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Random linear extensions of posets

Mathematical Physics and Probability Seminar

UC Davis

Talk Content

0) Combinatorics & Complexity Background

1) Sorting probability

2) Our results for Young diagrams

3) Inequalities for numbers of linear extensions

4) Our new results on the cross-product conjecture

Linear extensions of posets

Def $\mathcal{P} = (X, \prec)$, $|X| = n$ ← finite

$f : X \to \{1 .. n\}$ is a linear extension

if (1) f is a bijection

(2) f is order-preserving

$x \prec y \Rightarrow f(x) < f(y) \quad \forall x, y \in X$

$\mathcal{E}(\mathcal{P})$ — set of LE of $\mathcal{P}$

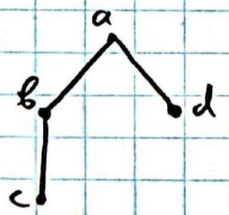
$e(\mathcal{P}) := |\mathcal{E}(\mathcal{P})|$ — number of LE of $\mathcal{P}$

Ex 1 $\mathcal{P} = (X, \prec)$, $|X| = n$ poset w/ no relations

$\Rightarrow e(\mathcal{P}) = n!$

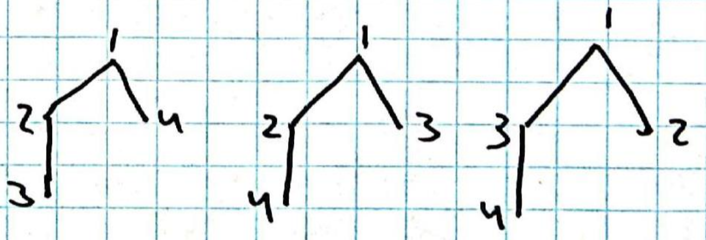
Ex 2

人

$\mathcal{P} =$ 

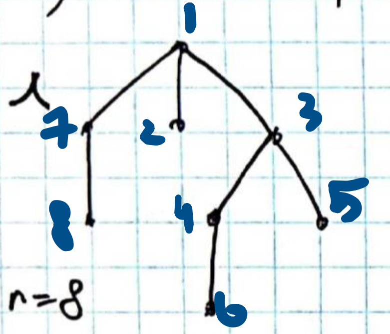
$\begin{cases} a \prec b \prec c \\ a \prec d \end{cases}$

$X = \{a, b, c, d\}$ $n = 4$

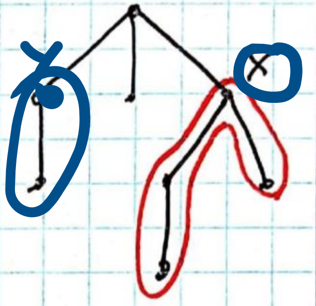
$\mathcal{E}(\mathcal{P}) =$ 

Large Examples

1) $\mathcal{P} = \mathcal{P}_T$, T – tree on n vertices



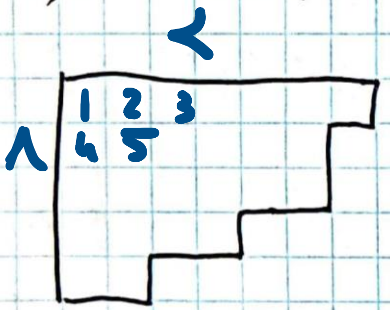
Th /Exc/ $\boxed{e(\mathcal{P}_T) = \dfrac{n!}{\prod\limits_{x \in X} b(x)}}$ ✓



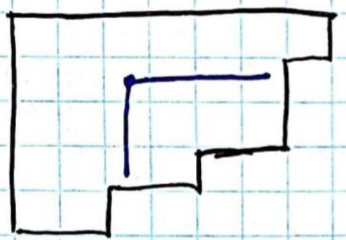
$b(x) = 4$

where $b(x) = \#\{y \in X,\ y \succeq x\}$

2) $\mathcal{P} = \mathcal{P}_\lambda$, $\lambda \vdash n$ Young diagram



Th [FRT'54] $\boxed{e(\mathcal{P}_\lambda) = \dfrac{n!}{\prod\limits_{x \in \lambda} h(x)}}$



$h(x) = 6$

hook-length formula

$e(\mathcal{P}_\lambda) = SYT(\lambda)$

3) $\mathcal{P} = \mathcal{P}_n$ Catalan poset

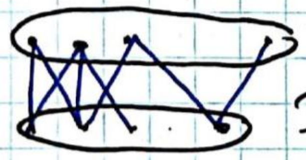
$e(\mathcal{P}_n) = |SYT(n,n)| = \boxed{\dfrac{1}{n+1}\dbinom{2n}{n}}$

Catalan number

Complexity aspects

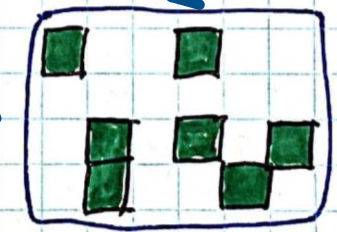
Th [Brightwell–Winkler '91]
Computing $e(P)$ is #P-c

Th [Dittmer – P. '18+]
Computing $e(P)$ is #P-c
even for
- P of height $= 2$
- P of $\dim = 2$



P

$\text{height}(P) = 2$



P

$(i,j) \preccurlyeq (k,\ell) \Leftrightarrow i \le k,\ j \le \ell$

$\dim(P) = 2$

Th [Karzanov–Khachiyan '91, Matthews '91]
$e(P)$ can be approx $(1 \pm \varepsilon)\, e(P)$ in time $\text{poly}(n, 1/\varepsilon)$

FPRAS

Idea: Markov chain "switch $i \leftrightarrow i+1$ if possible"

Bubley–Dyer '99

Complexity aspects (cont'd)

Th $e(\mathcal{P})$ can be computed in poly time for

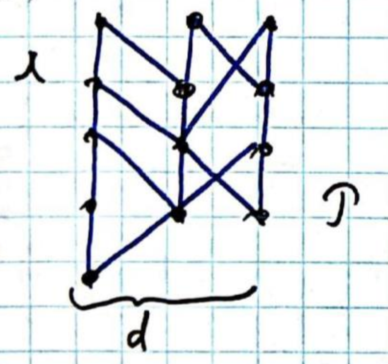
(1) posets $\mathcal{P}$, $\text{width}(\mathcal{P}) = d$ fixed — dynamic programming

(2) posets $\mathcal{P}_{\lambda/\mu}$ — Feit det formula

(3) series-parallel posets — induction

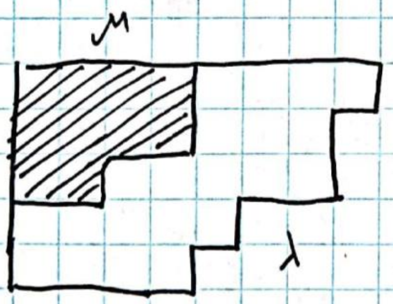
$\vdots$

(1)

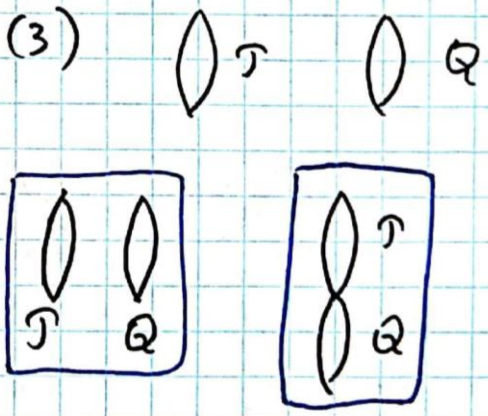


$\text{width}(\mathcal{P}) = d$

(2) $|\lambda/\mu| = n$



(3)



parallel series

includes $\mathcal{P}_T$

Sorting Probability

Def $\mathcal{P}=(X,\prec)$, $|X|=n$ fixed

$$\delta(\mathcal{P}) := \min_{x,y\in X} \big|\, P[f(x)<f(y)] - P[f(x)>f(y)] \,\big|$$

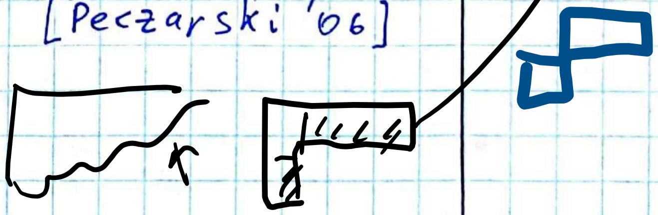
sorting probability

Conj [1/3 - 2/3 conjecture, Kislitsyn '68, Fredman '75]

$\boxed{\delta(\mathcal{P})\le \frac{1}{3}} \iff \exists\, x,y\in X$ s.t. $\frac{1}{3}\le P[f(x)<f(y)]\le \frac{2}{3}$

Th 1/3 - 2/3 conj holds for

- posets $\mathcal{P}$ of width 2 [Linial '84]
- posets $\mathcal{P}$ of height 2 [Trotter-Gehrlein-Fishburn '92]
- series - parallel posets [Zaguia '12]
- skew Young diagrams $\mathcal{P}_{\lambda/\mu}$ [Olson-Sagan '18]
- posets w/ $n\le 11$ elt's [Peczarski '06]
- $\vdots$



Sorting Probability (cont'd)

Th [Kahn-Saks '84] $\forall \mathcal{P}$

$\boxed{\delta(\mathcal{P}) \leq \frac{5}{11}} \approx 0.4545 \iff \frac{3}{11} \leq P[\ldots] \leq \frac{8}{11}$

$f(x) < f(y)$

Th [Kahn-Kim '95]

Sorting w/ $\mathcal{P}$ as partial information can be done in $O(\log e(\mathcal{P}))$ comparisons

main motivation

Th [Brightwell-Felsner-Trotter '95] $\forall \mathcal{P}$

$\delta(\mathcal{P}) \leq \frac{1}{\sqrt{5}} \approx 0.4472$

Conj [Kahn-Saks '84]

$\forall \{\mathcal{P}_n\}$, width$(\mathcal{P}_n) \to \infty$

we have $\delta(\mathcal{P}_n) \to 0$

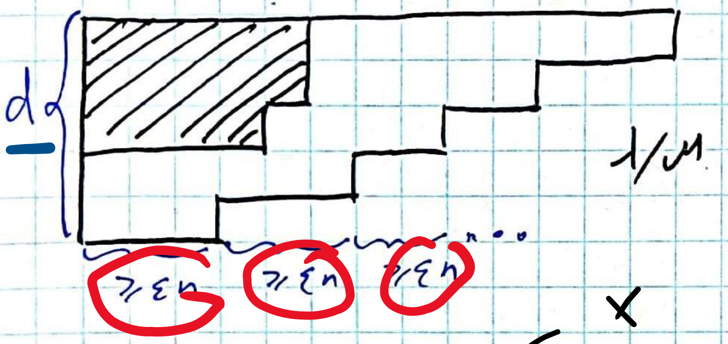
Case of Skew Young Diagrams

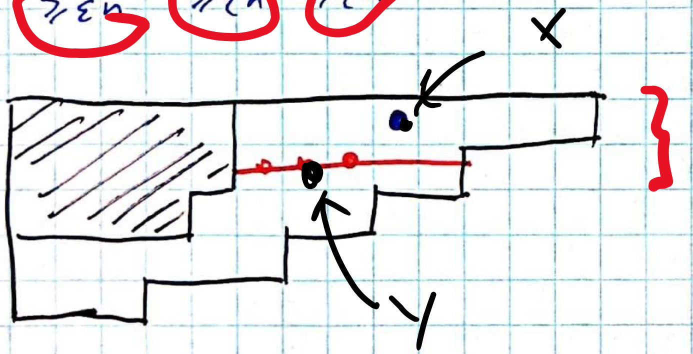
Th Let $\{\lambda/\mu,\ |\lambda|-|\mu|=n\}$ a seq of partitions

Suppose $\left\{\begin{array}{lll} \exists\ \varepsilon>0 & \text{s.t.} & \boxed{\lambda_i-\lambda_{i+1} > \varepsilon n} \\ \exists\ d & \text{s.t.} & \ell(\lambda)=d \ \leftarrow \lambda \text{ has } d \text{ parts} \end{array}\right\}$ fixed ε, d

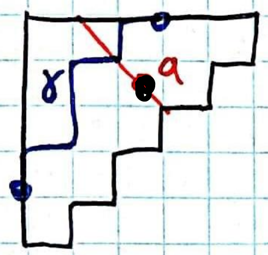
Then $\delta(\mathcal{T}_{\lambda/\mu}) = \boxed{O_{d,\varepsilon}\left(\frac{1}{\sqrt{n}}\right)}$ as $n\to\infty$

[Chan - P. - Panova '20+]



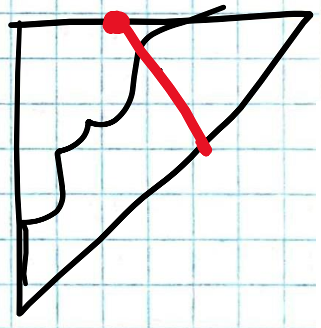


Proof idea

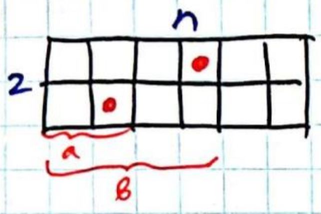


$\delta = P[\gamma \text{ above } a] - P[\gamma \text{ below } a]$

- asymptotics of Schur functions
- Naruse HLF after [Morales - P. - Panova '18]

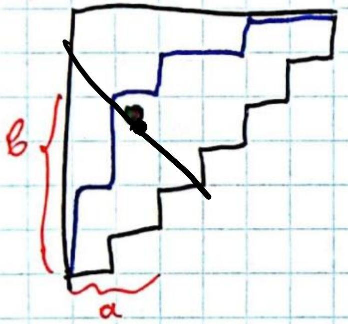


Catalan Posets



$\mathcal{P}_n$

$$e(\mathcal{P}_n) = \boxed{\frac{1}{n+1}\binom{2n}{n}}$$



Th [Chan - P. - Panova , 20+]

$$\delta(\mathcal{P}_n) = O\left(\frac{1}{n^{5/4}}\right)$$

$$O\left(\frac{1}{\sqrt{n}}\right)$$

Conj (—"—) $\forall \varepsilon > 0$

$$\delta(\mathcal{P}_n) = \Omega\left(\frac{1}{n^{5/4+\varepsilon}}\right)$$

$O(\sqrt{n})$

Conj

$$\delta(\mathcal{P}_\lambda) = O\left(\frac{1}{n}\right)$$

Inequalities for #LE's

Th [Stanley '81] $\mathcal{P}=(X,\prec)$, $x\in X$ fixed

$a(k) := \#\{f\in\mathcal{E}(\mathcal{P}) \text{ s.t. } f(x)=k\}$

Then $\boxed{a(k)^2 \geq a(k-1)\,a(k+1)}$

log-concavity of $\{a(k)\}$

Proof uses Alexandrov-Fenchel inequality

Th [Kahn-Saks '84] $\mathcal{P}=(X,\prec)$, $x,y\in X$ fixed

$a(k) := \#\{f\in\mathcal{E}(\mathcal{P}) \text{ s.t. } f(x)-f(y)=k\}$

Then $\boxed{a(k)^2 \geq a(k-1)\,a(k+1)}$

(—) Main Lemma in $\delta(\mathcal{P})\leq\frac{5}{11}$ thm

Conj [= cross-product conjecture, BFT '95]

$\mathcal{P}=(X,\prec)$, $x,y,z\in X$ fixed

$F(k,\ell) := \#\left\{f\in\mathcal{E}(\mathcal{P}) \text{ s.t. } \boxed{\begin{array}{l} f(x)-f(y)=k \\ f(y)-f(z)=\ell \end{array}}\right\}$

Then $F(k,\ell)\,F(k+1,\ell+1) \leq F(k,\ell+1)\,F(k+1,\ell)$

CPC

[BFT '95] proves CPC for $k=\ell=1$

Generalized Cross-Product Conjecture

Conj [Chan – P. – Panova, 21+] $= \text{GCPC}$

$\mathcal{P} = (X, \prec)$, $x, y, z \in X$ fixed

$$F(k,\ell) = \#\left\{ f \in \mathcal{E}(\mathcal{P}) \ : \ \begin{matrix} f(x) - f(y) = k \\ f(y) - f(z) = \ell \end{matrix} \right\}$$

Then
$$\boxed{F(k,\ell)\, F(k+i,\ell+j) \le F(k,\ell+j)\, F(k+i,\ell)}$$
$\forall\, i, j, k, \ell \ge 1$

Th [–1+] / GCPC has super powers /

(1) GCPC $\Rightarrow$ Kahn-Saks inequality $\Rightarrow$ Stanley inequality

(2) GCPC for posets of width 2 $\Rightarrow$ GYY inequality

/ GYY = Graham-Yao-Yao for posets of width 2 /

(3) GCPC $\Rightarrow$ XYZ inequality

Th [Shepp '82] $\forall\, x, y, z \in X$

$$\boxed{P[f(x) < f(y)] \le P[f(x) < f(y) \mid f(x) < f(z)]}$$

Our results on GCPC

Th [Chan-P.-Panova '21+]
GCPC holds for all posets of width two

Proof 1

Def (q-analogue) $\mathcal{P} = (X, \prec)$, width$(\mathcal{P}) = 2$

$X = C_1 \sqcup C_2$, $C_1 \cap C_2 = \emptyset$ chains in $\mathcal{P}$

$\forall f \in \mathcal{E}(\mathcal{P})$ let $\boxed{w(f) := \sum_{x \in C_2} f(x)}$

Define $\boxed{F_q(k,\ell) := \sum_{f} q^{w(f)}}$ $\quad q \leftarrow 1$

$F_1(k,\ell) = F(k,\ell)$

Th [—||—] $\Leftarrow$ q-CPC $\boxed{\forall \mathcal{P} \text{ width two}}$

$$F_q(k,\ell)\, F_q(k+1,\ell+1) \leqslant F_q(k,\ell+1)\, F_q(k+1,\ell)$$

where "$\leqslant$" is coeff-wise.

Proof?

More Conjectures

① q-GCPC ?

② ⊜ in GCPC $\Longleftrightarrow \begin{cases} F(k,\ell) = F(k,\ell+1) \\ F(k+1,\ell) = F(k+1,\ell+1) \end{cases}$ equality

or viceversa ⑪ , ⑪

③ $\mathcal{F} = \Big(F(k,\ell)\Big)_{\substack{k=1..n \\ \ell=n...1}}$ is non-negative definite

Thank you!