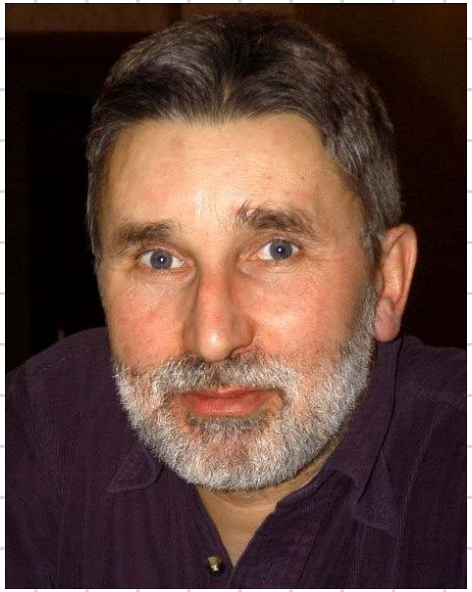
Igor Pak, UCLA

August 23, 2021

Hook formulas and their generalizations, a survey

Based on joint work with ***Alejandro Morales*** and ***Greta Panova***

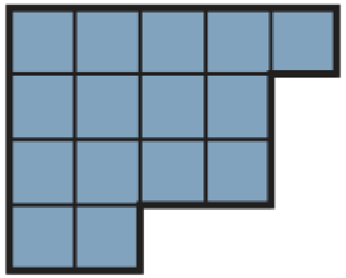


Sergei V. Kerov Memorial Conference

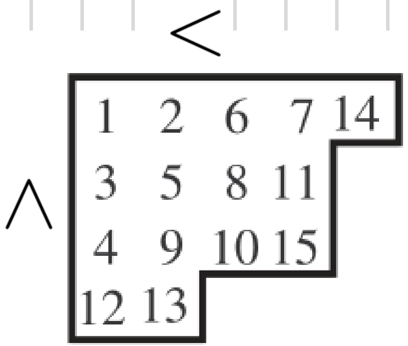
POMI, St. Petersburg, Russia

arXiv:2108.10140

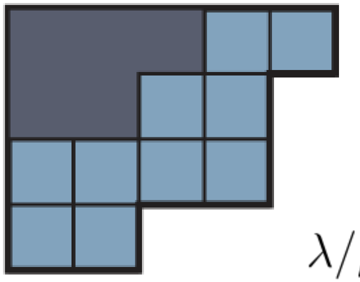
Standard Young tableaux



$\lambda = 5442 \vdash 15$

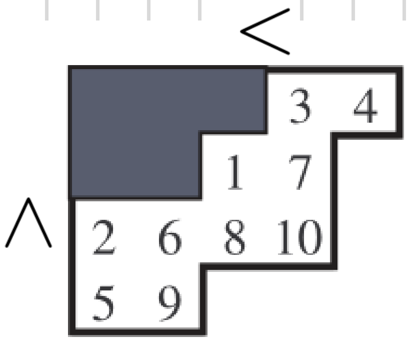


$A \in \mathrm{SYT}(\lambda)$



λ/μ

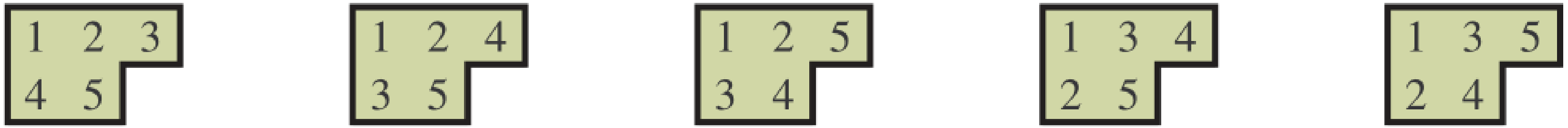
$\mu = 21$



$B \in \mathrm{SYT}(\lambda/\mu)$

$$f^\lambda := |\mathrm{SYT}(\lambda)|$$

$$f^{\lambda/\mu} := |\mathrm{SYT}(\lambda/\mu)|$$



$$f^{32} = 5$$

Note: $f^\lambda = \dim \mathbb{S}^\lambda \Rightarrow f^\lambda \,|\, n!$

Theorem [Aitken'43, Feit'53]

$$f^{\lambda/\mu} = n! \det\left(\frac{1}{(\lambda_i - \mu_j - i + j)!}\right)_{i,j=1}^{\ell(\lambda)}$$

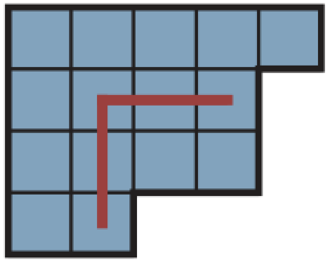
Hook-length formula

Theorem [Frame–Robinson–Thrall'54]

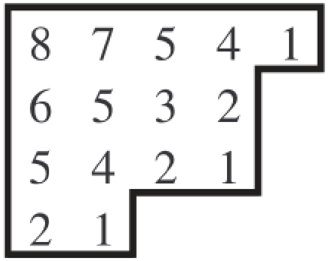
$$f^\lambda = n! \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)}$$

$$h(i,j) := \lambda_i + \lambda'_j - i - j + 1$$

$$f^{5442} = \frac{15!}{8\cdot 7\cdot 6\cdot 5^3\cdot 4^2\cdot 3\cdot 2^3\cdot 1^3} = 81081$$



$h(2,2) = 5$



$\{h(i,j) : (i,j)\in\lambda\}$

Note: dozens of proofs, notably *NPS bijection* and *GNW hook walk*

Note: variations for *shifted Young tableaux* and *increasing trees.*

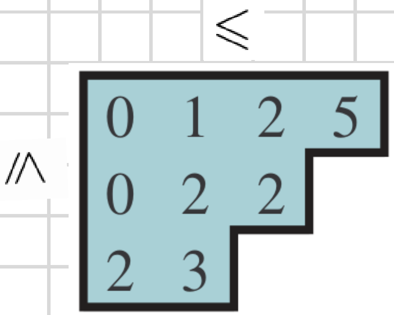
Stanley's *q*-HLF

Theorem [Stanley'71]

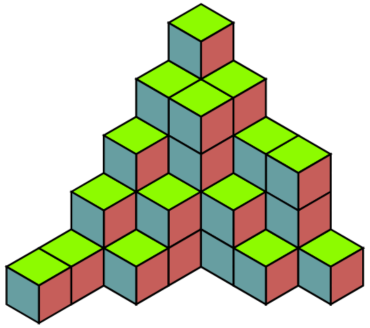
$$\sum_{A \in \mathrm{RPP}(\lambda)} t^{|A|} = \prod_{(i,j)\in\lambda} \frac{1}{1 - t^{h(i,j)}}$$

$$|A| := \sum_{(i,j)\in\lambda} A(i,j)$$

RPP = *reverse plane partitions*



$|A| = 17$



Bijective proof: *Hillman-Grassl bijection*'76, *Geometric RSK* [Gansner'80], [P.'01]

Corollary [MacMahon, 1915]

$$\sum_{A \in \mathrm{PP}} t^{|A|} = \prod_{k=1}^{\infty} \frac{1}{(1 - t^k)^k}$$

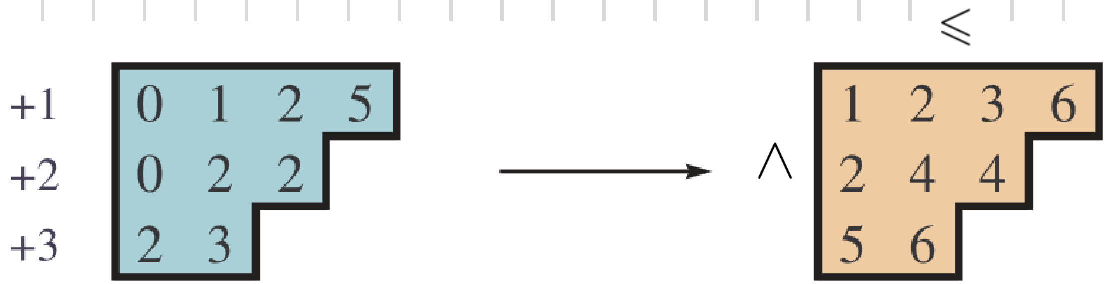
Note: Stanley's thm => **HLF**, *MacMahon's formula*
Stanley's hook-content formula is a generalization
=> product formula for #(*lozenge tilings of a hexagon*)

Super analogue of Stanley's *q*-HLF

Theorem [Littlewood'50]

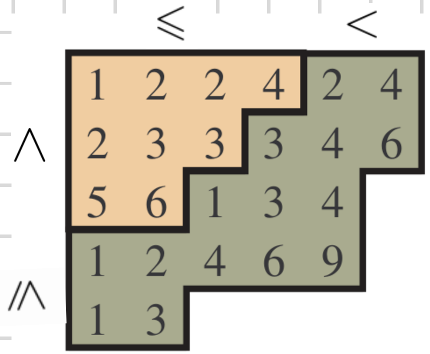
$$\sum_{A\in \mathrm{SSYT}(\lambda)} t^{|A|} = \prod_{(i,j)\in\lambda} \frac{t^i}{1-t^{h(i,j)}}$$

SSYT = *semistandard Young tableaux*



Theorem [Kirillov–P.'90]

$$\sum_{A\in \mathrm{SuperSSYT}(\lambda)} t^{|A|}\, s^{\xi(A)} = \prod_{(i,j)\in\lambda} \frac{t^i + s t^j}{1-t^{h(i,j)}}$$



$\xi(A) = 15$

SuperSSYT = *semistandard Young supertableaux* (symplectic tableaux)

$\xi(A) :=$ number of super entries

Note: *jeu-de-taquin* type combinatorial proof, algebraic proof in [Molchanov'92], [Thibon'92]

RHS = λ-colored HOMFLY polynomial of the unknot [Gorsky-Gukov-Stošić'18]

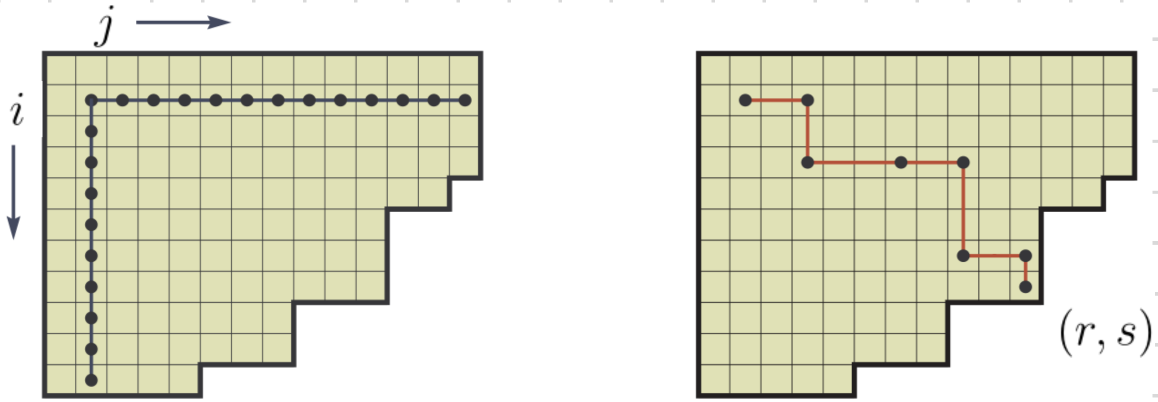
GNW hook walk

Hook walk construction:

- Start at random $(i, j) \in \lambda$
- Move to a random $(a, b) \in \mathrm{Hook}(i, j)$
- Repeat until at a corner of λ

Theorem [Greene–Nijenhuis–Wilf'79]

$$\mathbf{P}\big[\text{HW stops at } (r, s)\big] = \frac{f^{\lambda-(r,s)}}{f^{\lambda}}$$



Corollary:

Iterated HW samples uniform $A \in \mathrm{SYT}(\lambda)$

Note: GNW used HW to give a new proof of HLF. The 2-dim bubble sorting (NPS bijection) gives another way to sample uniform SYTs

Kerov's *q*-hook walk

A q-Analog of the Hook Walk Algorithm for Random Young Tableaux

S. KEROV*

Theorem [Stanley'68]

$$\sum_{A \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(A)} = (n!)_q \prod_{(i,j)\in\lambda} \frac{q^{j-1}}{h(i,j)_q}$$

$$(n!)_q = 1_q \cdot 2_q \cdots n_q$$

$$i_q = 1 + q + \ldots + q^{i-1}$$

$\mathrm{maj}(A)$ = *major index* of A

Kerov's q-hook walk construction:

- Start at (i,j) with $\mathbf{P} \propto q^{i-j}$
- Move along the hook to (a,b) with $\mathbf{P} \propto q^{a-b}$
- Repeat until at a corner of λ

Theorem [Kerov'93]

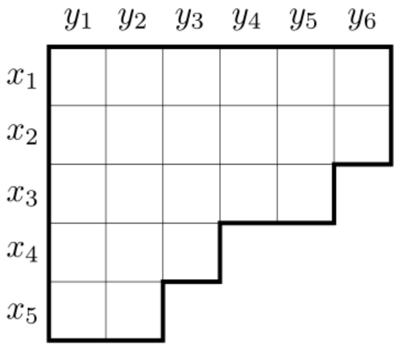
$\mathbf{P}[q\text{-HW stops at } (r,s)] = [\text{formula}]$

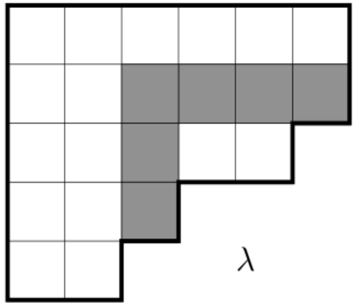
Corollary:

Iterated q-HW samples $A \in \mathrm{SYT}(\lambda)$ with $\mathbf{P} \propto q^{\mathrm{maj}(A)}$

Note: Garsia-Haiman'98 defined a (q,t)-HW to sample according to *Macdonald polynomials*

CKP weighted hook walk





CKP weighted hook walk construction:

- Start at (i,j) with $\mathbf{P} \propto x_i y_j$
- Move along the hook to (a,b) with $\mathbf{P} \propto x_a y_b$
- Repeat until at a corner of λ

Theorem [Ciocan-Fontanine, Konvalinka, P., 2011] $\mathbf{P}\big[\text{WHW stops at } (r,s)\big] =$

$$\frac{x_r y_s}{\sum_{(p,q)\in[\lambda]} x_p y_q} \prod_{i=1}^{r-1} \left(1 + \frac{x_i}{x_{i+1}+\ldots+x_r+y_{s+1}+\ldots+y_{\lambda_i}}\right) \prod_{j=1}^{s-1} \left(1 + \frac{y_j}{x_{r+1}+\ldots+x_{\lambda'_j}+y_{j+1}+\ldots+y_s}\right)$$

Note: For $x_i = y_j = 1$ we get the (usual) HW, for $x_i = q^i,\ y_j = q^{-j}$ we get Kerov's *q*-HW

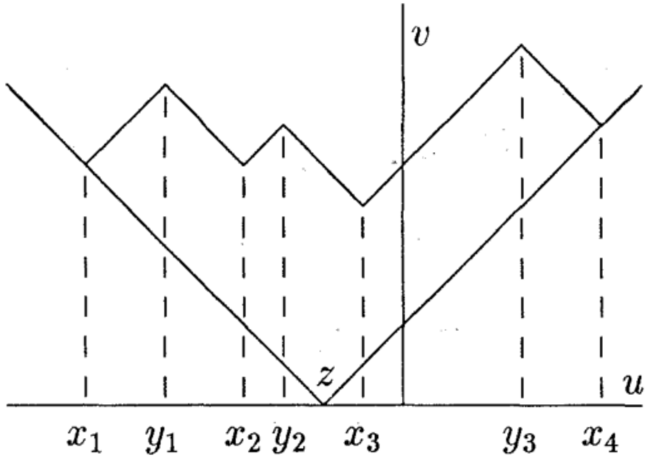
Note: For $x_i = q^i,\ y_j = t^j$ we get a (*q,t*)-identity which computes *equivariant quantum cohomology* of the Hilbert scheme $QH^*_{(\mathbb{C}^*)^2}(\mathrm{Hilb}_n)$ [Okounkov-Pandharipande'04]

Note: Related multivariate formulas appear in proofs of HLF by [Kirillov'92], [Vershik'92], [Zeilberger'84]

Kerov's segment sampling

Transition Probabilities for Continual Young Diagrams and the Markov Moment Problem

S. V. Kerov

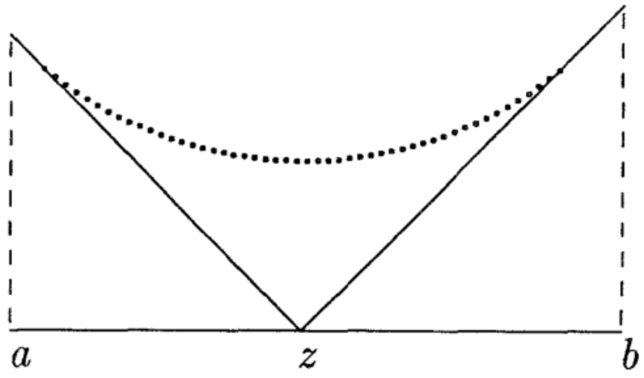


Let $x_1 < y_1 < x_2 < y_2 < x_3 < \ldots < y_{m-1} < x_m$

Segment sampling algorithm. Start with an interval $I = [x_1, x_m]$. Repeat the following procedure. Suppose $I = [x_i, x_j]$. Pick a uniform point ξ on the interval I.

- If $\xi \in [x_k, y_k]$ for some $k \in \{i, i+1, \ldots, j-1\}$, let $I \leftarrow [x_i, x_k]$.
- If $\xi \in [x_{\ell-1}, y_\ell]$ for some $\ell \in \{i+1, \ldots, j-1, j\}$, let $I \leftarrow [x_\ell, x_j]$.

Stop when $I = [x_i, x_i]$ for some $i \in \{1, \ldots, k\}$. Output x_i.



Theorem [Kerov'92]

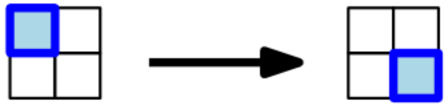
The algorithm defined above outputs x_k with probability γ_k.

$$\gamma_k = \frac{\prod_{j=1}^{m-1}(x_k - y_j)}{\prod_{i=1}^{m}(x_k - x_i)}$$

Note: Variation on *Pittel's hook walk*, and *external hook walk* by **GNW**.
Can be used to sample from the *Plancherel distribution* $(f^\lambda)^2/n!$

Naruse hook-length formula (NHLF)

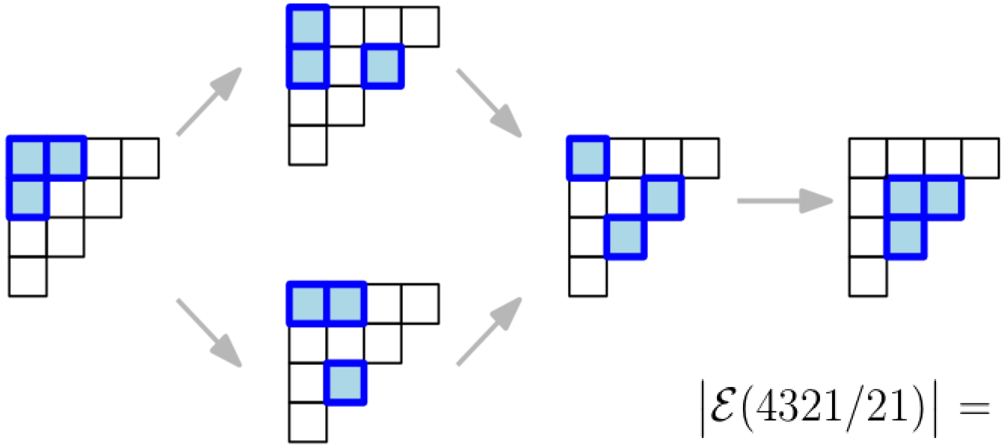
Definition [Ikeda–Naruse'92, Kreiman'05]
Excited diagrams are subsets $D \subseteq \lambda$ obtained from μ by a sequence of *excited moves*. Let $\mathcal{E}(\lambda/\mu) := \{D\}$



excited move

Theorem [Naruse'14]

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{1}{h(i,j)}$$



$\big|\mathcal{E}(4321/21)\big| = 5$

Theorem [Morales–P.–Panova'18] (*q*-NHLF)

$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{T \in \mathrm{SSYT}(\lambda/\mu)} q^{|T|} \overset{\downarrow}{=} \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in \lambda \setminus D} \frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}}$$

Note: algebraic proof via *factorial Schur functions*.
NHLF has no direct proof,
q-NHLF has HG-type bijection

K-hook-length formula (K-HLF)

Definition: ***Standard increasing tableaux*** (SIT) are *increasing tableaux* which do not skip integer entries

$<$, $\wedge$

1	2
3	4

1	3
2	4

1	2
2	3

$$|\mathrm{SIT}(22)| = 3$$

Theorem [Morales–P.–Panova'21] $\leftarrow$ *yesterday!*

For all $\lambda \subseteq d \times (n-d)$

$$\sum_{T \in \mathrm{SIT}(\lambda)} \prod_{k=0}^{\max(T)-1} \left(\left[\prod_{i=1}^{d} \frac{1+\beta\left(\mathrm{shape}(T_{\leq k})_i + d - i + 1\right)}{1+\beta\left(\lambda_i + d - i + 1\right)} \right] - 1 \right)^{-1}$$

$$= \frac{1}{(-\beta)^n} \prod_{i=1}^{\ell(\lambda)} \left(\beta(\lambda_i + d - i + 1) + 1\right)^{\lambda_i} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)}$$

Note: $A \in \mathrm{SIT}(\lambda),\ \max(A) = n \Leftrightarrow A \in \mathrm{SYT}(\lambda)$

Proposition: When $\beta \to 0$ K-HLF $\Rightarrow$ HLF

Note: Algebraic proof via *factorial Grothendieck polynomials* = *double Grothendieck polynomials* for Grassmannian permutations

Note: Originally motivated by the *K-theory Schubert calculus* of the Grassmannian [Lascoux-Schützenberger'82]

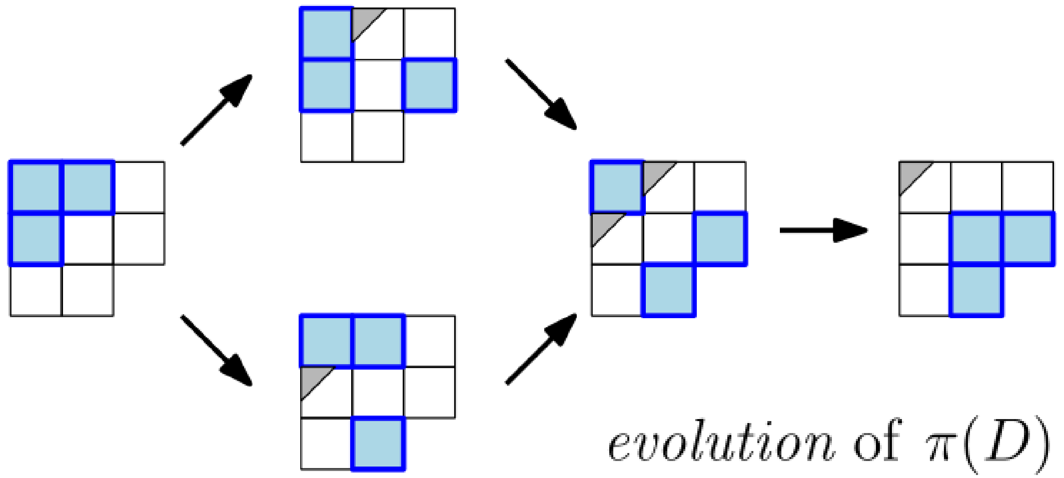
q-K-Naruse hook-length formula (q-K-NHLF)

Definition [Naruse–Okada'19]

Generalized excited diagrams:

$$\mathcal{D}(\lambda/\mu) := \bigcup_{D\in\mathcal{E}(\lambda/\mu)} \{D \cup S \ : \ S \subseteq \pi(D)\}$$

where $\pi(D) \subseteq \lambda/D$ is the set of ***high peaks*** of D



evolution of $\pi(D)$

Theorem [Morales–P.–Panova'21] For all $\mu \subseteq \lambda \subseteq d \times (n-d)$

$$\sum_{T\in \mathrm{SIT}(\lambda/\mu)} \prod_{k=0}^{\max(T)-1} \left(\left[\prod_{i=1}^{d} \frac{1 + \beta q^{\mathrm{shape}(T_{\leq k})_i + d - i + 1}}{1 + \beta q^{\lambda_i + d - i + 1}} \right] - 1 \right)^{-1}$$

$$= \sum_{D\in\mathcal{D}(\lambda/\mu)} \beta^{|D|-|\lambda|} \prod_{(i,j)\in\lambda\setminus D} \frac{\beta q^{\lambda_i + d - i + 1} + 1}{q^{d+j-\lambda'_j}(1 - q^{h(i,j)})}$$

Proposition/Summary:

q-K-NHLF $\Rightarrow$ K-HLF when $q \to 1, \ \mu = \varnothing$

q-K-NHLF $\Rightarrow$ q-NHLF when $\beta \to 0$

q-K-NHLF $\Rightarrow$ NHLF when $q \to 1, \ \beta \to 0$

q-K-NHLF $\Rightarrow$ Stanley's q-HLF when $\beta \to 0, \ \mu = \varnothing$

q-K-NHLF $\Rightarrow$ HLF when $q \to 1, \ \beta \to 0, \ \mu = \varnothing$

Thank you!