Igor Pak, UCLA

September 23, 2021

Random linear extensions of posets

(joint with *Swee Hong Chan* and *Greta Panova*)

Permutations and Probability

Banff workshop, BIRS Banff, CA

Plan:

1) Linear extensions
2) Sorting probability
3) Poset inequalities
4) Proof of one RW inequality

Linear extensions of posets

Def $P = (X, \prec)$, $|X| = n$ ← finite

$f : X \to \{1 .. n\}$ is a linear extension

if (1) f is a bijection

(2) f is order-preserving

$x \prec y \Rightarrow f(x) < f(y) \quad \forall x, y \in X$

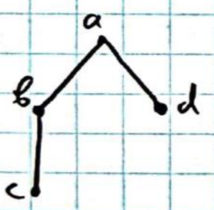
$\mathcal{E}(P)$ — set of LE of P

$e(P) := |\mathcal{E}(P)|$ — number of LE of P

Ex 1 $P = (X, \prec)$, $|X| = n$ poset w/ no relations

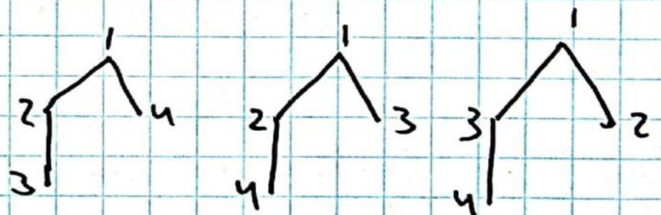
$\Rightarrow e(P) = n!$

Ex 2

$P =$ 

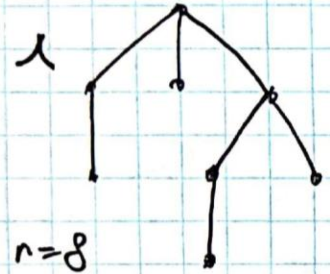
$\begin{cases} a \prec b \prec c \\ a \prec d \end{cases}$

$X = \{a, b, c, d\}$ $n = 4$

$\mathcal{E}(P) =$ 

Large Examples

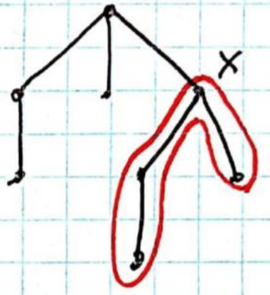
1) $\mathcal{P} = \mathcal{P}_T$, T - tree on n vertices



n=8

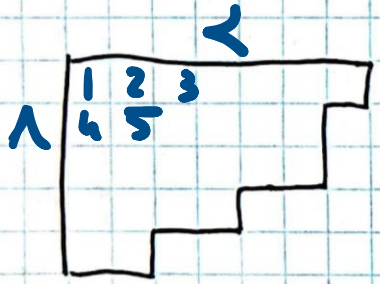
Th /Exc/ $$e(\mathcal{P}_T) = \frac{n!}{\prod_{x \in X} b(x)}$$

where $b(x) = \#\{y \in X,\ y \succcurlyeq x\}$



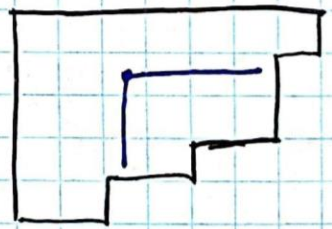
$b(x) = 4$

2) $\mathcal{P} = \mathcal{P}_\lambda$, $\lambda \vdash n$ Young diagram



Th [FRT '54] $$e(\mathcal{P}_\lambda) = \frac{n!}{\prod_{x \in \lambda} h(x)}$$

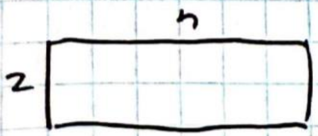
hook-length formula



$h(x) = 6$

$e(\mathcal{P}_\lambda) = SYT(\lambda)$

3) $\mathcal{P} = \mathcal{P}_n$ Catalan poset



$$e(\mathcal{P}_n) = |SYT(n,n)| = \frac{1}{n+1}\binom{2n}{n}$$

Catalan number

Sorting Probability

Def $\mathcal{P} = (X, \prec)$, $|X| = n$, f fixed

$$\delta(\mathcal{P}) := \min_{x,y \in X} \Big| P[f(x) < f(y)] - P[f(x) > f(y)] \Big|$$ sorting probability

Conj [$\frac{1}{3}$-$\frac{2}{3}$ conjecture, Kislitsyn'68, Fredman'75]

$$\boxed{\delta(\mathcal{P}) \leq \frac{1}{3}} \iff \exists x, y \in X \text{ s.t. } \frac{1}{3} \leq P[f(x) < f(y)] \leq \frac{2}{3}$$

Th 1/3 - 2/3 Conj holds for

- posets $\mathcal{P}$ of width 2 [Linial '84]
- posets $\mathcal{P}$ of height 2 [Trotter-Gehrlein-Fishburn '92]
- series-parallel posets [Zaguia '12]
- skew Young diagrams $\mathcal{P}_{\lambda/\mu}$ [Olson-Sagan '18]
- posets w/ $n \leq 11$ elt's [Peczarski '06]
- ⋮

Sorting Probability (cont'd)

Th [Kahn-Saks '84] $\forall \mathcal{P}$

$$\delta(\mathcal{P}) \le \frac{5}{11} \approx 0.4545 \iff \frac{3}{11} \le P[\underset{f(x)<f(y)}{..}] \le \frac{8}{11}$$

Th [Kahn-Kim '95]

Sorting w/ $\mathcal{P}$ as partial information can be done in $O(\log e(\mathcal{P}))$ comparisons

main motivation

Th [Brightwell-Felsner-Trotter '95] $\forall \mathcal{P}$

$$\delta(\mathcal{P}) \le \frac{1}{\sqrt{5}} \approx 0.4472$$

Conj [Kahn-Saks '84]

$\forall \{\mathcal{P}_n\}$, $\operatorname{width}(\mathcal{P}_n) \to \infty$

we have $\delta(\mathcal{P}_n) \to 0$

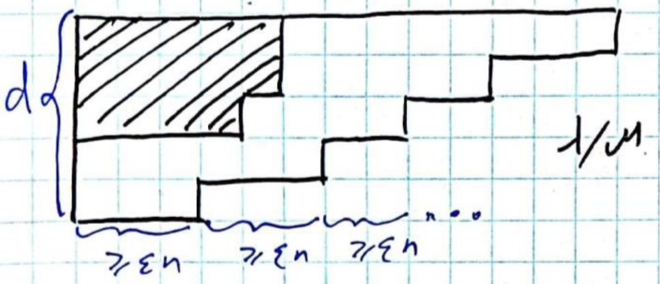
Case of Skew Young Diagrams

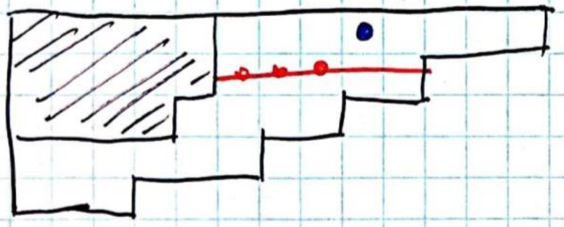
Th Let $\{\lambda/\mu,\ |\lambda|-|\mu|=n\}$ seq of partitions

Suppose $\exists\, \varepsilon>0$ s.t. $\lambda_i-\lambda_{i+1}\geq\varepsilon n$
$\exists\, d$ s.t. $\ell(\lambda)=d$ $\leftarrow$ λ has d parts
$\Big\}$ fixed ε, d

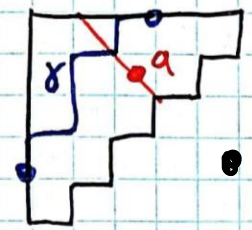
Then $\delta(\mathcal{J}_{\lambda/\mu}) = O_{d,\varepsilon}\left(\frac{1}{\sqrt{n}}\right)$ as $n\to\infty$

[Chan - P. - Panova '20+]





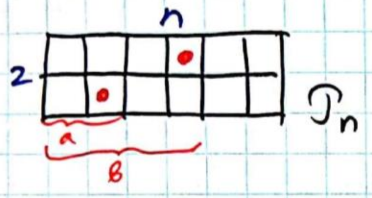
Proof idea



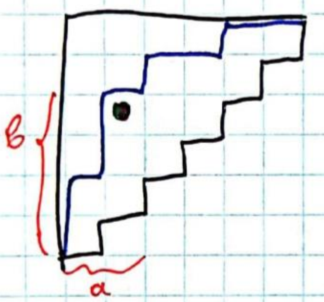
$\delta = P[\gamma \text{ above } a] - P[\gamma \text{ below } a]$

- asymptotics of Schur functions
- Naruse HLF after [Morales - P. - Panova '18]

Catalan Posets



$$e(\mathcal{P}_n) = \frac{1}{n+1}\binom{2n}{n}$$



Th [Chan-P.-Panova, 20+]

$$\delta(P_n) = O\left(\frac{1}{n^{5/4}}\right)$$

Conj $\forall \varepsilon > 0$

$$\delta(P_n) = \Omega\left(\frac{1}{n^{5/4+\varepsilon}}\right)$$

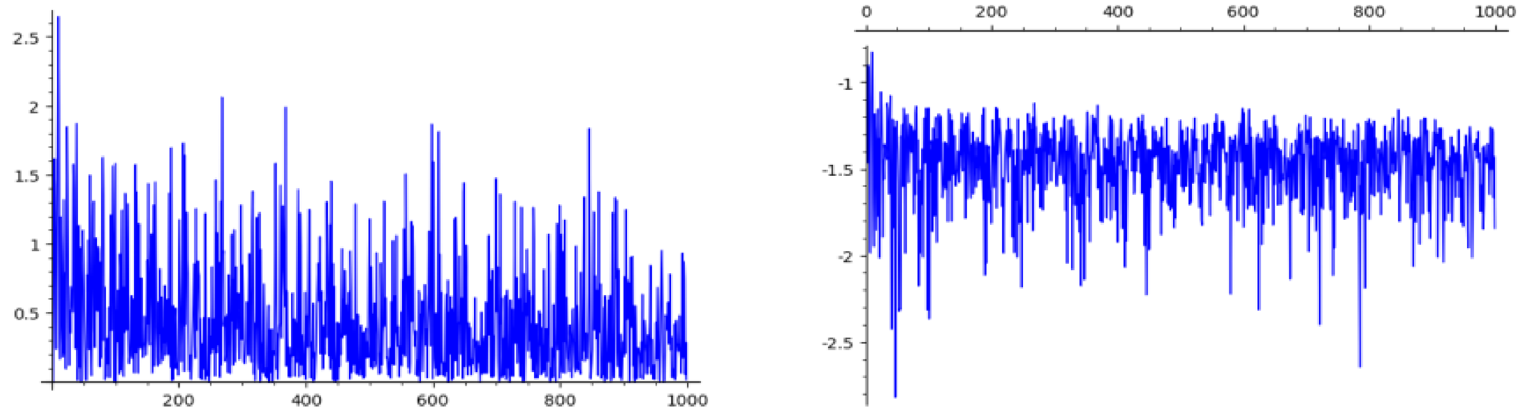


FIGURE 6. Graphs of $\delta(P_n)\,n^{5/4}$ and $\log_n \delta(P_n)$, for $3 \le n \le 1000$.

Inequalities for #LE's

Th [Stanley '81] $\mathcal{P}=(X,\prec)$, $x\in X$ fixed

$$a(k) := \#\{f\in\mathcal{E}(\mathcal{P}) \text{ s.t. } f(x)=k\}$$

Then $a(k)^2 \geq a(k-1)\, a(k+1)$

log-concavity of $\{a(k)\}$

Proof uses Alexandrov-Fenchel inequality

Th [Kahn-Saks '84] $\mathcal{P}=(X,\prec)$, $x,y\in X$ fixed

$$a(k) := \#\{f\in\mathcal{E}(\mathcal{P}) \text{ s.t. } f(x)-f(y)=k\}$$

Then $a(k)^2 \geq a(k-1)\, a(k+1)$

(— II —)

Main Lemma in $\delta(\mathcal{P})\leq \frac{5}{11}$ thm

Conj [= cross-product conjecture, BFT '95]

$\mathcal{P}=(X,\prec)$, $x,y,z\in X$ fixed

$$F(k,\ell) := \#\left\{f\in\mathcal{E}(\mathcal{P}) \text{ s.t. } \begin{matrix} f(x)-f(y)=k \\ f(y)-f(z)=\ell \end{matrix}\right\}$$

Then $F(k,\ell)\, F(k+1,\ell+1) \leq F(k,\ell+1)\, F(k+1,\ell)$

CPC

[BFT '95] proves CPC for $k=\ell=1$

Generalized Cross-Product Conjecture

Conj [Chan - P. - Panova, 21+] $= \text{GCPC}$

$\mathcal{P} = (X, \prec)$, $\quad x, y, z \in X$ fixed

$$F(k,\ell) = \#\left\{ f \in \mathcal{E}(\mathcal{P}) : \begin{array}{l} f(x) - f(y) = k \\ f(y) - f(z) = \ell \end{array} \right\}$$

Then $$F(k,\ell)\, F(k+i, \ell+j) \le F(k, \ell+j)\, F(k+i, \ell) \quad \forall\, i\, j\, k\, \ell \geq 1$$

Th [–'1+] /GCPC has super powers/

(1) GCPC $\Rightarrow$ Kahn-Saks inequality $\Rightarrow$ Stanley inequality

(2) GCPC for posets of width 2 $\Rightarrow$ GYY inequality

/ GYY = Graham-Yao-Yao for posets of width 2 /

(3) GCPC $\Rightarrow$ XYZ inequality

Th [Shepp '82] $\forall\, x, y, z \in X$

$$P[\, f(x) < f(y) \,] \le P[\, f(x) < f(y) \mid f(x) < f(z) \,]$$

Our results on GCPC

<u>Th</u> [Chan-P.-Panova '21+]

GCPC holds for all posets of width two

<u>Def</u> (q-analogue) $\mathcal{P} = (X, \prec)$, $\text{width}(\mathcal{P}) = 2$

$X = C_1 \sqcup C_2$, $C_1 \cap C_2 = \emptyset$ chains in $\mathcal{P}$

$\forall f \in \mathcal{E}(\mathcal{P})$ <u>let</u> $w(f) := \sum_{x \in C_2} f(x)$

<u>Define</u> $F_q(k, \ell) := \sum_f q^{w(f)}$

<u>Th</u> [—||—] $\leftarrow$ <u>q-CPC</u> $\boxed{\forall \mathcal{P} \text{ width two}}$

$$F_q(k, \ell)\, F_q(k+1, \ell+1) \leqslant F_q(k, \ell+1)\, F_q(k+1, \ell)$$

<u>where</u> "$\leqslant$" is coeff-wise.

Equality conditions

Th [Shenfeld – van Handel '20]

$P = (X, \prec)$, $x \in X$ fixed

$a(k) := \#\{ f \in \mathcal{E}(P) \text{ s.t. } f(x) = k \}$ $\qquad$ $\underline{a(k) > 0}$

TFAE:
- $a(k)^2 = a(k-1)\, a(k+1)$
- $a(k-1) = a(k) = a(k+1)$

[CPP '21]: $\underline{\text{width}(P) = 2}$

1) q-analogue of (Stanley ineq) SvH-thm

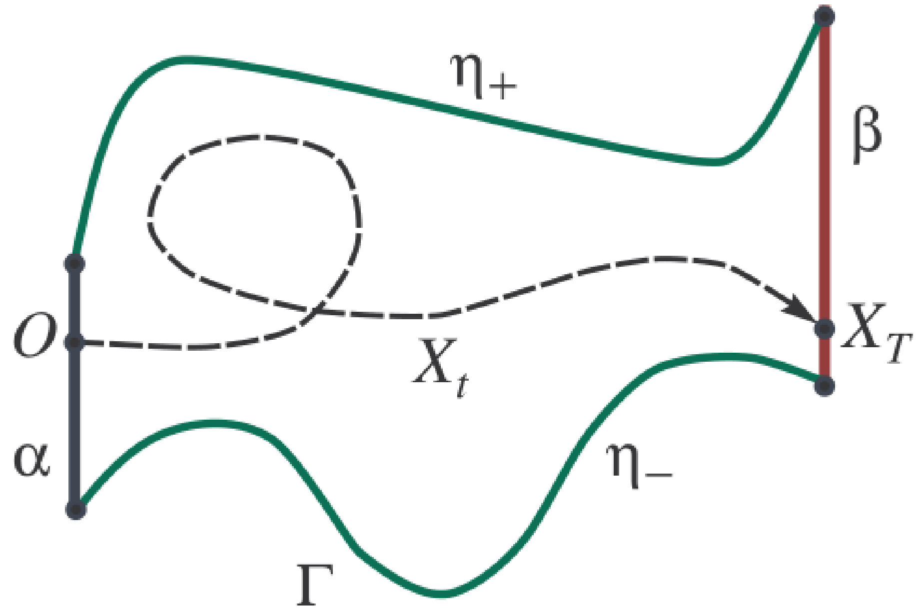
2) q-analogue of (KS ineq) equality conditions

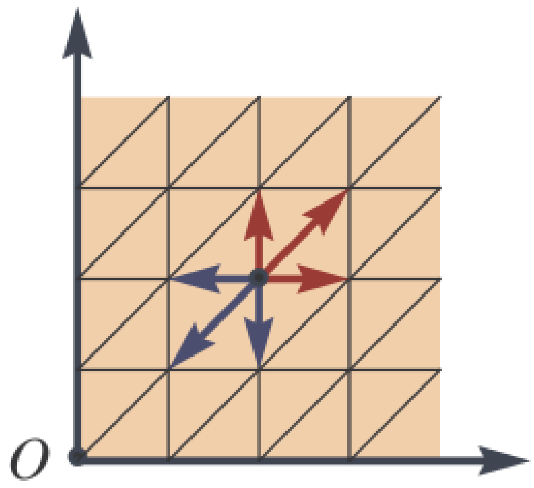
Log-concavity of exist probabilities

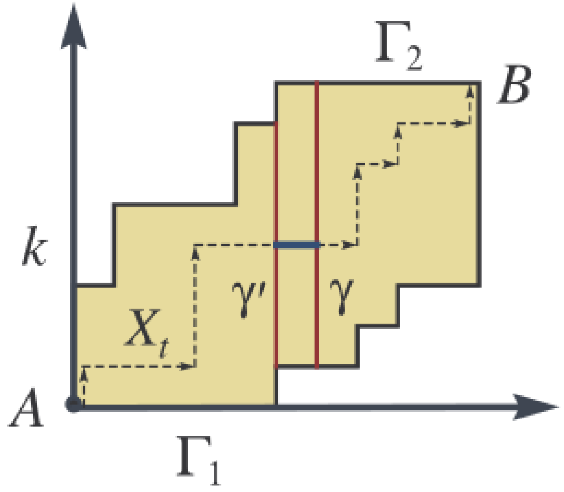
Theorem 1 [Chan-P.-Panova'21]

Let $\{X_t\}$ be the nearest neighbor lattice random walk which starts at the origin $X_0 = O \in \alpha$, and is absorbed whenever X_t tries to exit the region Γ. Denote by T the first time t such that $X_t \in \beta$, and let $p(k)$ be the probability that $X_T = (m,k)$. Then $\{p(k)\}$ is log-concave:

$$p(k)^2 \geq p(k+1)\,p(k-1) \qquad \text{for all } \ k \in \mathbb{Z}, \ \text{ such that } \ (m, k \pm 1) \in \beta.$$



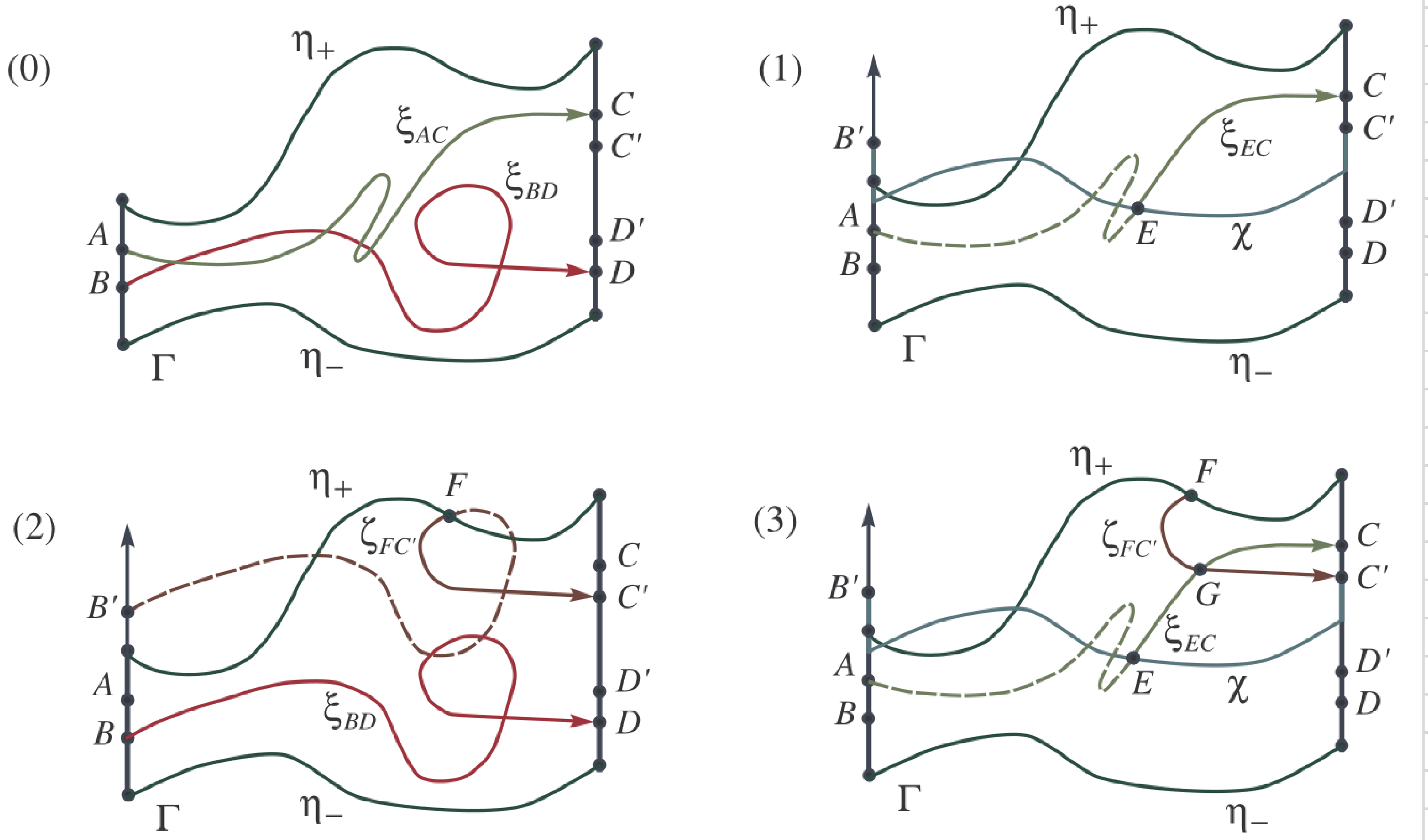




$$p(k)^2 \geq p(k+1)\,p(k-1)$$

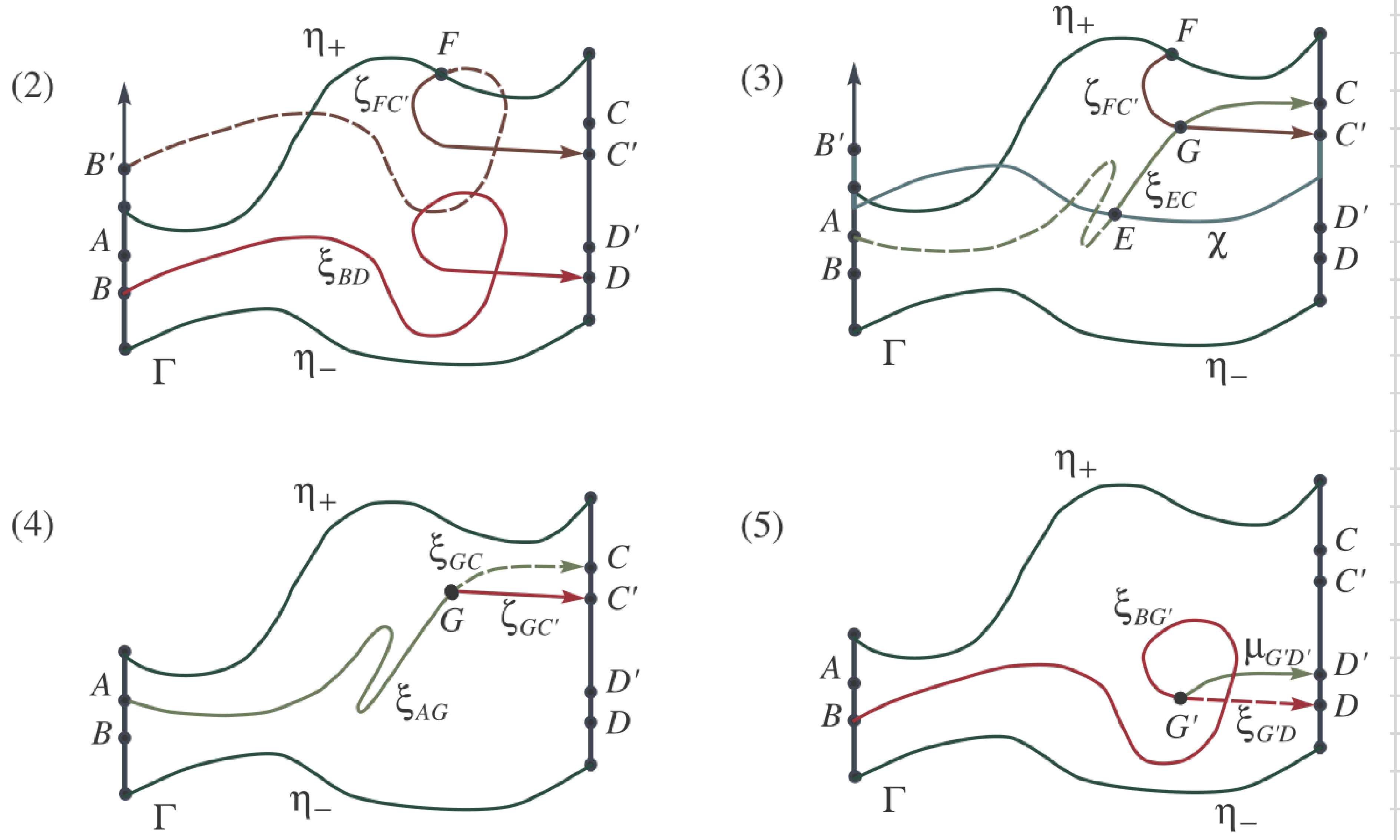
there is an injection

$$\Phi : \{(\xi_{AC}, \xi_{BD})\} \longrightarrow \{(\xi_{AC'}, \xi_{BD'})\},$$



there is an injection

$$\Phi : \{(\xi_{AC}, \xi_{BD})\} \longrightarrow \{(\xi_{AC'}, \xi_{BD'})\},$$



Thank you!