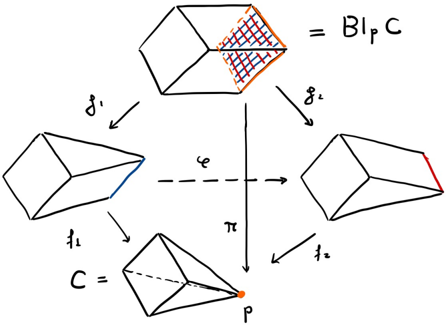
$$C := \{x^2 + y^2 + z^2 + w^2 = 0\}.$$

Then, C is simply the anti-canonical cone over $\mathbb{P}^1 \times \mathbb{P}^1$. The blow-up of C at the point $p := (0,0,0,0)$ is a smooth variety:



An example in dimension three

The ordinary double point admits two other resolutions of singularities induced by the two projections of $\mathbb{P}^1 \times \mathbb{P}^1$:



The birational map ϕ between smooth threefolds is known as the *Atiyah flop*. By taking dual fans, the Atiyah flops corresponds to a change of triangulations on the cone over a square.

Strong factorization conjecture

The *strong factorization conjecture* states that any birational map $f : X \dashrightarrow Y$ between smooth projective varieties can be factored as $\pi_2 \circ \pi_1^{-1}$, where each π_i is the composition of smooth blow-ups.

The conjecture consists of two parts:

1. The *toroidalization conjecture*, and
2. *Oda's conjecture*.

The first conjecture, roughly speaking, states that any birational map can be transformed (using blow-ups) into one that is locally described by polytopes (or fans).
The second conjecture states that two triangulations of the same polytope admit a common stellar subdivision.

Factorization of birational maps between 3-folds

The toroidalization conjecture in dimension 3 was solved in 2004 by Cutkosky. Recently, Adiprasito and Pak proved Oda's conjecture in arbitary dimension. These two together, lead to the following theorem.

Theorem (Cutkosky 2004 + Adiprasito-Pak, 2024)

Let $f\colon X \dashrightarrow Y$ be a birational map between smooth projective threefolds. Then, there is a commutative diagram

$$\begin{tikzcd} & Z \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[rr, dashed, "f"] & & Y \end{tikzcd}$$

where both π_1 and π_2 are compositions of blow-ups of points and curves.