The story of MacMahon’s Master Theorem

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MacMahon’s Master Theorem

Let $A = (a_{ij})_{m \times m}$, $a_{ij} \in \mathbb{C}$, be a complex matrix, $x_1, \ldots, x_m$ be a set of commuting variables,

$$G(k_1, \ldots, k_m) = \left[ x_1^{k_1} \cdots x_m^{k_m} \right] \prod_{i=1}^{m} (a_{i1}x_1 + \ldots + a_{im}x_m)^{k_i}.$$ 

Let $t_1, \ldots, t_m$ be another set of commutative variables, and $T = (\delta_{ij}t_i)_{m \times m}$ a diagonal matrix. Then:

$$\sum_{(k_1, \ldots, k_m)} G(k_1, \ldots, k_m) \ t_1^{k_1} \cdots t_m^{k_m} = \frac{1}{\det(I - TA)},$$

where the summation is over all nonnegative integer vectors $(k_1, \ldots, k_m)$. 
Main result:

Let $q_{ij} \in \mathbb{C}$, $q_{ij} \neq 0$, where $1 \leq i < j \leq m$. Suppose variables $x_1, \ldots, x_m$ are $q$-commuting:

$$x_j x_i = q_{ij} x_i x_j, \quad \text{for all } i < j,$$

Suppose also that the variables $a_{ij}$ $q$-commute within columns:

$$a_{jk} a_{ik} = q_{ij} a_{ik} a_{jk}, \quad \text{for all } i < j,$$

commute with $x_s$, $1 \leq s \leq m$, and satisfy:

$$a_{jk} a_{il} - q_{ij} a_{ik} a_{jl} + q_{kl} a_{jl} a_{ik} - q_{kl} q_{ij} a_{il} a_{jk} = 0, \quad \text{for all } i < j, \ k < l.$$

Define

$$\det_q(I - A) = \sum_{J \subseteq [m]} (-1)^{|J|} \det_q A_J,$$

where

$$\det_q A = \sum_{\sigma \in S_m} \left( \prod_{p<r: \sigma(p) > \sigma(r)} q_{rp}^{-1} \right) a_{\sigma(1)1} \cdots a_{\sigma(k)k}.$$

Theorem [Konvalinka–P:] Let

$$G(k_1, \ldots, k_m) = [x_1^{k_1} \cdots x_m^{k_m}] \prod_{i=1..m} (a_{i1} x_1 + \ldots + a_{im} x_m)^{k_i}.$$

Then

$$\sum_{(k_1, \ldots, k_m)} G(k_1, \ldots, k_m) = \frac{1}{\det_q(I - A)}.$$
This talk

**Question:** How can prove such a generalization of MMT?

**Answer:** You need to find a really good proof of MMT – the rest is easy (both the statement and the proof of the theorem).

**Followup Question:** How can one come up with such a proof?

**Answer:** You need to see and understand the underlying algebra. To do that, I need to tell the full story of MacMahon’s Master Theorem, as we understand it now.

**Hint:** Such proof is part bijective and part algebraic.
Starting point:

About the author:

Major Percy Alexander MacMahon (1854–1929)

- Second son of Brigadier-General, served in British Army in India
- Taught at Royal Military Academy, retired a Major
- Fellow of the Royal Society
- Sylvester Medal, Morgan Medal, Queen’s Medal by the Royal Society
- President of the London Mathematical Society

MacMahon is the father of combinatorics.
— [Jonathan Borwein]

MacMahon’s expertise lay in combinatorics, a sort of glorified dicethrowing, and in it he had made contributions original enough to be named a Fellow of the Royal Society.
— [certain MIT professor]
MMT:

\[ A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{pmatrix}, \quad T = \begin{pmatrix} t_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & t_m \end{pmatrix} \]

\[ G(k_1, \ldots, k_m) := \begin{bmatrix} x_1^{k_1} \cdots x_m^{k_m} \end{bmatrix} \prod_{i=1}^{m} (a_{i1}x_1 + \ldots + a_{im}x_m)^{k_i} \]

\[ G(k_1, \ldots, k_m) = \begin{bmatrix} t_1^{k_1} \cdots t_m^{k_m} \end{bmatrix} \frac{1}{\det(I - TA)} \]
First quick example:

\[
A = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1
\end{pmatrix}, \quad I - TA = \begin{pmatrix}
1 - t_1 & 0 & \ldots & 0 \\
0 & 1 - t_2 & \ldots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 1 - t_m
\end{pmatrix}
\]

Then

\[
G(k_1, \ldots, k_m) = \left[ x_1^{k_1} \ldots x_m^{k_m} \right] (x_1)^{k_1} \cdots (x_m)^{k_m} = 1,
\]

\[
\det(I - TA) = (1 - t_1)(1 - t_2) \cdots (1 - t_m).
\]

The MMT now says:

\[
\sum_{(k_1, \ldots, k_m)} t_1^{k_1} \cdots t_m^{k_m} = \frac{1}{(1 - t_1) \cdots (1 - t_m)}.
\]
Second quick example:

\[ A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \]

Then

\[ G(n, n, n) = \left[ x_1^n x_2^n x_3^n \right] (x_2 + x_3)^n (x_1 + x_3)^n (x_1 + x_2)^n \]

is the number of derangements of \( x_1^n x_2^n x_3^n \), i.e. permutations of the letters with no fixed points. Observe (see the figure):

\[ G(n, n, n) = \sum_{k=0}^{n} \binom{n}{k}^3. \]
On the other hand, the MMT gives:

\[
G(n, n, n) = \left[ t_1^n t_2^n t_3^n \right] \det^{-1} \begin{pmatrix} 1 & -t_1 & -t_1 \\ -t_2 & 1 & -t_2 \\ -t_3 & -t_3 & 1 \end{pmatrix}
\]

\[
= \left[ t_1^n t_2^n t_3^n \right] \frac{1}{1 - t_1 t_2 - t_1 t_3 - t_2 t_3 - 2 t_1 t_2 t_3}
\]
Third quick example:

\[
A = \begin{pmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{pmatrix}
\]

Then

\[
G(n, n, n) = \left[x_1^n x_2^n x_3^n\right] (x_2 - x_3)^n (x_1 - x_3)^n (x_1 - x_2)^n
\]

and we similarly have

\[
G(n, n, n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3,
\]

which implies that \(G(n, n, n) = 0\) for odd \(n\).
On the other hand, for \( n = 2m \), the MMT gives:

\[
G(n, n, n) = \left[ t_1^n t_2^n t_3^n \right] \det^{-1} \begin{pmatrix}
1 & -t_1 & t_1 \\
t_2 & 1 & -t_2 \\
-t_3 & t_3 & 1
\end{pmatrix}
\]

\[
= \left[ t_1^n t_2^n t_3^n \right] \frac{1}{1 + t_1 t_2 + t_1 t_3 + t_2 t_3}
\]

\[
= \left[ t_1^{2m} t_2^{2m} t_3^{2m} \right] (-1)^m (t_1 t_2 + t_1 t_3 + t_2 t_3)^{3m}
\]

\[
= (-1)^m \binom{3m}{m, m, m}
\]

We obtain Dixon’s identity:

\[
\sum_{k=0}^{2m} (-1)^k \binom{2m}{k}^3 = (-1)^m \binom{3m}{m, m, m}.
\]
How about $q$–MMT?

Finding $q$–MMT was an open problem for over 50 years.

Here is the $q$–Dixon identity:

$$
\sum_{k=0}^{m} (-1)^k q^{k(3k+1)/2} \left[ \frac{2m}{m+k} \right]_q^3 = \left[ \frac{3m}{m,m,m} \right]_q,
$$

where

$$
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \frac{n!_q}{k!_q(n-k)!_q}, \quad n!_q = (n)_q \cdots (1)_q, \quad (r)_q = \frac{q^r - 1}{q - 1}.
$$
Lagrange Inversion Theorem (1768)

Let \( f(t) = t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \ldots \) Suppose
\[
f(t) = t \left( 1 + c_1 f(t) + c_2 f(t)^2 + c_3 f(t)^3 + \ldots \right)
\]
Then:
\[
a_n = \frac{1}{n} \cdot \left[ z^{n-1} \right] \Phi(z)^n,
\]
where \( \Phi(z) = 1 + c_1 z + c_2 z^2 + \ldots \)

Remark: This is a small special case of what is known in the literature as Lagrange Inversion Theorem. Also, \( \Phi(z) \) is uniquely determined by \( f(t) \).
Quick examples:

(1) \( a_n = \# \) binary trees with \( n \) vertices, \( f(t) = t(1 + f(t))^2 \),

\[
\Phi(z) = (1 + z)^2, \quad a_n = \frac{1}{n} \left[ z^{n-1} \right] (1 + z)^{2n} = \frac{1}{n} \binom{2n}{n-1}.
\]

[**Catalan numbers**]

(2) \( a_n = \frac{1}{n!} \# \) rooted labeled trees with \( n \) vertices, \( f(t) = t \exp f(t) \),

\[
\Phi(z) = e^z, \quad a_n = \frac{1}{n} \left[ z^{n-1} \right] e^{nz} = \frac{1}{n} \cdot \frac{n^{n-1}}{(n-1)!} = \frac{1}{n!} \cdot n^{n-1}.
\]

[**Cayley formula**]
Life of MMT as an analytic result

I. J. Good observation (1962): MMT immediately follows from the multivariate Lagrange Inversion Theorem.

Fallout: For generations, combinatorialists looked for $q$–MMT via the $q$-Lagrange inversion results.

Minor obstacle: There are many inequivalent (multivariate) $q$–LIT. There are even several non-commutative LIT. None gives a $q$–MMT.
Timeline of $q$–Lagrange Inversion theorems:

- L. Carlitz (1974, stated as open problem)
- I. Gessel (1980, + a non-commutative version)
- A. Garsia (1981)
- J. Hofbauer (1984)
- C. Krattenthaler (1984)
- A. Garsia, J. Remmel (1986)
- D. Singer (1995, unification result)
Further generalizations:

- C. Lenart (2000, symmetric functions and $q$-version)
- J.-C. Novelli, J.-Y. Thibon (2008, non-commutative symmetric functions + unification)
One application

For a tree $\tau$ in $K_n$ with root at 1, define the number of inversions
\[ \text{inv}(\tau) = \#(i, j) \text{ such that } [i \to_{\tau} j \to_{\tau} 1], \ 1 < i < j \leq n. \]

Let
\[ J_n(q) = \sum_{T \in K_n} q^{\text{inv}(T)}, \text{ so that } J_n(1) = n^{n-2}. \]

**Theorem** [Mallows, Riordan, 1968]
\[ \sum_{n=1}^{\infty} (q-1)^{n-1} J_n(q) \frac{z^n}{n!} = \log \left[ \sum_{k=0}^{\infty} q^k \frac{z^k}{k!} \right] \]

*Proof idea:* Use recurrence relations to show that $J_n(q) = T_{K_n}(1, q)$. Use that $T_{K_n}(1, 1+y)$ counts connected subgraphs of $K_n$ by the number of edges.
**Theorem** [Gessel, 1982]

\[ \sum_{n=1}^{\infty} J_n(q) \frac{z^n}{n!} = \frac{\sum_{k=0}^{\infty} q^{-\binom{k+1}{2}} (1 + q + \cdots + q^k) k \frac{z^k}{k!}}{\sum_{k=0}^{\infty} q^{-\binom{k+1}{2}} (1 + q + \cdots + q^{k-1}) k \frac{z^k}{k!}}. \]

**Proof idea:** Use Gessel’s q–Lagrange inversion applied to

\[ J(z) = \sum_{k=1}^{\infty} q^{\binom{k}{2}} J(z) \cdot J(qz) \cdots J(q^{k-1}z), \]

where

\[ J(z) = \sum_{n=1}^{\infty} J_n(q) \frac{z^n}{n!}. \]
**Theorem** [P., Postnikov, Retakh, 1995]

Let \( a_1, a_2, a_3, \ldots \) be non-commuting formal variables, and let \( R \) be the ring generated by them. Suppose \( f \in R \) satisfies

\[
f = a_0 + a_1 f + a_2 f^2 + a_3 f^3 + \ldots
\]

Then \( f = P \cdot Q^{-1} \), where

\[
P = a_0 + \sum_{i_1+i_2+\cdots+i_{\ell+1}=\ell} a_{i_1}a_{i_2}\cdots a_{i_{\ell+1}}
\]

and

\[
Q = 1 + \sum_{i_1+i_2+\cdots+i_\ell=\ell} a_{i_1}a_{i_2}\cdots a_{i_\ell}.
\]

**Remark:** To obtain Gessel’s formula, use \( a_i = g_i x^i y \), where \( g_i \) commute with everything, and \( xy = qyx \).
Proof idea:

Take a nearly upper triangular Toeplitz matrix:

\[
A := \begin{pmatrix}
  a_1 & a_2 & a_3 & a_4 & \cdots \\
  a_0 & a_1 & a_2 & a_3 & \cdots \\
  0 & a_0 & a_1 & a_2 & \cdots \\
  0 & 0 & a_0 & a_1 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Prove:

\[
f = \left( \frac{1}{I - A} \right)_{11} a_0 = (I + A + A^2 + A^3 + \ldots)_{11} a_0
\]

For that, use tools for working with non-commutative (quasi-)determinants and explicit combinatorial calculations.
Back to MMT

• P. Cartier, D. Foata (1968, partially commutative matrix entries)
• D. Foata, D. Zeilberger (1988, powers of matrices)
• C. Krattenthaler, M. Schlosser (1999, $q$-multiplying operators)

Breakthrough:

• S. Garoufalidis, T. Lê, D. Zeilberger (2006, quantum MMT)

Remark: In 1980, Zeilberger introduced a highly technical “operator elimination” technique which he used to prove MMT. In [GLZ], the authors extend this technique to get a quantum extension.
Back story:

- J. Bernstein, M. Sato (1971, independently, $D$-modules and holonomic systems)
- V. Jones (1983, Jones polynomial of knots)
- L. Faddeev, M. Jimbo, V. Drinfeld, etc. (late 1980’s, quantum groups)
- N. Reshetikhin, V. Turaev (1990, Jones polynomial from quantum groups)
- D. Zeilberger (1990, holonomic systems and binomial identities)
- S. Garoufalidis, T. Lê (2005, colored Jones polynomial is $q$-holonomic)
The algebraic proof of MMT:

Observe that:

$$\sum_{k_1+\ldots+k_m=n} G(k_1, \ldots, k_m) = \text{tr} S^n A,$$

Now MMT can be reduced to a basic result in linear algebra:

$$(\text{MMT}) \quad \sum_{n=0}^{\infty} \text{tr} S^n A = \frac{1}{\det(I - A)}$$

Follow the standard scheme:

1) MMT is trivial for diagonal matrices (see first quick example).
2) Check that $$(\text{MMT})$$ is invariant under conjugation $BAB^{-1}$.
3) Extend to general matrices by continuity.
Life of MMT as an algebraic result

- D. Foata, G.-N. Han (2007-8, 3 papers, 2 new algebraic proofs, various quantum extensions)
- P. Hai and M. Lorenz (2007, Koszul duality proof)
- M. Konvalinka, IP (2007, quantum extensions to Manin matrices)
- P. Etingof, IP (2008, extension to Berger’s $N$-homogeneous algebras, proof uses known generalized Koszul duality)
- P. Hai, B. Kriegk, M. Lorenz (2008, extension to $N$-homogeneous superalgebras)
- A. Molev, E. Ragoucy (2009, right quantum superalgebras)
Combinatorial proof of MMT:

Recall that:

\[(B^{-1})_{11} = \frac{\det B^{11}}{\det B}.\]

Use \(B = I - A\) to rewrite the r.h.s. of \((MMT)\):

\[
\frac{1}{\det(I - A)} = \frac{\det(I - A^{11})}{\det(I - A)} \cdot \frac{\det(I - A^{12,12})}{\det(I - A^{11})} \cdot \frac{\det(I - A^{123,123})}{\det(I - A^{12,12})} \cdots
\]

\[
= \left(\frac{1}{I - A}\right)_{11} \left(\frac{1}{I - A^{11}}\right)_{22} \left(\frac{1}{I - A^{12,12}}\right)_{33} \cdots \frac{1}{1 - a_{mm}}
\]

\[
= (I + A + A^2 + \ldots)_{11} (I + A^{11} + (A^{11})^2 + \ldots)_{22} \times (I + A^{12,11} + (A^{12,12})^2 + \ldots)_{33} \cdots
\]

Now both l.h.s. and r.h.s. of \((MMT)\) are positive sums of monomials in \(\{a_{ij}\}\). Konvalinka and IP provide an explicit bijection proving MMT.

**Remark:** To obtain the proof of quantum generalizations we extend this bijection and use the theory of non-commutative and quantum determinants by Gelfand-Retakh, Etingof-Retakh and Manin.
One final application

Denote by $L_{m,k}(n)$ the number of sequences $(i_1 \ldots i_n)$, $i_r \in \{1, \ldots, m\}$, such that no $k$ subsequent indices are strictly decreasing.

**Corollary** [Etingof, P.]

\[
1 + \sum_{n=1}^{\infty} L_{m,k}(n) t^n =
\]

\[
\left( 1 - mt + \binom{m}{k} t^k - \binom{m}{k+1} t^{k+1} + \binom{m}{2k} t^{2k} - \binom{m}{2k+1} t^{2k+1} + \ldots \right)^{-1}
\]

**Question:** An easy combinatorial proof for $k > 2$ ?
Thank you!