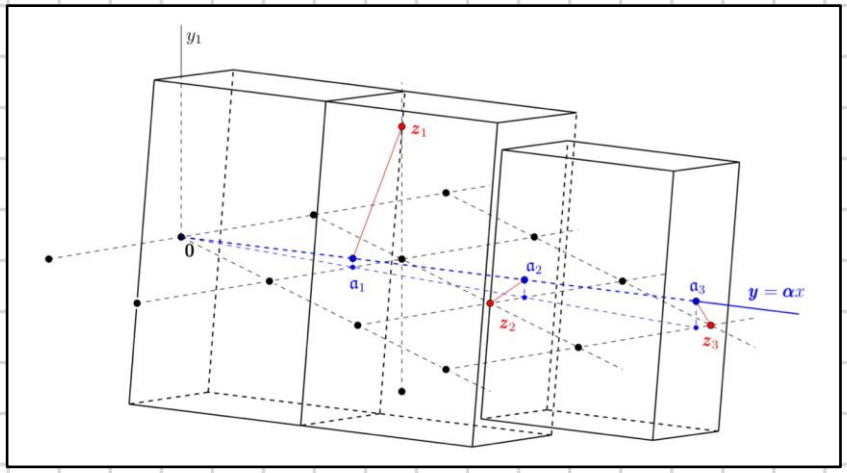
Igor Pak, UCLA

January 30, 2025

Counting trees and matchings via continuous fractions

Diophantine Approximation and Related Topics Online Seminar

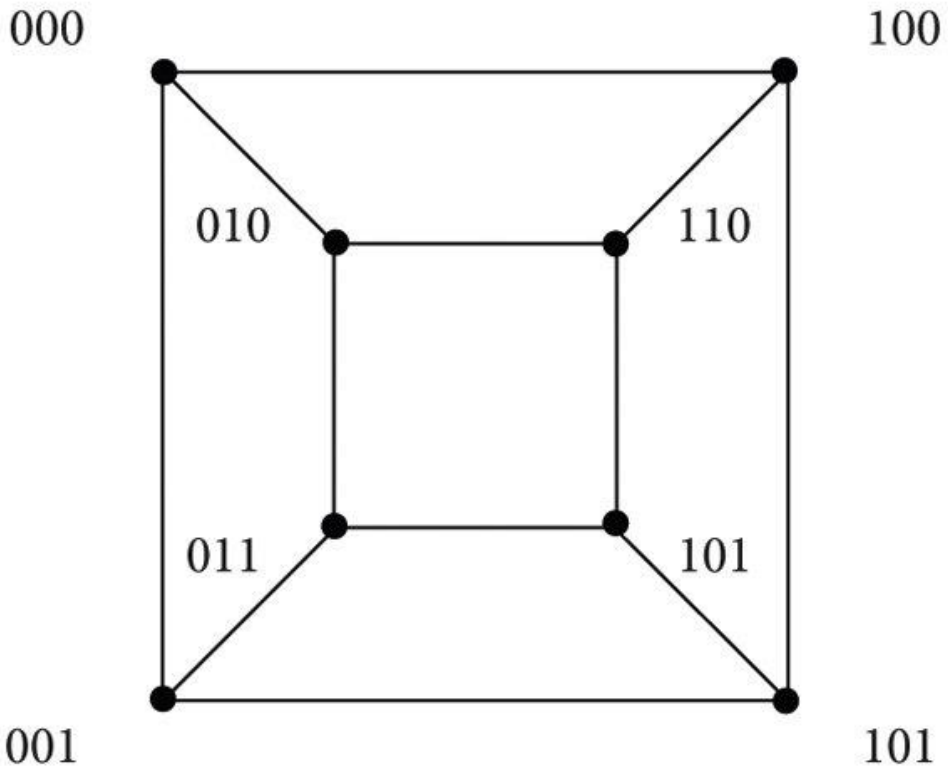


What is a counting problem?

x ← *combinatorial object*

f(x) := # *combinatorial objects indexed by* *x*

What is a counting problem?

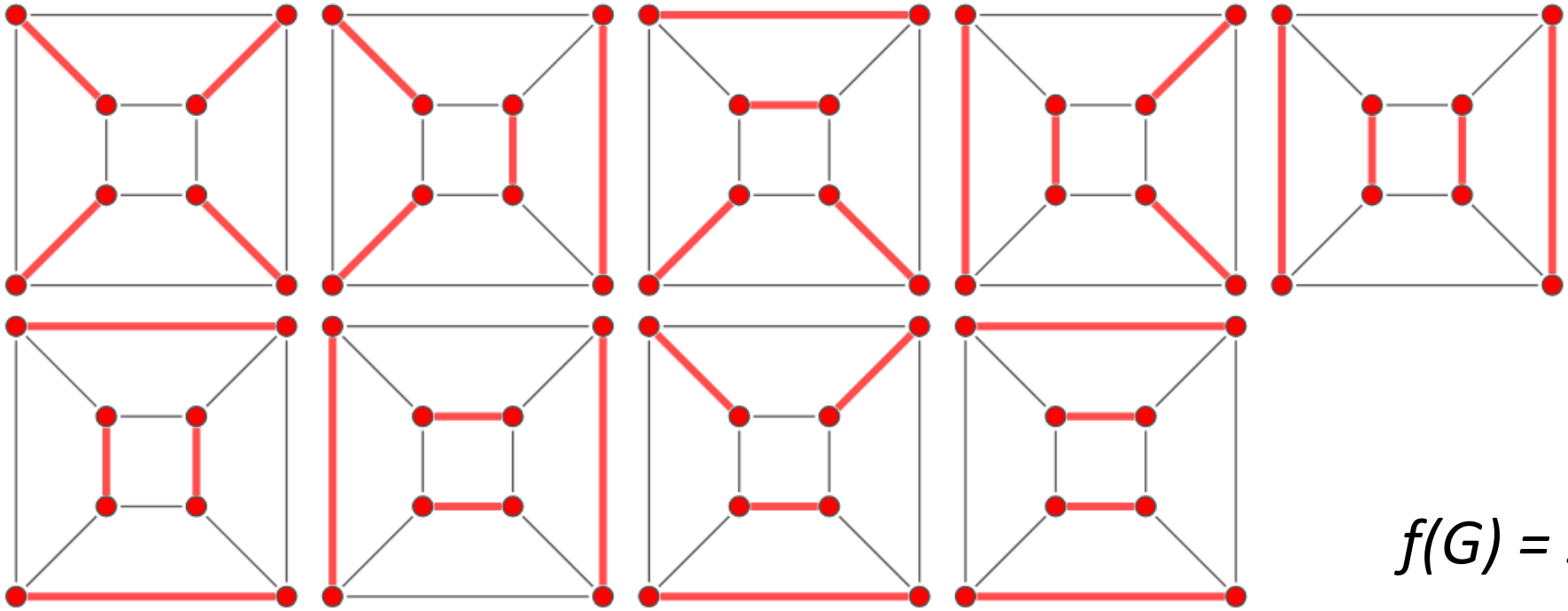


$G=(V,E)$ graph on $n=2k$ vertices

Perfect matching – *collection of k pairwise disjoint edges of G*

f(G) := the number of perfect matchings in graph G

What is a counting problem?



f(G) = 9

f(G) := the number of perfect matchings in graph G

What is a counting problem?

f(d-cube) =

1

2

9

272

589185

16332454526976

391689748492473664721077609089

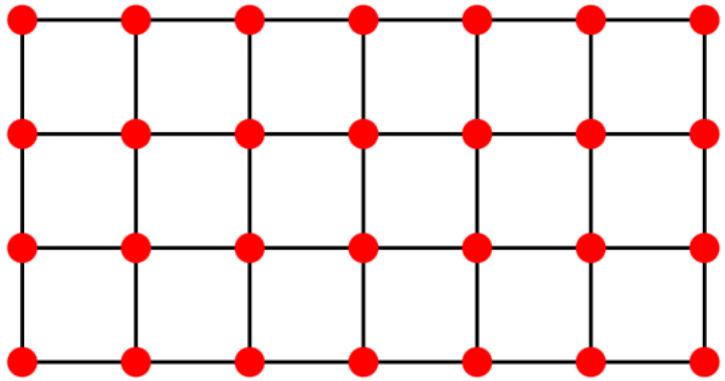
factor	391 689 748 492 473 664 721 077 609 089

Result

$3 \times 7^2 \times 34981 \times 515929 \times 255681973 \times 577434331$ (7 prime

f(G) := the number of perfect matchings in graph G

What is a counting problem?

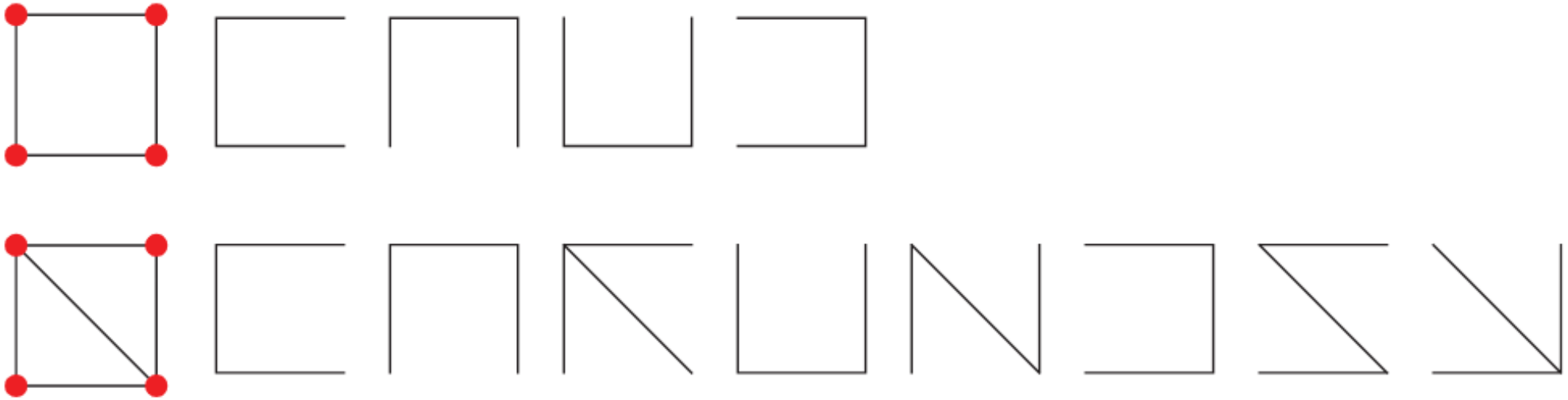


$G=(V,E)$ graph on n vertices

Spanning tree – collection of n-1 edges of G with no cycles

f(G) := the number of spanning trees in graph G

What is a counting problem?



f(G) := the number of spanning trees in graph G

What is a counting problem?

A007341 Number of spanning trees in n X n grid.
(Formerly M3721)

1, 4, 192, 100352, 557568000, 32565539635200, 198723693018409861 12,
126231322912498539682594816, 832662766169181854512184490039705 6,
569431900407909779595721572576532837171200 0,
403250217214041185132768595134976792491836235935907 84,
295454099395278800622876498708444322681581419009948478603264000 0

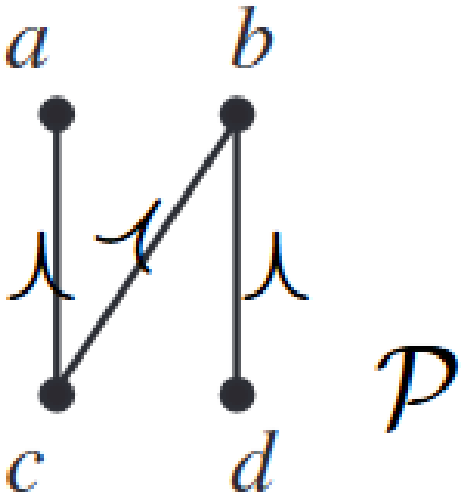
factor 2 954 540 993 952 788 006 228 764 987 084 443 226 815 814 190 ·. 099 484 786 032 640 000

Result

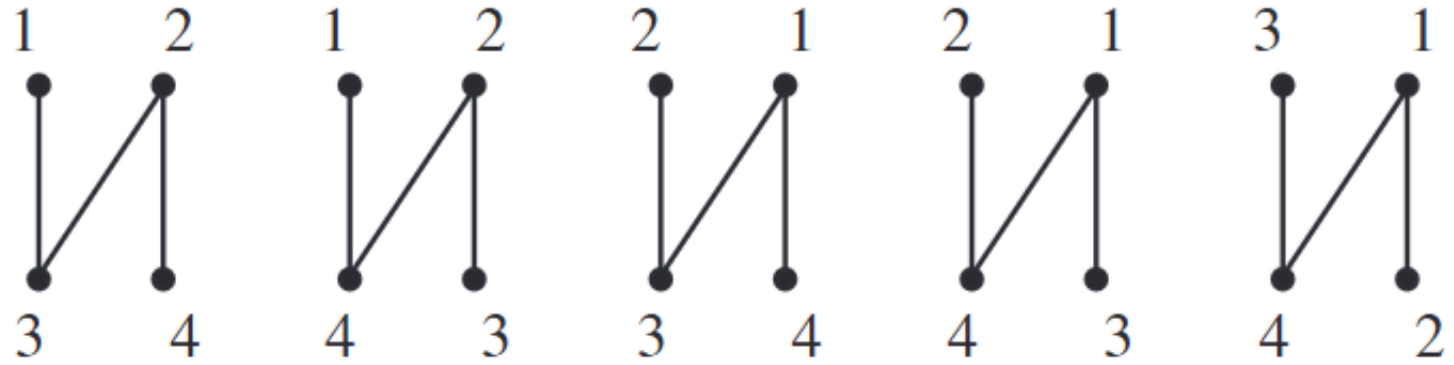
$2^{51} \times 3^5 \times 5^4 \times 7^6 \times 11^2 \times 13^2 \times 23^6 \times 71^2 \times 73^2 \times 97^2 \times 193^4 \times 263^2$

f(G) := the number of spanning trees in graph G

What is a counting problem?

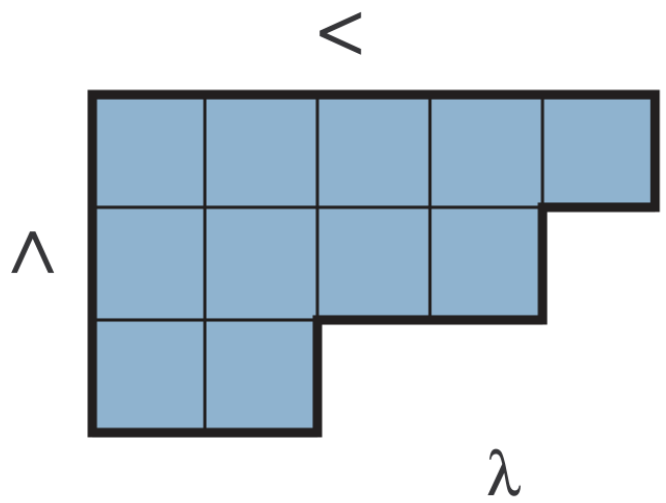


f(P) = 5



f(P) := the number of linear extensions in poset P

What is a counting problem?



1	2	4	5	10
3	6	9	11	
7	8			

f(λ) = number of standard Young tableaux of shape λ

Inverse counting problem

Computational Complexity

Fix $f : X \rightarrow \{0,1,2,\ldots\}$

Given k find a combinatorial object x in X of smallest size, such that f(x) = k.

Main Definition:

Let $X = \cup_n X_n$ be set of combinatorial objects.

Let $\mathcal{T}_f := \{f(x) : x \in X\}$ set of all values of f.

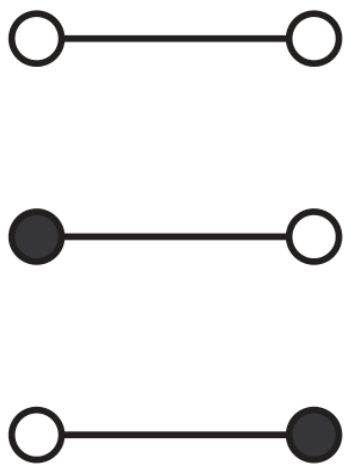
Function $f : X \to \mathbb{N}$ is *concise* if $\exists C, c > 0$, s.t.

for all $k \in \mathcal{T}_f$ there is $x \in X_n$ with $f(x) = k$ and $n < C(\log k)^c$.

Easy example: independent sets

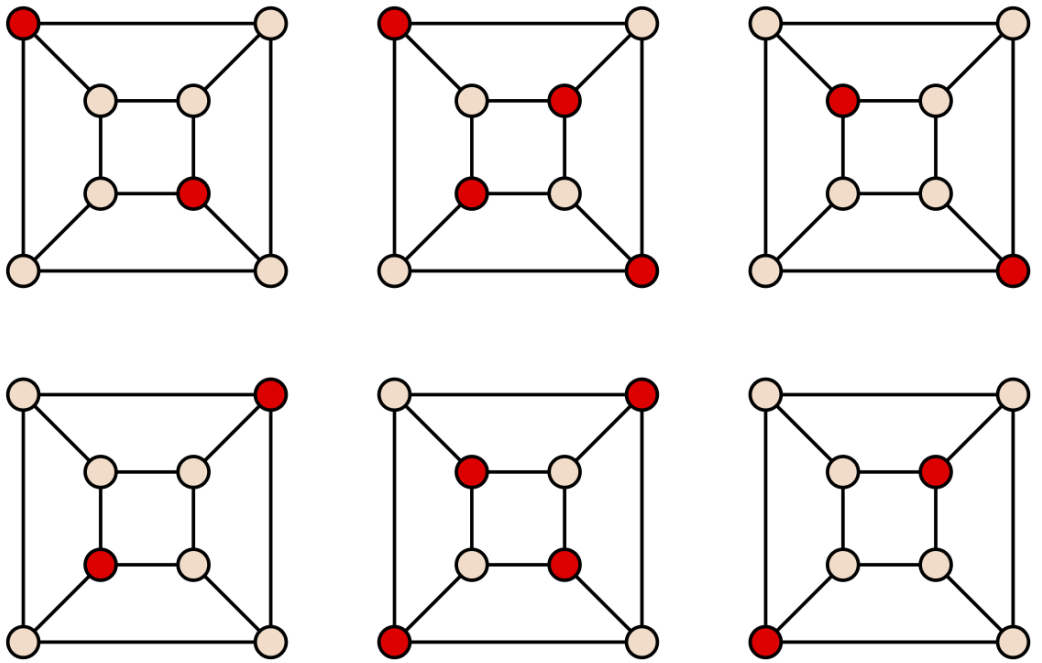
Let $f : G=(V,E) \rightarrow$ *number of independent sets in G*

independent set := *subset of nonadjacent vertices in graph G*



e

$f(e) = 3$



Easy example: independent sets

Let $f : G=(V,E) \rightarrow$ *number of independent sets in G*

independent set := *subset of nonadjacent vertices in graph G*

$$\mathcal{T}(n) := \{ f(G), \text{ where } G = (V, E), \ |V| \leq n \} \subseteq \{2, \ldots, 2^n\}$$

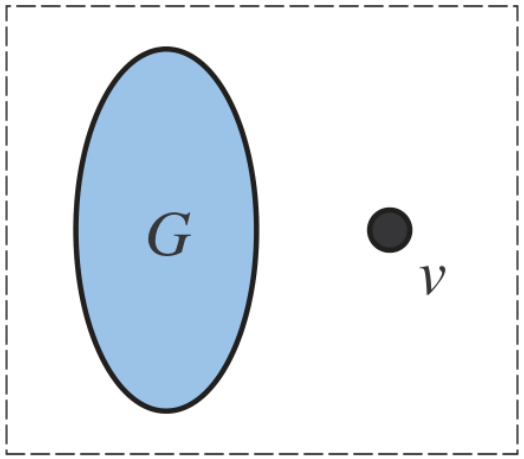
Theorem [easy] $(\Rightarrow f$ is concise$)$

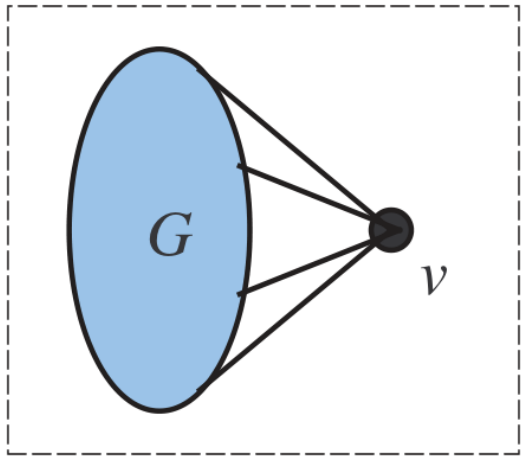
We have $\mathcal{T}(n) \supseteq \{2, 3, \ldots, 2^n\}$, for all $n \geq 1$.

Easy example: independent sets

Let $f : G=(V,E) \rightarrow$ *number of independent sets in G*

independent set := *subset of nonadjacent vertices in graph G*

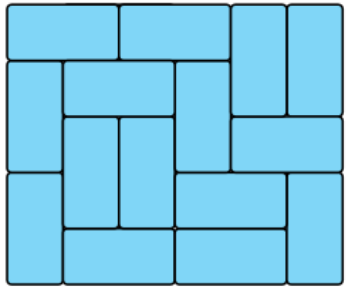




$f(G + v) = 2\, f(G)$

$f(G \times v) = f(G) + 1$

Domino tilings



Let $\Gamma \subset \mathbb{Z}^2$ be a finite region of area $2n$, $\tau(\Gamma)$ number of domino tilings of Γ.

Let $\mathcal{T}(n)$ the set of numbers of domino tilings over all regions of size $2n$:

$$\mathcal{T}(n) := \{\tau(\Gamma), \text{ where } \Gamma \subset \mathbb{Z}^2, |\Gamma| = 2n\} \subseteq \{0, 1, \ldots, 4^n\}$$

Theorem [Chan–P'23] $(\Rightarrow \tau$ is concise$)$

There is a constant $c > 1$, such that $\mathcal{T}(n) \supseteq \{0, 1, \ldots, c^n\}$, for all $n \geq 1$.

Proof idea

Denote by $\mathcal{D}(a,b)$ the set of regions Γ such that $\tau(\Gamma) = a$ and $\tau(\Gamma - x - y) = b$

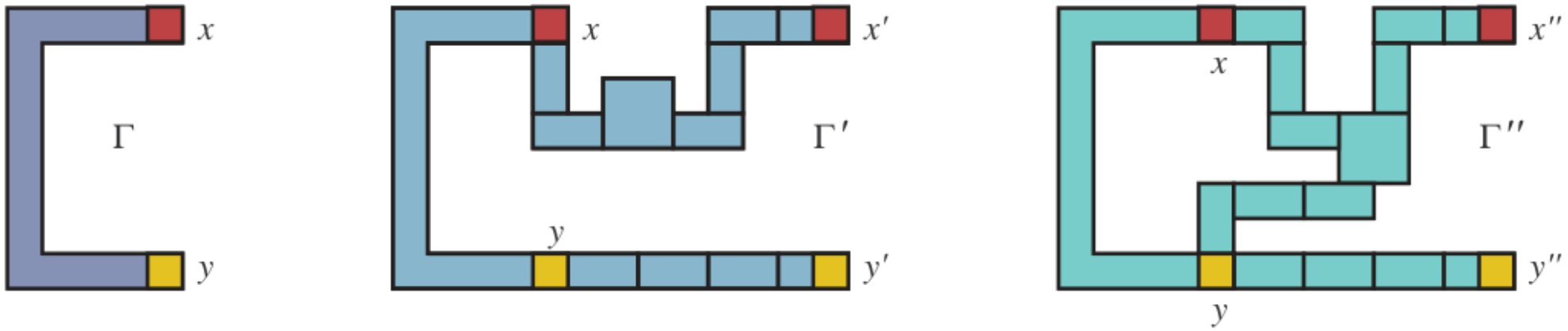
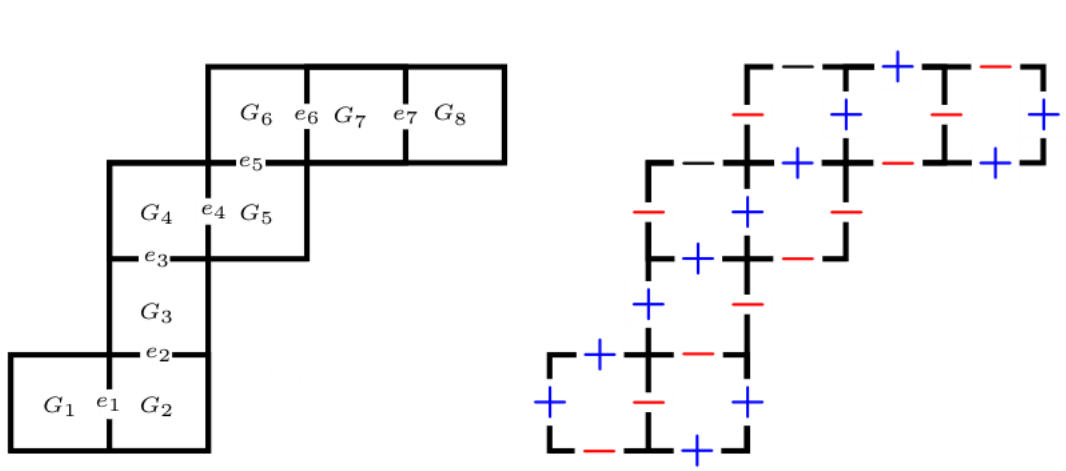


FIGURE 3.1. Region $\Gamma \in \mathcal{D}(1,1)$, and two transformations $\Gamma \in \mathcal{D}(a,1) \Rightarrow \Gamma' \in \mathcal{D}(2a,1)$, and $\Gamma \in \mathcal{D}(a,1) \Rightarrow \Gamma'' \in \mathcal{D}(2a+1,1)$.

Note: τ is concise on *simply-connected regions*. Proof is *much* harder.

Conjecture: The exponential bound in the theorem still holds.

Perfect matchings of snake regions



sign sequence = (+ − − + + + + − −)

$$(a_1, a_2, a_3, a_4) = (1, 2, 4, 2)$$

$$[a_1, a_2, \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

$$1 + \cfrac{1}{2 + \cfrac{1}{4 + \cfrac{1}{2}}} = \frac{29}{20}$$

$$\frac{\#\mathrm{PM}(a_1, \ldots, a_n)}{\#\mathrm{PM}(a_2, \ldots, a_n)} = [a_1, \ldots, a_n]$$

Perfect matching problems in cluster algebras and number theory

Ralf Schiffler

Perfect matchings of snake regions

$$\frac{\#\mathrm{PM}(a_1,\ldots,a_n)}{\#\mathrm{PM}(a_2,\ldots,a_n)} = [a_1,\ldots,a_n]$$

$$[a_1,a_2,\ldots,a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}$$

Observation [Chan-P.'24] *Zaremba's conjecture implies*

$\mathcal{T}_{\mathrm{snake}}(n) \supseteq \{0,1,2,\ldots,c^n\}$ *for some* $c>1$ *and all* $n \geq 1$.

Conjecture 1.13 (*Zaremba's Conjecture* [Zar72, p. 76]). *There is a universal constant* $A>0$ *so that, for every integer* $u \geq 1$, *there is a coprime integer* $1 \leq t < u$, *such that* $t/u = [a_1,\ldots,a_\ell]$ *and* $a_1,\ldots,a_\ell \leq A$.

Perfect matchings of snake regions

Theorem [Chan-P.'24] *There is* $c > 1$, *such that*

$$\mathcal{T}_{\text{snake}}(n) \supseteq \left\{0, 1, \ldots, c^{n/(\log n)}\right\}, \text{ for all } n \geq 1.$$

Lemma [Larcher'86] *For every integer* $d \geq 1$, *there exists an integer* $1 \leq c < d$, *such that* $\gcd(c, d) = 1$, $c/d = [0, b_1, \ldots, b_m]$ *and*

$$b_1 + \ldots + b_m \leq C \frac{d}{\phi(d)} \log d \log\log d,$$

where $\phi(n)$ *is Euler's totient function, and* $C > 0$ *is a universal constant.*

Perfect matchings of snake regions

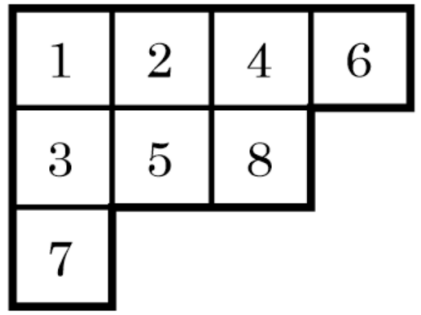
Theorem [Chan-P.'24] *There is* $c > 1$, *such that*

$\mathcal{T}_{\text{snake}}(n) \supseteq \{0, 1, \ldots, c^{n/(\log n)}\}$, *for all* $n \geq 1$.

Conjecture: [Chan-P.'24]

$\mathcal{T}_{\text{s.c.}}(n) \supseteq \{0, 1, 2, \ldots, c^n\}$ *for some* $c > 1$ *and all* $n \geq 1$.

Linear extensions



Theorem [Tenner'09] $\mathcal{T}_e(n) \supseteq \{1, \ldots, cn^2\}$ for some $c > 1$.

Theorem [Kravitz–Sah'21] $(\Rightarrow e$ is concise$)$

Let $\mathcal{T}_e(n)$ the set of numbers of linear extensions of all posets of size n. Then:

$$\mathcal{T}_e(n) \supseteq \{1, \ldots, c^{n/(\log n)}\} \quad \text{for some} \quad c > 1.$$

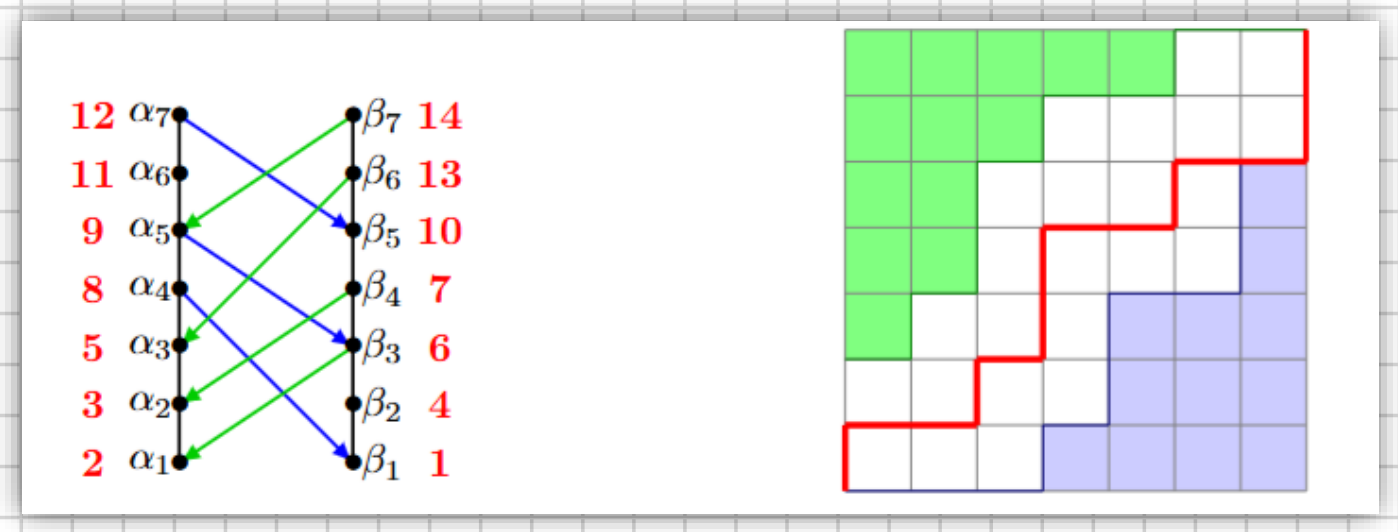
Linear Extension Numbers of *n*-Element Posets

Noah Kravitz[1] · Ashwin Sah[2]

Linear extensions

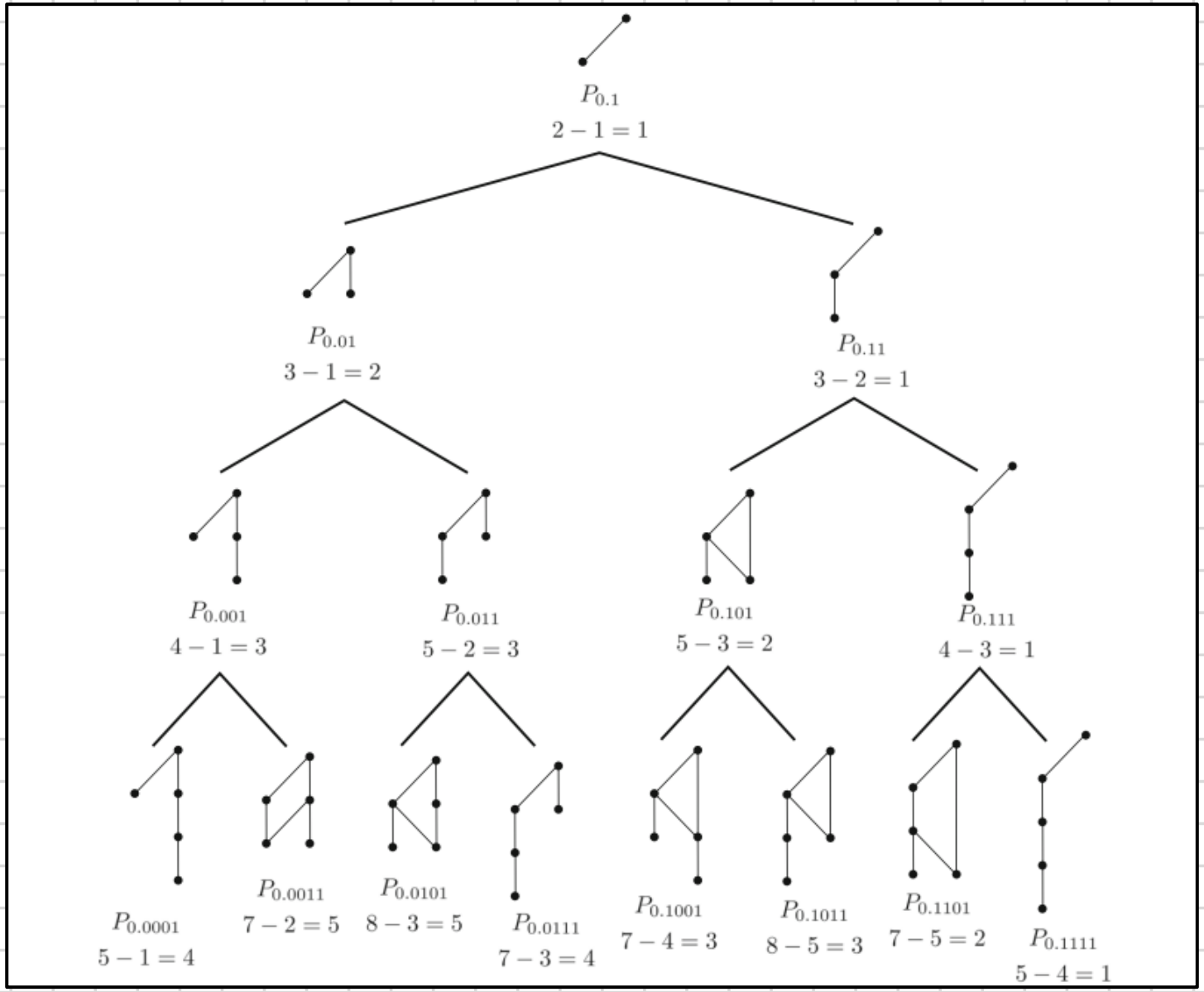
Lemma [Kravitz–Sah]:

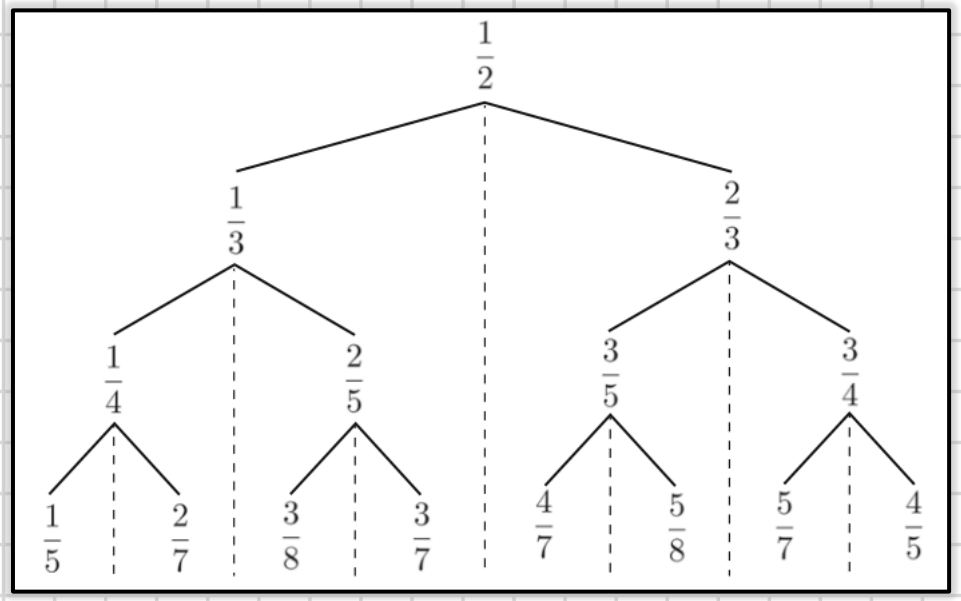
recursive construction of width two posets $P = P(a_1, \ldots, a_m)$ *such that* $[a_1, \ldots, a_m] = e(P)/e(P - x)$ *for some maximal element* x *in* P.



+ Larcher's lemma implies KS theorem

Proof idea of KS Lemma: *Stern-Brocot tree*





Calkin-Wilf tree

Linear extensions

Conjecture: [Kravitz-Sah'21]

$\mathcal{T}_e(n) \supseteq \{1, 2, \ldots, c^n\}$ *for some* $c > 1$ *and all* $n \geq 1$.

Observation [Chan-P.'24] *Zaremba's conjecture implies KS Conjecture.*

Theorem [Chan-P.'24] *For some* $c > 1$, *we have* $|\mathcal{T}_e(n)| > c^n$. *Moreover,*

$$\frac{1}{c^n}\left|\mathcal{T}_e(n) \cap \{1, 2, \ldots, c^n\}\right| \to 1 \quad as \quad n \to \infty.$$

Linear extensions

Theorem [Chan-P.'24] *For some $c > 1$, we have $|\mathcal{T}_e(n)| > c^n$. Moreover,*

$$\frac{1}{c^n}\left|\mathcal{T}_e(n) \cap \{1, 2, \ldots, c^n\}\right| \to 1 \quad as \quad n \to \infty.$$

On Zaremba's conjecture

By Jean Bourgain and Alex Kontorovich

Theorem [Bourgain–Kontorovich'14]

Almost every positive integer is the denominator which satisfies Zaremba's conjecture.

Linear extensions

Let $m \geq 0$, $a_1, \ldots, a_m \in \mathbb{P}$, $b_0, \ldots, b_m \in \mathbb{P}$. A *generalized continued fraction*

$$[a_1, \ldots, a_m; b_0, \ldots, b_m] := b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_m}{b_m}}}}.$$

$$G(a_1, \ldots, a_m; b_0, \ldots, b_m) := (b_0 + \cdots + b_m) - (a_1 + \cdots + a_m) + m,$$

$$\frac{20}{7} = 2 + \frac{1}{1 + \frac{1}{6}} = 2 + \frac{2}{2 + \frac{1}{3}}$$

G(2,1;2,2,3) = (2+2+3)-(2+1)+2 = 6
< 2+1+6 = 9

Linear extensions

Let $m \geq 0$, $a_1, \ldots, a_m \in \mathbb{P}$, $b_0, \ldots, b_m \in \mathbb{P}$. A *generalized continued fraction*

$$[a_1, \ldots, a_m; b_0, \ldots, b_m] := b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots + \cfrac{a_m}{b_m}}}}.$$

$$G(a_1, \ldots, a_m; b_0, \ldots, b_m) := (b_0 + \cdots + b_m) - (a_1 + \cdots + a_m) + m,$$

For $\alpha \in \mathbb{Q}_{\geq 1}$, define

$$g(\alpha) := \min\{ G(a_1, \ldots, a_m; b_0, \ldots, b_m) : [a_1, \ldots, a_m; b_0, \ldots, b_m] = \alpha \},$$

Conjecture [Chan-P.'24]

For every prime b, there is an integer $1 \leq a < b$, such that $g\left(\frac{a}{b}\right) \leq C \log b$, where $C > 0$ is a universal constant.

Linear extensions

Conjecture [Chan-P.'24]

For every prime b, there is an integer $1 \le a < b$, such that $g\left(\frac{a}{b}\right) \le C \log b$, where $C > 0$ is a universal constant.

Theorem [Chan-P.'24]

CP Conjecture implies KS Conjecture.

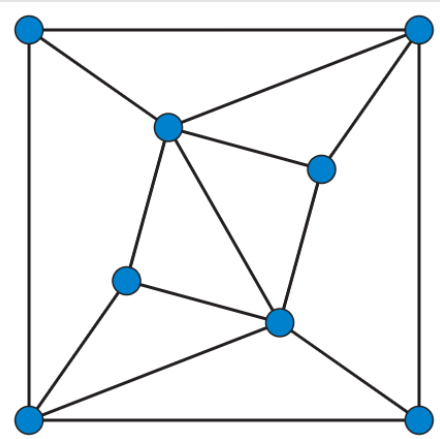
Linear extensions and continued fractions

Swee Hong Chan [a], Igor Pak [b]

Spanning trees

Sedláček's First Problem (1966)

Describe the set of spanning tree numbers



$$\mathcal{T}_s(5) = \{0, 1, 3, 4, 5, 8, 9, 11, 12, 16, 20, 21, 24, 40, 45, 75, 125\}$$

ON THE SPANNING TREES OF FINITE GRAPHS

Jiří Sedláček, Praha

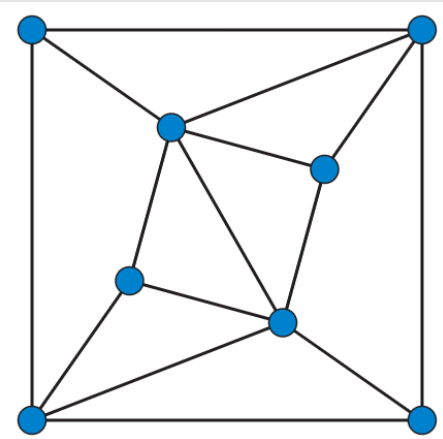
ОБ ОСНОВАХ КОНЕЧНЫХ ГРАФОВ

ЙИРЖИ СЕДЛАЧЕК (Jiří Sedláček), Прага

O KOSTRÁCH KONEČNÝCH GRAFŮ

Jiří Sedláček. Praha

Spanning trees



Sedláček's First Problem (1966)

Describe the set of spanning tree numbers

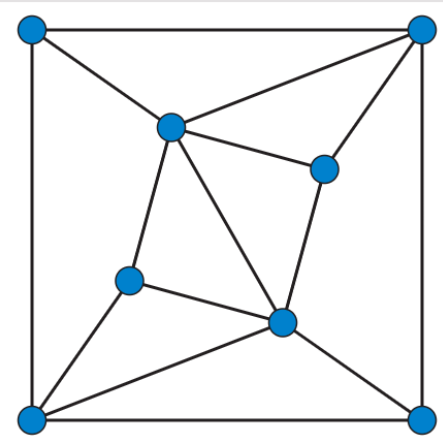
Theorem [Stong'22] $(\Rightarrow$ number of spanning trees is concise$)$

Let $\mathcal{T}_s(n)$ the set of numbers of spanning trees of all graphs with n vertices. Then:

$$\mathcal{T}_s(n) \supseteq \{0, 1, 3, 4, \ldots, c^{n^{2/3}}\} \quad \text{for some} \quad c > 1.$$

Conjecture [Sedláček'67, Azarija–Škrekovski'12]: $\mathcal{T}_s(n) \supseteq \{3, \ldots, c^n\}$ for some $c > 1$.

Spanning trees



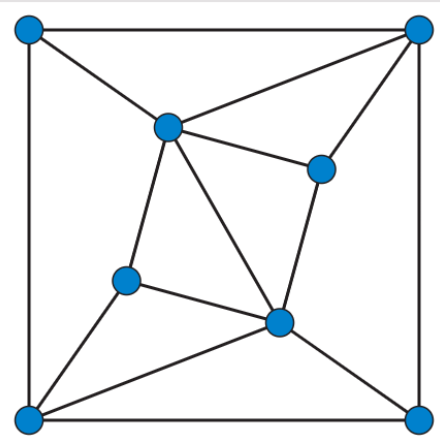
Sedláček's Second Problem (1966)
Describe the set of spanning tree numbers of planar graphs.

Conjecture [Sedláček'67]: $|\mathcal{T}_{sp}(n)| \geq c^n$ for some $c > 1$.

Note: This is best possible since # planar graphs with n vertices is $O(30.061^n)$.

Additionally, the maximal number of spanning trees of a planar graph is $O(5.2852^n)$.

Spanning trees



Conjecture [Sedláček'67]: $|\mathcal{T}_{sp}(n)| \geq c^n$ for some $c > 1$.

Theorem [Chan-Kontorovich-P.'24]

For some $c > 1$, we have $|\mathcal{T}_{sp}(n)| > c^n$. Moreover,

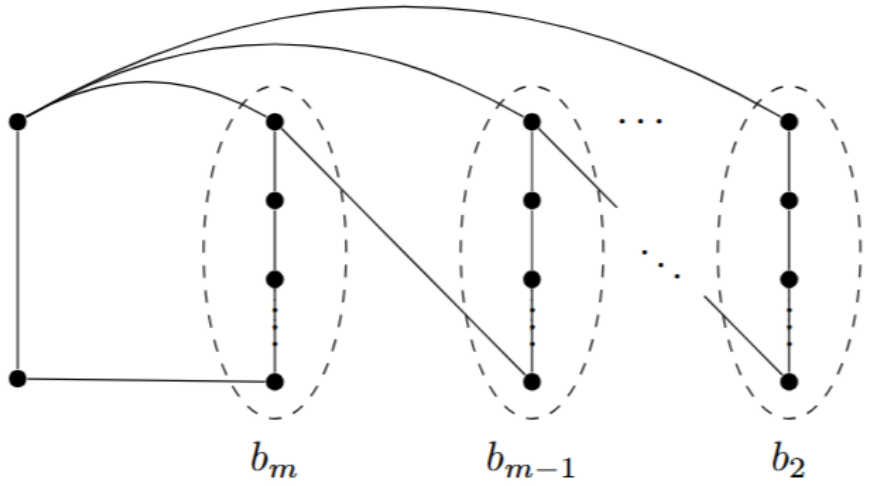
$$\frac{1}{c^n}\left|\mathcal{T}_{sp}(n) \cap \{1, 2, \ldots, c^n\}\right| \to 1 \quad as \quad n \to \infty.$$

Proof idea of the CKP theorem

Lemma [Chan-Kontorovich-P.'24] *Let* $1 \le p < q, \quad \gcd(p,q) = 1 \quad$ *and*

$$\frac{p}{q} = [b_1, 1, b_2, 1, \ldots, b_m, 1].$$

Then there exists a simple planar graph G *s.t.* $\tau(G) = q$ *and* $|V| = b_2 + \ldots + b_m + 2$.



Proof idea of the CKP theorem

Theorem: *Bourgain–Kontorovich technology extends to fractal with* Hdim

- $\delta > 5/6 \approx 0.833$ [Frolenkov–Kan'14], [Huang'15],
- $\delta > 4/5 = 0.800$ [Kan'15],
- $\delta > \frac{1}{4}(\sqrt{17} - 1) \approx 0.781$ [Kan'17],
- $\delta > \frac{1}{3}(\sqrt{40} - 4) \approx 0.775$ [Kan'21].

Hančl–Turek's fractal: $\bar{\mathfrak{C}} := \{[b_1, 1, b_2, 1, \ldots] : b_i \geq 1, \ \forall i\}$

Theorem [Hančl–Turek'23] Let $\bar{\zeta} := \mathrm{Hdim}(\bar{\mathfrak{C}})$. Then:

$$0.732 < \bar{\zeta} < 0.819.$$

Proof idea of the CKP theorem

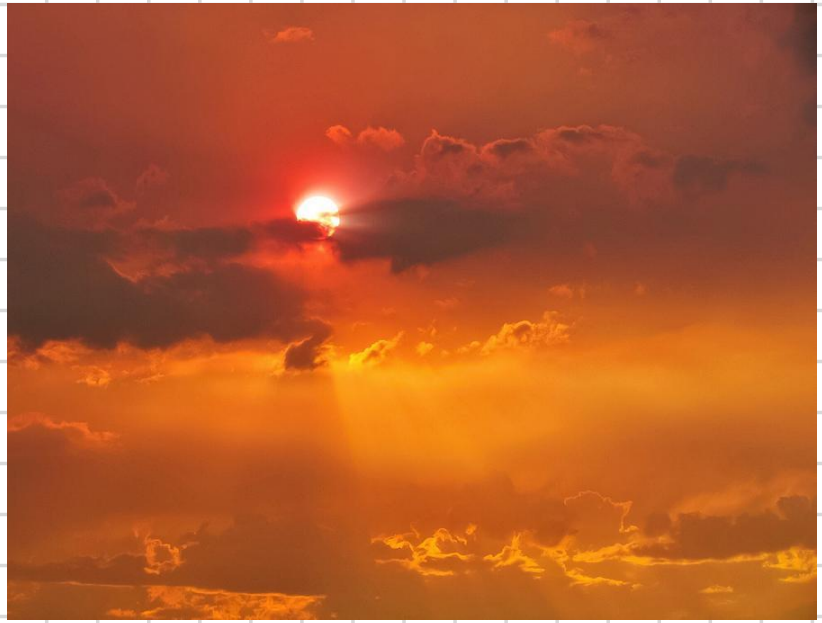
Theorem: *Bourgain–Kontorovich technology extends to fractal with* Hdim

- $\delta > \frac{1}{3}(\sqrt{40} - 4) \approx 0.775$ [Kan21].

Hančl–Turek's fractal: $\bar{\mathfrak{C}} := \{[b_1, 1, b_2, 1, \ldots] : b_i \geq 1, \quad \forall i\}$

Lemma [CKP'24] *Let* $\bar{\zeta} := \mathrm{Hdim}(\bar{\mathfrak{C}})$. *Then:* $0.775 < \bar{\zeta} < 0.799$.

Moreover, taking all $b_i \leq 110$ *suffices.*

The proof is computer assisted and uses the *Pollicott–Vytnova algorithm* (2022)

Thank you!