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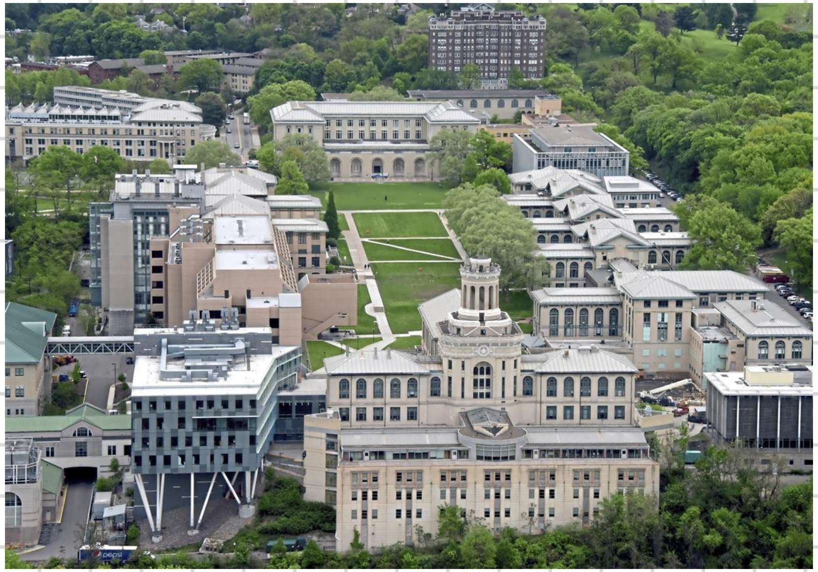
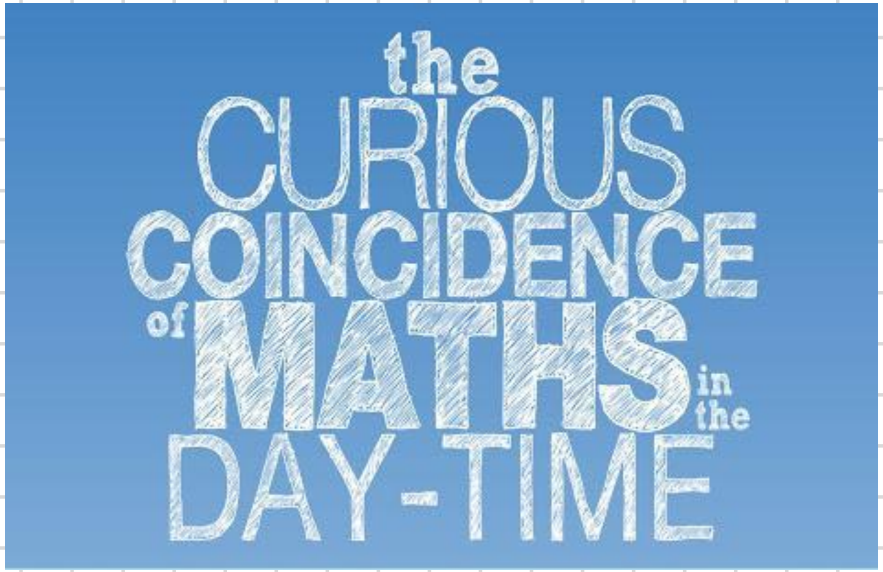
March 17, 2024

Combinatorics and computational complexity of counting coincidences

GSCC, CMU, Pittsburgh, PA



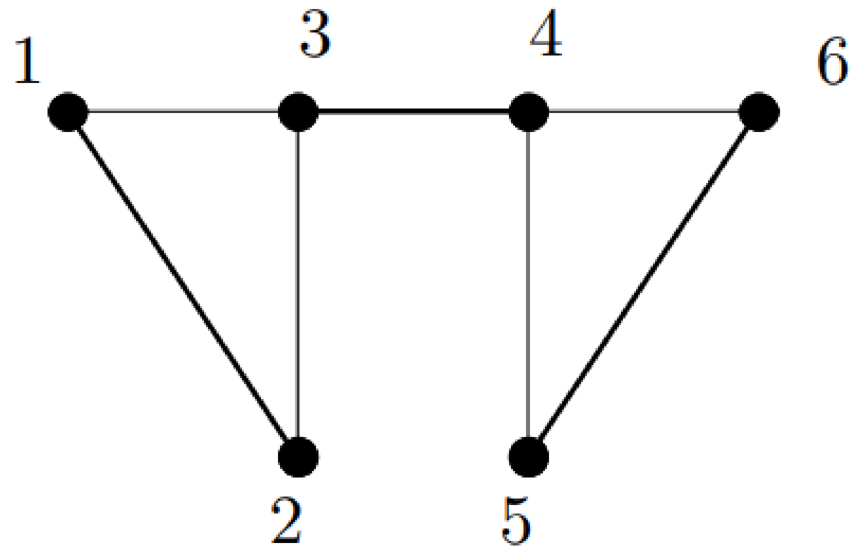


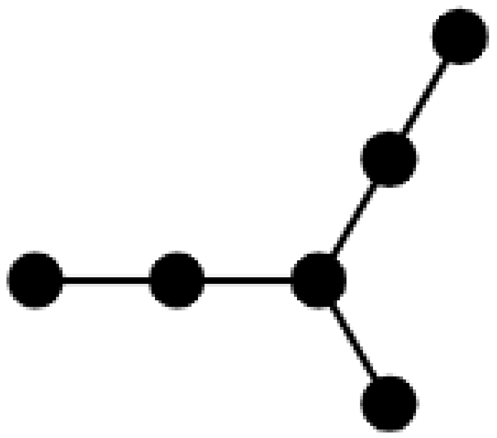
What’s a counting coincidence?

$$\pi^4 + \pi^5 \approx e^6$$

← Not that!

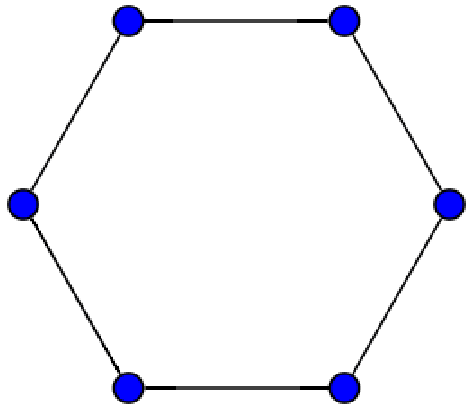
What's a counting coincidence?

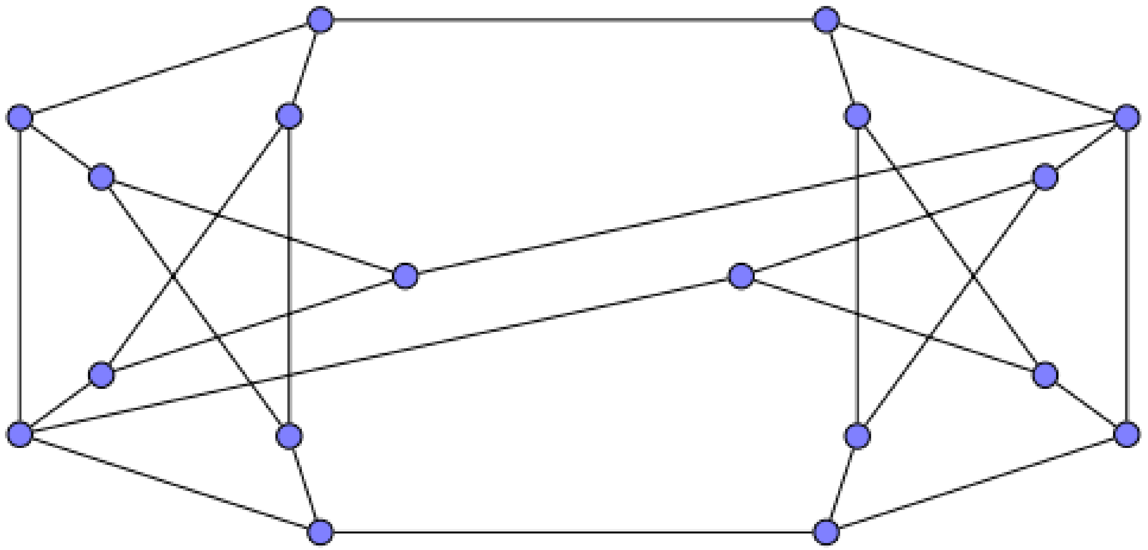




Two graphs with the same number of perfect matchings

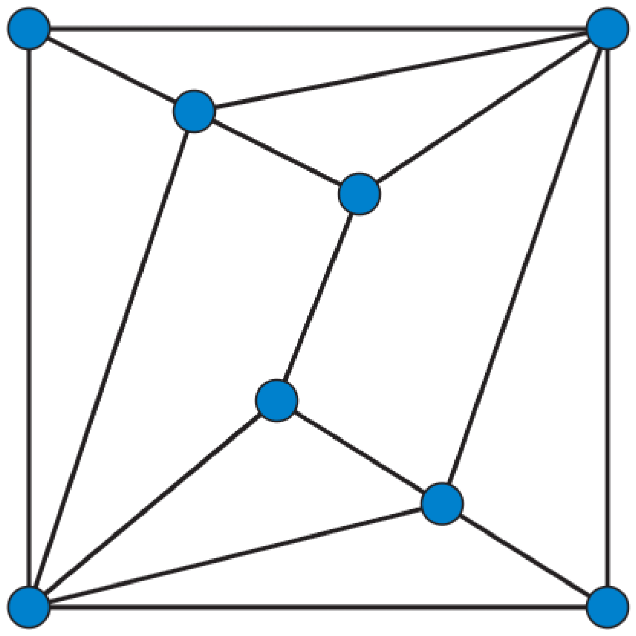
What’s a counting coincidence?

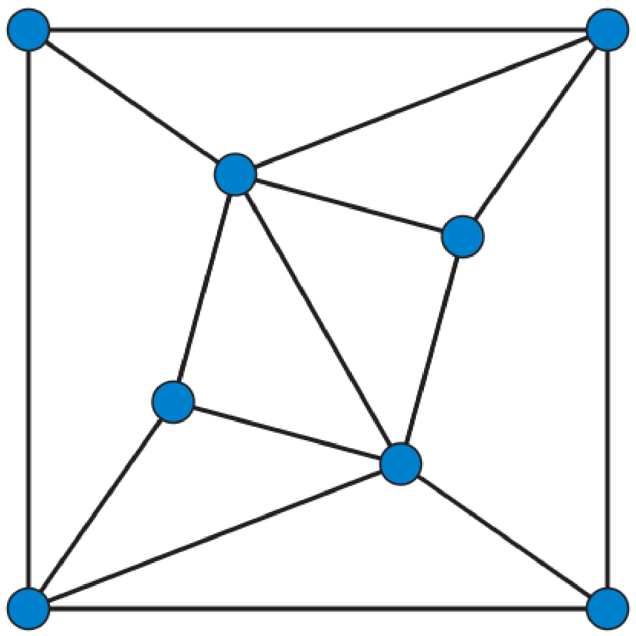




Two graphs with the same number of Hamiltonian cycles

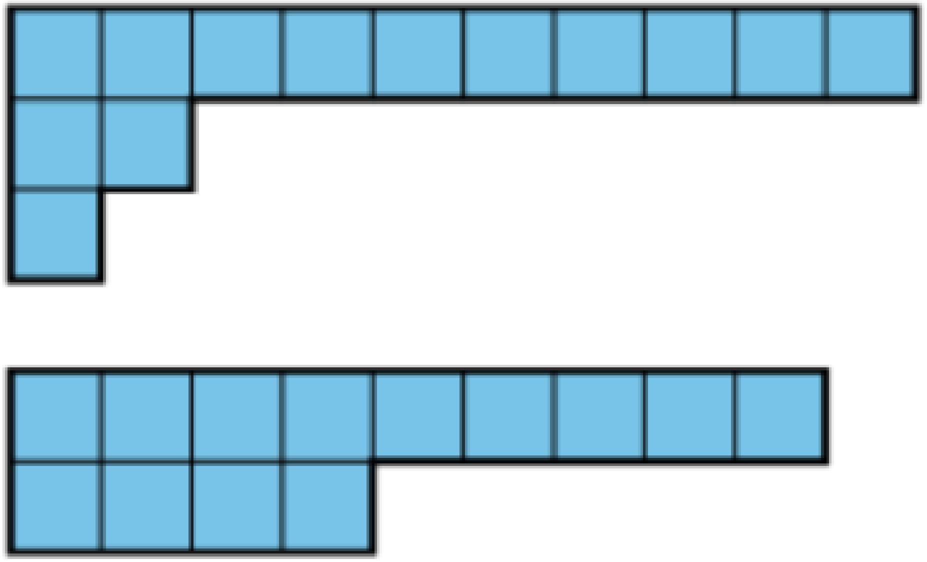
What’s a counting coincidence?

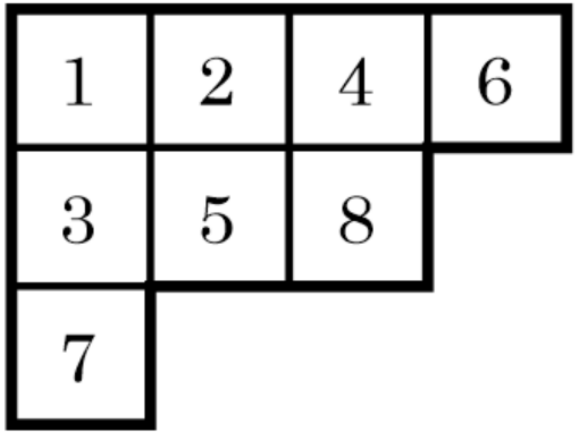




Two graphs with the same number of spanning trees

What’s a counting coincidence?





Two partitions with the same number of standard Young tableaux:

#SYT(10,2,1) = #SYT(9,4) = 429

Definition of counting coincidences

Let $f \in \#\mathrm{P}$ counting function. The *coincidence problem* for f is defined as:

$$\mathrm{C}_f := \{ f(x) =^? f(y) \}$$

Observation 1: $f \in \mathrm{FP} \implies \mathrm{C}_f \in \mathrm{P}$.

Observation 2: $\{ f(x) \geq 1 \} \in \mathrm{NP}\text{-complete} \implies \mathrm{C}_f \in \mathrm{coNP}\text{-hard}$.

Definition of counting coincidences

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Definition:

$$\mathrm{C}_=\mathrm{P} := \{f(x) =^? g(y) \; : \; f, g \in \#\mathrm{P}\}$$

Note:

$$\mathrm{C}_f \in \mathrm{C}_=\mathrm{P} \quad \textit{and} \quad \mathrm{C}_{\#3\mathrm{SAT}} \in \mathrm{C}_=\mathrm{P}\text{-complete}$$

Theorem:

$$\mathrm{C}_=\mathrm{P} \subseteq \mathrm{coNP} \implies \mathrm{PH} = \Sigma_2$$

Problems with parsimonious reductions

- number of Hamiltonian cycles in a graph
- number of 3-colorings of a planar graph
- number of π-patterns in a permutation $\sigma \in S_n$
- Kronecker coefficients $g(\lambda, \mu, \nu)$

In all these cases $f \in$ #P-complete via parsimonious reduction from #3SAT

Therefore, $\mathrm{C}_f \notin \mathrm{PH}$ unless PH collapses

Problems without parsimonious reductions

- number of perfect matchings in a simple bipartite graph
- number of independent sets in a planar graph
- number of order ideals of a poset
- number of linear extensions $e(P)$ of a poset
- number of bases of a rational matroid

In all these cases $f \in$ #P-complete

Theorem [Chan–P.'23]

For all f as above, $\mathrm{C}_f \notin \mathrm{PH}$ unless PH collapses

Proof idea: these functions are *concise*

Main Definition:

Let $X = \cup_n X_n$ be set of combinatorial objects.

Let $\mathcal{T}_f := \{f(x) : x \in X\}$ set of all values of f.

Function $f : X \to \mathbb{N}$ is *concise* if $\exists C, c > 0$, s.t.

for all $k \in \mathcal{T}_f$ there is $x \in X_n$ with $f(x) = k$ and $n < C(\log k)^c$.

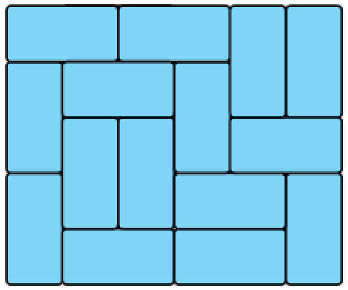
Main Lemma:

f is #P-complete and concise $\implies$ $\mathrm{C}_f \notin \mathrm{PH}$ unless PH collapses

Meta Lemma:

All functions f above are concise.

Concise functions all over the place



Let $\Gamma \subset \mathbb{Z}^2$ be a finite region of area $2n$, $\tau(\Gamma)$ number of domino tilings of Γ.

Let $\mathcal{T}(n)$ the set of numbers of domino tilings over all regions of size $2n$:

$$\mathcal{T}(n) \ := \ \{\tau(\Gamma), \ \text{ where } \ \Gamma \subset \mathbb{Z}^2, \ |\Gamma| = 2n\} \ \subseteq \ \{0, 1, \ldots, 4^n\}$$

Theorem [Chan–P'23] $(\Rightarrow \tau$ is concise$)$

There is a constant $c > 1$, such that $\mathcal{T}(n) \supseteq \{0, 1, \ldots, c^n\}$, for all $n \geq 1$.

Proof idea

Denote by $\mathcal{D}(a,b)$ the set of regions Γ such that $\tau(\Gamma) = a$ and $\tau(\Gamma - x - y) = b$

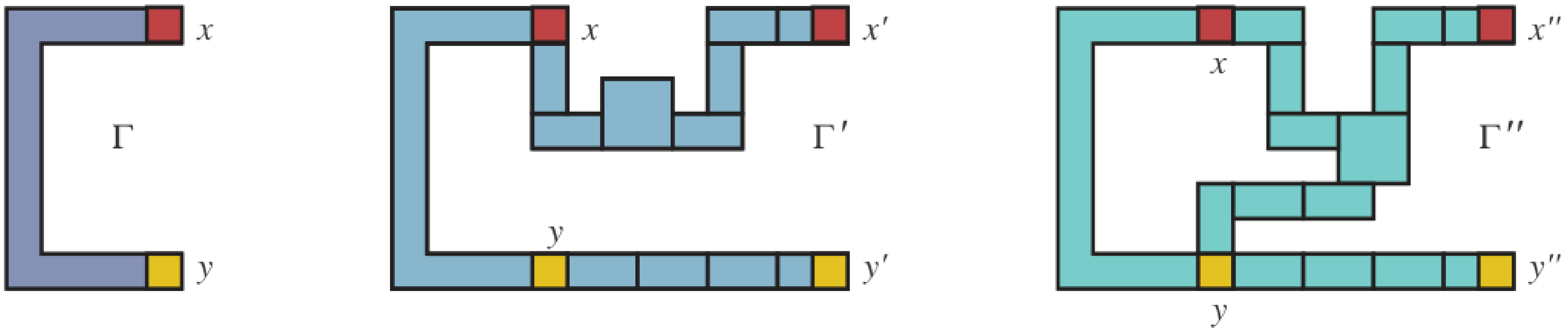


FIGURE 3.1. Region $\Gamma \in \mathcal{D}(1,1)$, and two transformations $\Gamma \in \mathcal{D}(a,1) \Rightarrow \Gamma' \in \mathcal{D}(2a,1)$, and $\Gamma \in \mathcal{D}(a,1) \Rightarrow \Gamma'' \in \mathcal{D}(2a+1,1)$.

Note: τ is concise on *simply-connected regions*. Proof is *much* harder.

Conjecture: The exponential bound in the theorem still holds.

Concise functions all over the place

1	2	4	6
3	5	8	
7			

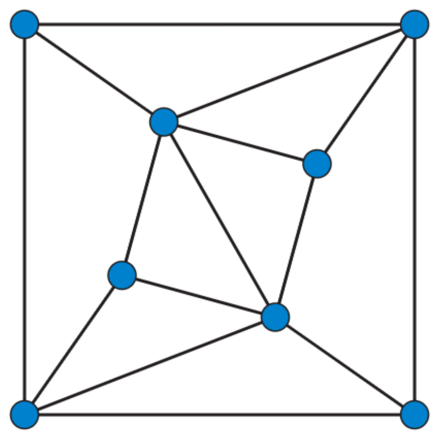
Theorem [Kravitz–Sah'21] ($\Rightarrow e$ is concise)

Let $\mathcal{T}_e(n)$ the set of numbers of linear extensions of all posets of size n. Then:

$$\mathcal{T}_e(n) \supseteq \{1, \ldots, c^{n/(\log n)}\} \quad \text{for some} \quad c > 1.$$

Conjecture: e is concise on posets of height 2.

Concise functions all over the place



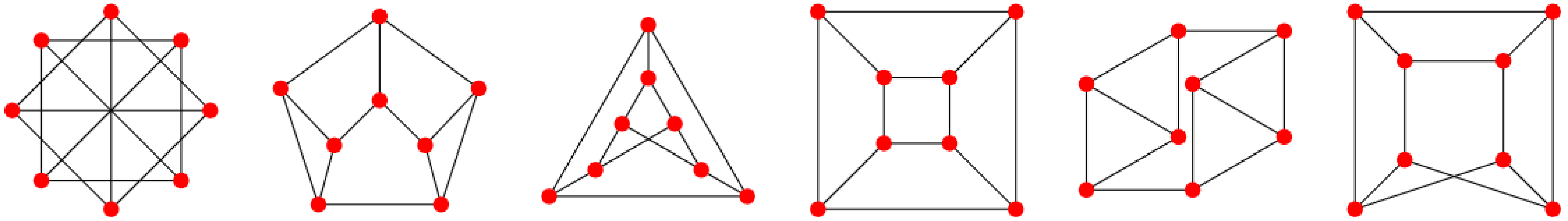
Theorem [Stong'22] ($\Rightarrow$ number of spanning trees is concise)

Let $\mathcal{T}_s(n)$ the set of numbers of spanning trees of all graphs with n vertices. Then:

$$\mathcal{T}_s(n) \supseteq \{0, \ldots, c^{n^{2/3}}\} \quad \text{for some} \quad c > 1.$$

Conjecture [Azarija–Škrekovski'12]: $\mathcal{T}_s(n) \supseteq \{0, \ldots, c^n\}$.

Concise functions all over the place



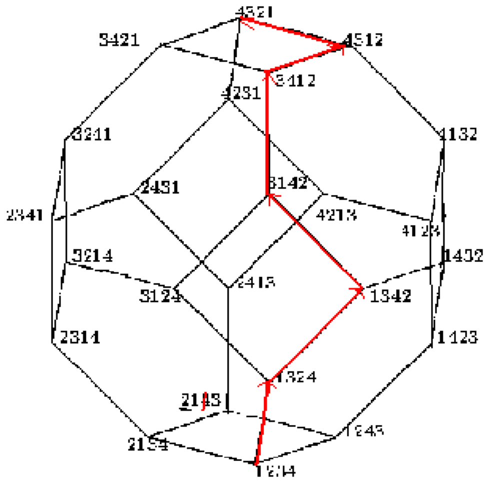
Definition:

Let $\boldsymbol{d} = (d_1, \ldots, d_n) \in \mathbb{N}^n$, s.t. $0 \leq d_1, \ldots, d_n \leq n-1$.

Denote by $g(\boldsymbol{d})$ the number of simple graphs with *degree sequence* $\boldsymbol{d}$.

Conjecture: Function g is concise (in unary).

Concise functions all over the place



Definition:

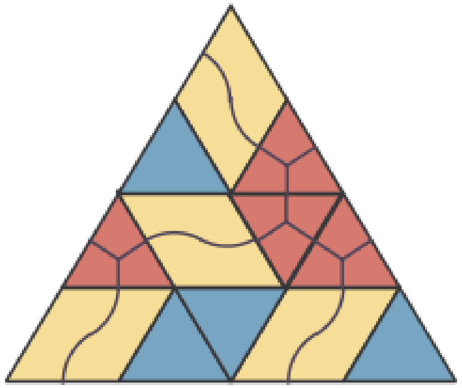
Let $\sigma \in S_n$. A *reduced factorization* is a shortest product $\sigma = (i_1, i_1 + 1) \cdots (i_\ell, i_\ell + 1)$

Denote by $\mathrm{red}(\sigma)$ the number of reduced factorizations of σ, for example:

$$\mathrm{red}(2, 1, n, 3, 4, \ldots, n-1) \;=\; n-2 \quad \text{for} \quad n \geq 3.$$

Conjecture: red is concise.

Concise functions all over the place



$$s_\mu \cdot s_\nu = \sum_\lambda c^\lambda_{\mu\nu} \, s_\lambda$$

Conjecture:

Littlewood–Richardson coefficients $c^\lambda_{\mu\nu}$ is a concise function (in unary).

$$c^3_{1,1^2} = 0, \quad c^2_{1,1} = 1 \quad \text{and} \quad c^{k(321)}_{k(21),k(21)} = k+1 \ \text{ for all } \ k \geq 1$$

Thank you!