Arrangements of m things of one sort and n things of another sort, under certain conditions of prio...

$$-4\frac{r^3}{a^3}\cos\theta\cos\frac{s\pi}{2n}+\frac{r^4}{a^4}\Bigg\}$$

$$-\frac{\omega a^2}{4\pi}\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\cos 2\theta\Sigma\sin\frac{s\pi}{2}\cos\frac{s\pi}{n}$$

$$\times\log\frac{1-2\frac{r}{a}\cos\left(\frac{s\pi}{2n}-\theta\right)+\frac{r^2}{a^2}}{1+2\frac{r}{a}\cos\left(\frac{s\pi}{2n}+\theta\right)+\frac{r^2}{a^2}}$$

$$-\frac{\omega a^2}{2\pi}\left(\frac{r^2}{a^2}-\frac{a^2}{r^2}\right)\sin 2\theta\Sigma\sin\frac{s\pi}{2}\cos\frac{s\pi}{n}$$

$$\times\tan^{-1}\frac{\sin\frac{s\pi}{2n}\left(\cos\frac{s\pi}{2n}-\frac{r}{a}\cos\theta\right)}{\cos\frac{s\pi}{n}-2\frac{r}{a}\cos\theta\cos\frac{s\pi}{2n}+\frac{r^2}{a^2}}.$$

ARRANGEMENTS OF *m* THINGS OF ONE SORT AND *n* THINGS OF ANOTHER SORT, UNDER CERTAIN CONDITIONS OF PRIORITY.

By *W. Allen Whitworth, M.A.*

Of the two integers m and n, let n (throughout the paper) be one which is not greater than the other.

The conditions of priority which I propose to consider are such as place limits on the proportion of the two classes of things in any part of the arrangement reckoned from its initial extremity.

For example, if an urn contain black balls and white balls which are to be drawn out in succession, the order in which they are drawn out may be limited by the condition that the number of white balls drawn must never exceed the number of black balls; or, again, by the condition that the excess of black balls over white must never be more than a given number.

I use the symbol $c_{r,\,n-r}$ in preference to c_r^n to denote the number of combinations of n things taken r at a time, and it is convenient to notice that

$$c_{m+1,\,n-1}=\frac{n}{m+1}\,c_{m,\,n};$$

and, therefore,

$$c_{m,n} - c_{m+1,n-1} = \frac{m-n+1}{m+1}c_{m,n} = \frac{m-n+1}{m+n+1}c_{m+1,n},$$

and, as a particular case,

$$c_{n,n} - c_{n+1,n-1} = \frac{1}{n+1}c_{n,n} = 2c_{n,n} - \tfrac{1}{2}c_{n+1,n+1}.$$

Also that $$c_{n,n-1} = \tfrac{1}{2}c_{n,n}.$$

1. From any origin O (fig. 13) draw OX horizontally and OY vertically, and let P be the point which would be reached by starting from O and taking m paces horizontally in the direction OX and n paces vertically parallel to OY.

Let A be the point reached by taking one pace horizontally and one pace vertically, B the point reached by taking two paces in each direction, C by taking three paces in each direction, and so on; so that ABC... lie in the straight line bisecting the right angle XOY.

Let o, a, b, c... be the points reached by taking one horizontal pace from each of the points O, A, B, C... and ω, α, β, γ... the points reached by taking one pace vertically from O, A, B, C... respectively.

In considering the number of routes from one point to another we shall proceed on the understanding that each route is to be traversed by paces taken only horizontally and vertically, and without retrogression.

2. The total number of routes from O to P must be the same as the number of different orders in which m horizontal paces and n vertical paces can be arranged. For in *any* order m horizontal paces and n vertical paces will take us from O to P and every different order will give a different route. Hence, the number of routes from O to P is

$$c_{m,n} \equiv \frac{\lfloor m+n}{\lfloor m \lfloor n} \equiv \frac{(m+n)(m+n-1)(m+n-2)\ldots(n+1)}{\lfloor m}$$
$$\equiv \frac{(m+n)(m+n-1)(m+n-2)\ldots(m+1)}{\lfloor n}.$$

3. Let $f_{m,n}$ denote the number of routes from O to P which do not cross the diagonal line OAB..., and let $x_{m,n}$ denote the number of routes which do cross this diagonal; so that

$$f_{m,n} + x_{m,n} = c_{m,n}.$$

4. All the routes included in the number $x_{m,n}$ must pass along one or other of the paces $O\omega$, $A\alpha$, $B\beta$...; some of the routes may pass along more than one, but we will classify all the routes according to the point at which they *first* cross the diagonal whether they recross more than once or not.

Let us consider the routes which cross the diagonal at K (after making say r horizontal and r vertical paces) without having crossed before, but independently of any consideration of the number of times they may cross afterwards.

Any such route $OK\kappa P$ may be divided into three portions OK, $K\kappa$, κP, of which

OK can be made in $f_{r,r}$ ways (Art. 3),

$K\kappa$ in one way,

κP in $c_{m-r,\, n-r-1}$ ways (Art. 2).

Hence, the whole number of ways in which the route can be made, or the number of routes through K, is

$$f_{r,r}\, c_{m-r,\, n-r-1}.$$

Giving r all values from 0 to $n-1$ we shall get all the routes comprehended in the number $x_{m,n}$.

Hence,

$$x_{m,n} = f_{0,0}c_{m,\, n-1} + f_{1,1}c_{m-1,\, n-2} + f_{2,2}c_{m-2,\, n-3} + \dots + f_{n,\, n-1}c_{m-n+1,\, 0},$$

or, in virtue of the relation,

$$f_{m,n} + x_{m,n} = c_{m,n},$$

$$f_{m,n} = c_{m,n} - f_{0,0}c_{m,\, n-1} - f_{1,1}c_{m-1,\, n-2} - f_{3,3}c_{m-2,\, n-3} - \dots - f_{n-1,\, n-1}c_{m-n+1,\, 0}.$$

5. Considering the particular case when $m = n$ we have

$$f_{n,n} = c_{n,n} - f_{0,0}c_{n,\, n-1} - f_{1,1}c_{n-1,\, n-2} - f_{2,2}c_{n-2,\, n-3} - \dots - f_{n-1,\, n-1}\, c_{1,0},$$

or, in virtue of the identity

$$c_{r,\, r-1} = \tfrac{1}{2}c_{r,r},$$

$$f_{n,n} = c_{n,n} - \tfrac{1}{2}\{f_{0,0}c_{n,n} + f_{1,1}c_{n-1,\, n-1} + f_{2,2}c_{n-2,\, n-2} + \dots + f_{n-1,\, n-1}c_{1,1}\}.$$

PROPOSITION I.

6. *To prove that*

$$f_{n,n} = \frac{c_{n,n}}{n+1} \equiv 2c_{n,n} - \tfrac{1}{2}c_{n+1,\, n+1}.$$

Observing that $c_{0,0}=1$, $c_{1,1}=2$, $c_{2,2}=6$, $c_{3,3}=20$, $c_{4,4}=70$, and actually counting the routes for the first five values of n,

$$f_{0,0}=1,\ f_{1,1}=1,\ f_{2,2}=2,\ f_{3,3}=5,\ f_{4,4}=14,$$

we see that the proposition is true as long as n does not exceed 4.

To prove it generally, we shall show that if it be true when $n=0, 1, 2, \ldots$ or $x-1$, it will also be true when $n=x$.

By Art. 5 we have

$$f_{x,x}=c_{x,x}-\tfrac{1}{2}\{f_{0,0}c_{x,x}+f_{1,1}c_{x-1,x-1}+f_{2,2}c_{x-2,x-2}+\ldots+f_{x-1,x-1}c_{1,1}\}.$$

But, since the theorem is by hypothesis true when $n=0$, $1, 2, \ldots$ or $x-1$, we have

$$\begin{aligned} f_{0,0}&=2c_{0,0}-\tfrac{1}{2}c_{1,1}, \\ f_{1,1}&=2c_{1,1}-\tfrac{1}{2}c_{2,2}, \\ \&c. &= \&c., \\ f_{x-1,x-1}&=2c_{x-1,x-1}-\tfrac{1}{2}c_{x,x}. \end{aligned}$$

Hence, substituting, we get

$$\begin{aligned} f_{x,x}=c_{x,x}-&\{c_{0,0}c_{x,x}+c_{1,1}c_{x-1,x-1}+c_{2,2}c_{x-2,x-2}+\ldots+c_{x-1,x-1}c_{1,1}\} \\ &+\tfrac{1}{4}\{c_{1,1}c_{x,x}+c_{2,2}c_{x-1,x-1}+c_{3,3}c_{x-2,x-2}+\ldots+c_{x,x}c_{1,1}\}. \end{aligned}$$

But we know that

$$c_{0,0}c_{x,x}+c_{1,1}c_{x-1,x-1}+\ldots+c_{x,x}c_{0,0}\equiv 2^{2n}$$

(or we may obtain it by expanding $(1-4x)^{-\frac{1}{2}}$ by Bin. Theor.; squaring, and equating coeffs. of x^n), therefore

$$f_{x,x}=c_{x,x}-\{2^{2n}-c_{x,x}c_{0,0}\}+\tfrac{1}{4}\{2^{2n+2}-2c_{x+1,x+1}c_{0,0}\},$$

or

$$f_{x,x}=2c_{x,x}-\tfrac{1}{2}c_{x+1,x+1},$$

which shows that if the theorem is true for the first x values of n, it is true for the next value.

But we have shown that it was true for initial values, therefore, universally,

$$f_{n,n}=2c_{n,n}-\tfrac{1}{2}c_{n+1,n+1},$$

or (which is the same thing),

$$f_{n,n}=\frac{1}{n+1}c_{n,n},$$

or again (which is the same thing),

$$f_{n,n}=c_{n,n}-c_{n+1,n-1}.$$

7. Since all the routes from O to P commence at the point O on the diagonal and terminate at the point P to the right of the diagonal, they may be classified according to the points at which they *first* pass to the right of the diagonal: *i.e.* according as they first pass along Oo or Aa or Bb, &c. Consider those which pass along Kk, any such route consists of three portions OK, kK, kP, of which OK lies altogether to the right of the diagonal, and can therefore be described in $\frac{1}{r+1}c_{r,\,r}$ ways; Kk can be described in one way, and kP can be described in $c_{m-r-1,\,n-r}$ ways. Hence there are

$$\frac{1}{r+1}c_{r,\,r}\,c_{m-r-1,\,n-r}$$

such routes, and giving r all values from 0 to n inclusive we must obtain all the $c_{m,\,n}$ routes from O to P. Therefore we must have identically

$$c_{m,\,n} \equiv c_{0,\,0}c_{m-1,\,n} + \tfrac{1}{2}c_{1,\,1}c_{m-2,\,n-1} + \tfrac{1}{3}c_{2,\,2}c_{m-3,\,n-2} + \dots$$
$$+ \frac{1}{n+1}c_{n,\,n}c_{m-n-1,\,0}.$$

This is true for all values of m and n; therefore, writing $m+1$ and $n-1$ for m and n respectively, we obtain

$$c_{m+1,\,n-1} \equiv c_{0,\,0}c_{m,\,n-1} + \tfrac{1}{2}c_{1,\,1}c_{m-1,\,n-2} + \tfrac{1}{3}c_{2,\,2}c_{m-2,\,n-3} + \dots$$
$$+ \frac{1}{n}c_{m,\,n-1}c_{m-n+1,\,0}.$$

PROPOSITION II.

8. *To prove that* $f_{m,\,n} = c_{m,\,n} - c_{m+1,\,n-1}$.

From Art. 4 we have

$$f_{m,\,n} = c_{m,\,n} - f_{0,\,0}c_{m,\,n-1} - f_{1,\,1}c_{m-1,\,n-2}$$
$$- f_{2,\,2}c_{m-2,\,n-3} - \dots - f_{n-1,\,n-1,}\,c_{m-n+1\;0}.$$

Substituting for $f_{0,\,0}$, $f_{1,\,1}$, $f_{2,\,2}$ their values as found in Art 6, in the form

$$f_{n,\,n} = \frac{1}{n+1}c_{n,\,n},$$

we obtain

$$f_{m,\,n} = c_{m,\,n} - c_{0,\,0}c_{m,\,n-1} - \tfrac{1}{2}c_{1,\,1}c_{m-1,\,n-2} - \tfrac{1}{3}c_{2,\,2}c_{m-2,\,n-3} - \dots$$
$$- \frac{1}{n}c_{n-1,\,n-1},\,c_{m-n+1,\,0},$$

or in virtue of the identity proved in Art. 7

$$f_{m,n} = c_{m,n} - c_{m+1,n-1}. \qquad \text{Q. E. D.}$$

N.B. We may also write the result in the form

$$f_{m,n} = \frac{m-n+1}{m+n+1} c_{m+1,n} \text{ or } f_{m,n} = \frac{m-n+1}{m+1} c_{m',n}.$$

9. If, in Art. 7, the routes instead of starting from O had started from the point J situated h paces horizontally to the left of O, the whole number of routes would have been $c_{m+h',n}$. And classifying them as in the case of Art. 7, any route passing along Kk would have consisted of three parts, JK, Kk, kP, of which

JK could be described in $\dfrac{h+1}{r+h+1} c_{r+h,r}$ ways (Art. 8),

Kk in one way,

kP in $c_{m-r-1,n-r}$ ways.

Hence there are $\dfrac{h+1}{r+h+1} c_{m-r-1,n-r}\, c_{r+h,r}$ such routes. And giving r all values from 0 to n we obtain the identity

$$c_{m+h',n} \equiv c_{h,0}\, c_{m-1',n} + \frac{h+1}{h+2} c_{h+1,1}\, c_{m-2,n-1} + \frac{h+1}{h+3} c_{h+2,2}\, c_{m-3,n-2}$$
$$+ \ldots + \frac{h+1}{h+n+1} c_{h+n,h}\, c_{m-n-1',0},$$

or

$$\frac{c_{m+h,n}}{h+1} \equiv \frac{c_{h,0}c_{m-1,n}}{h+1} + \frac{c_{h+1,1}c_{m-2,n-1}}{h+2} + \frac{c_{h+2,2}c_{m-3,n-2}}{h+3} + \ldots + \frac{c_{h+n,n}c_{m-n-1,0}}{h+n+1}.$$

Write h for $h+1$ and m for $m-1$, and we have

$$\frac{c_{m+h,n}}{h} \equiv \frac{c_{h-1,0}c_{m,n}}{h} + \frac{c_{h,1}\, c_{m-1,n-1}}{h+1} + \ldots + \frac{c_{h+n-1,n}c_{m-n,0}}{h+n}.$$

Now write $n-h$ for n, and we have

$$\frac{c_{m+h',n-h}}{h} \equiv \frac{c_{h-1',0}c_{m',n-h}}{h} + \frac{c_{h,1}c_{m-1,n-h-1}}{h+1} + \ldots + \frac{c_{n-1,n-h}c_{m-n+h',0}}{n}.$$

PROPOSITION III.

10. *To find the number of routes from O to P which touch or cross a line HR parallel to the diagonal OK at a vertical distance of h paces above it.*

The routes may be classified according to the points at which they first touch the line HR.

Let $O\lambda'LP$ be any route which first reaches this line at a point L, distant r horizontal paces and $r+h$ vertical paces from O, and let $\lambda'L$ be the pace by which the point L is approached.

The route may be divided into three parts $O\lambda'$, $\lambda'L$, and LP; of which

$$O\lambda' \text{ can be made in } \frac{h}{r+h} c_{r+h-1,\, r} \text{ ways (Art. 7),}$$

$$\lambda'L \text{ in one way,}$$

$$LP \text{ in } c_{m-r,\, n-r-h} \text{ ways;}$$

therefore the number of routes first touching at L is

$$\frac{h}{r+h} c_{r+h-1,\, r} c_{m-r,\, n-r-h},$$

and the whole number of routes will be got by giving r all values from 0 to $n-h$ inclusive, and adding.

The summation is that of the final series of Art. 9.

Hence, the whole number of routes required is $c_{m+h,\, n-h}$.

11. Corollary. It follows that *the number of routes from O to P which do not cross or touch the diagonal h paces above the diagonal through O, is*

$$c_{m,\, n} - c_{m+h,\, n-h},$$

and writing $h+1$ for h, *the number of routes from O to P which do not cross* (but may touch) *the diagonal h paces above the diagonal through O, is*

$$c_{m,\, n} - c_{m+h+1,\, n-h-1},$$

for if they do not *cross* this diagonal, they cannot *touch* the diagonal next beyond it.

Observe that if $h=0$ this reduces to the case of Prop. II. (Art. 8).

12. Examples.

(1) *A man drinks in random order n glasses of wine and n glasses of water (all equal); shew that the odds are n to 1 against his never having drunk throughout the process more wine than water* (*Educational Times*, June, 1878, question 5669).

Let glasses of water be represented by horizontal paces and the glasses of wine by vertical paces, then the chance required must be the same as the chance of keeping always to the right of the diagonal OV in passing at random from O to V.

The total number of routes (by Art. 2) is $c_{n,n}$.

The number which keep to the right of the diagonal (by Art. 6) is $c_{n,n} \div (n+1)$.

Therefore the chance is $1 \div (n+1)$, or the odds are n to 1.

Q. E. D.

(2) *If n men and their wives go over a bridge in single file, in random order, subject only to the condition that there are to be never more men than women gone over, prove that the chance that no man goes over before his wife is* $(n+1)\,2^{-n}$ (*Educational Times*, September 1878, Question 5744).

Regarding the different men as indifferent and the different women as indifferent, the number of orders in which they could go over would be (as in Art. 2) $c_{n,n}$ and the number of orders subject to the condition that never more men than women are gone over would be (as in Art. 6) $c_{n,n} \div (n+1)$.

But the men may be arranged among themselves in $\lfloor n$ ways, and the women in $\lfloor n$ ways.

Therefore the total number of possible orders, subject to the given condition, is

$$\lfloor n \lfloor n\, c_{n,n} \div (n+1) \text{ or } \lfloor 2n \div (n+1).$$

These include amongst them all the orders in which each man has gone over before his own wife. But amongst all the possible $\lfloor 2n$ orders in which the $2n$ persons could cross, any assigned man will be before his wife in $(\frac{1}{2})\lfloor 2n$ orders and all the n men will be before their wives in $(\frac{1}{2})^n \lfloor 2n$ orders.

Therefore the required chance is

$$\frac{(\frac{1}{2})^n \lfloor 2n}{\lfloor 2n \div (n+1)} \text{ or } (n+1)\,2^{-n}.$$

Q. E. D.

(3) *In how many orders can m positive units and n negative units be arranged so that the sum to any number of terms may never be negative?*

By taking horizontal paces to represent positive units and vertical paces to represent negative units, this question is seen to be equivalent to that of Art. 3. And, therefore, (by Art. 8) we have the result

$$c_{m,n} - c_{m+1,n-1} = (m-n+1)\frac{\lfloor m+n}{\lfloor m+1 \lfloor n}.$$

(4) *In how many orders can m even powers of x and n odd powers of x be arranged, so that when $x=-1$ the sum to any number of terms may never be negative?*

This question differs from the preceding in that the terms are all different. The positive terms may therefore be permuted amongst themselves in $\lfloor m$ orders and the negative terms in $\lfloor n$ orders. The required number of ways is therefore

$$\frac{(m-n+1)\lfloor m+n}{m+1}.$$

(5) *In how many orders can a man win m games and lose n games so as at no period to have lost more than he has won?*

The question is equivalent to question (3) and the answer is identically the same.

(6) *A man possessed of $a+1$ pounds plays even wagers for a stake of 1 pound. Find the chance that he is ruined at the $(a+2x+1)^{\text{th}}$ wager and not before.*

He necessarily loses the last wager.

Let $m=a+x$, the number of wagers he loses before the last one.

$n=x$, the number he wins.

The number of orders in which his gains and losses can be arranged will be represented by the number of routes from P to O (fig. 13) without crossing the diagonal OV. And this must be the same as the number of routes from O to P which is {by Art. 8)

$$\frac{m-n+1}{m+1}c_{m,n};$$

and, therefore, the chance that m loses and n gains are arranged in such order that the man is not ruined before the $(m+n)^{\text{th}}$ wager is

$$\frac{m-n+1}{m+1}.$$

But the chance that his first m wagers should give m losses and n gains is

$$\frac{\lfloor m+n}{\lfloor m \lfloor n} \left(\tfrac{1}{2}\right)^{m+n},$$

and the chance that the final wager gives a loss is $\frac{1}{2}$.

Therefore the chance that he is not ruined before the $(a+2x+1)^{\text{th}}$ wager, and that he is ruined then, is

$$\frac{m-n+1}{m+1} \frac{\lfloor m+n}{\lfloor m \lfloor n} \left(\tfrac{1}{2}\right)^{m+n+1} = \frac{(a+1) \lfloor a+2x}{\lfloor a+x+1 \lfloor x} \left(\tfrac{1}{2}\right)^{a+2x+1}.$$

(7) *In how many different orders can a man possessed of h pounds win m wagers and lose n wagers of 1 pound each without being ruined during the process?*

The number of ways must be the same as the number of routes from O to P without touching the diagonal JR. The result is therefore (by Art. 11)

$$c_{m,\,n} - c_{m+h,\,n-h}.$$

(8) *If a man playing for a constant stake, win $2n$ games and lose n games, the chance that he is never worse off than at the beginning and never better off than at the end is*

$$\frac{n^2+n+2}{4n^2+6n+2}.$$

(9) *If he win $2n+1$ games and lose $n+1$ games, the chance is*

$$\frac{n}{4n+6}.$$

(*Educational Times*, November, 1878, Question 5804).

ON CENTRES OF PRESSURE, METACENTRES, &c.

By *T. C. Lewis, M.A.*, Fellow of Trinity College, Cambridge.

THE centre of pressure of a plane area bounded by a closed conic may be found by the following elementary geometrical method.

(i) Through every point of the boundary of the given area draw vertical lines to the surface; then if the vertical