Lecture Notes in Mathematics

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PREFACE

These Lecture Notes are the work-out of a seminar hald at the Technological University Eindhoven (THE) in the years 1971-1972 and 1972-1973. As a guide for the seminar the book "Combinatorial Theory" by Marshall Hall, Jr. was chosen. Since this book is used by so many combinatorialists it was considered worthwhile to publish our notes as a service to the mathematical community. The contents fall into the following categories: anwers to questions which came up during the seminar, extensions and generalizations of theorems in Hall's book, references and reports on results which appeared after the book, and finally a number of research results of members of the group.
The members of the seminar were M.L.J. Hautus, H.J.L. Kamps, J.H. van Lint, K.A. Post, C.P.J. Schnabel, J.J. Seidel, H.C.A. van Tilborg, J.H. Timmermans and J.A.P.M. van de Wiel. The author of these notes acted as leader of the seminar. A number of valuable suggestions is due to N.G. de Bruijn.
The chapters in these notes have the same titles as those in Hall's book and the notation is the same. References to this book are preceded by H., e.g. H. Theorem 8.3.2 or (H.8.3.10); definitions and theorems are not repeated.
For her excellent typing of these lecture notes I thank Mrs. E. Baselmans-Weijers.

J.H. van Lint.

Eindhoven, November 1973

CONTENTS

III. GENERATING FUNCTIONS AND RECURSIONS

3.1. *The recursion* $u_n = \sum_{i=1}^{n-1} u_i\, u_{n-i}$.

In H. § 3.2 the combinatorial problem of counting the number of ways a sequence $x_1, x_2, \ldots, x_n$ may be combined in this order by a binary nonassociative product is treated. This leads in a natural way to the recursion in the title. The solution of the problem is

$$(3.1.1) \qquad u_n = \frac{1}{n}\binom{2n-2}{n-1}, \qquad n \geq 1 .$$

The same result, generally derived from the same recursion, is found for many other combinatorial problems. We shall list a number of these problems below and then give a number of combinatorial demonstrations that these problems indeed have the same solution. The sequence $(u_n)_{n\in\mathbb{N}}$ is known as the *Catalan* sequence. A bibliography of 243 papers and books in which the Catalan numbers occur can be found in [8].

PROBLEM 1. The nonassociative product problem mentioned above.

PROBLEM 2. Consider a random walk in the plane, where the steps are from (x,y) to (x+1,y+1) or (x+1,y-1), starting at a given point. In how many ways can the random walk go from (0,0) to (2n,0) through the upper halfplane without crossing the X-axis? Similarly we can demand that the walk does not meet the X-axis between (0,0) and (2n,0).

PROBLEM 3. A *tree* on n vertices is a connected graph with n vertices and n - 1 edges. Such a graph is planar. If the graph is drawn in the plane we refer to it as a *plane tree*. A *rooted tree* is a tree with a distinguished vertex r called the *root*. If the valency of the root is 1 we say the tree is a *planted tree*. How many planted plane trees are there with n vertices?

PROBLEM 4. A planted plane tree is called *trivalent* (or *binary* tree or *bifurcating* tree) if every vertex has valency 1 or 3. It is easily seen that if there are n vertices of valency 1 then there are n - 2 vertices of valency 3. How many trivalent planted plane trees are there with n vertices of valency 1 ?

PROBLEM 5. In how many ways can one decompose a convex (n+1)-gon into triangles by n - 2 nonintersecting diagonals?

PROBLEM 6. In how many ways can 2n points on a circle be joined by n nonintersecting chords?

PROBLEM 7. A less familiar problem is the following. Let A_n be the set of n-tuples $(a_1,a_2,\dots,a_n)$ of integers > 1 such that in the sequence $1,a_1,a_2,\dots,a_n,1$ every a_i divides the sum of its two neighbors. Let U_n be defined in the same way, replacing > 1 by ≥ 1. Determine $|A_n|$ and $|U_n|$.

Of the very many references for these seven problems we list a few. Problem 1: [1], [2]; Problem 2: [3] Ch. 3; Problems 3 and 4: [4], [5]; Problem 5: [6].
Before showing the equivalence of the problems 1 to 7 we solve Problem 2 by a combinatorial argument. The method used is due to D. André and is called the *reflection principle* (cf. [3]). Let A and B be two points in the upper halfplane as in figure 9, and consider a path from A to B, which meets (or crosses) the X-axis.

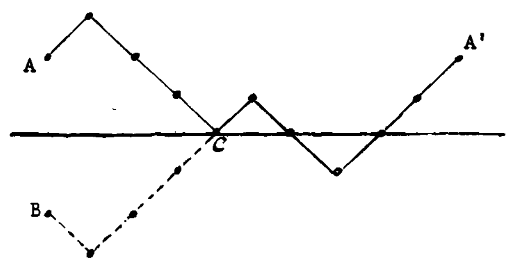

Fig. 9

By reflecting the part of the path between A and the first meeting with the X-axis (C in figure 9) with respect to the X-axis, we find a path from the reflected point A' to B. This establishes a 1-1 correspondence between paths from A' to B and paths from A to B which meet or cross the X-axis.
It follows that if $A = (0,k)$ and $B = (n,m)$, then there are $\binom{n}{\ell_1}$ paths from A to B which cross or meet the X-axis, where $2\ell_1 := n-k-m$. Since there are $\binom{n}{\ell_2}$ paths from A to B, where $2\ell_2 := n-m+k$, we find $\binom{n}{\ell_2} - \binom{n}{\ell_1}$ paths from A to B which do not meet the X-axis. Any path from (0,0) to (2n,0) through the upper halfplane which does not meet the X-axis between these points goes from (0,0) to (1,1) =: A, from A to B := (2n-1,1) without meeting the X-axis, and then from (2n-1,1) to (2n,0). By the argument above there are u_n such paths. If we allow the paths to meet the X-axis, without crossing, then there are u_{n+1} such paths. It seems very hard to find this number by a combinatorial argument which yields the factor n^{-1} (resp. $(n+1)^{-1}$) in a natural way.
We remark that the number of paths from (0,0) to (2n,0) through the upper halfplane which do not meet the X-axis between these points is equal to the number of sequences of zeros and ones

$$(x_1,x_2,\dots,x_{2n})$$

with

(3.1.2) $\quad x_1 + x_2 + \dots + x_j < \tfrac{1}{2}j\ , \qquad j = 1,2,\dots,2n-1\ ,$

(3.1.3) $\quad x_1 + x_2 + \dots + x_{2n} = n\ .$

The correspondence is given by letting a 1 correspond to a step $(x,y) \to (x+1,y-1)$ of the path.

We now turn to the problem of showing the equivalence of problems 1 to 7. In most cases we do not give a formal proof but simply illustrate the correspondence by a figure.

(i) *Problems 1 and 4*: The correspondence is shown by figure 10.

$x_1((x_2x_3)(x_4x_5)) \rightleftarrows$

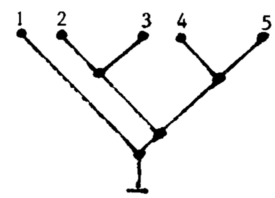

Fig. 10

It follows that the solution to Problem 4 is u_{n-1}.

(ii) *Problems 2 and 4*: Consider the trivalent planted plane tree in figure 11.

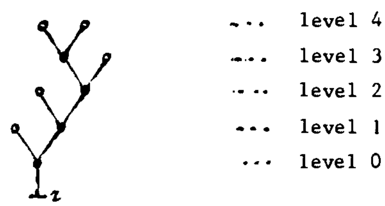

Fig. 11

We have ordered the vertices in *levels* and in each level we read from left to right. We describe the tree by a sequence of zeros and ones, taking a 0 for a vertex of valency 3, a 1 otherwise. We find 0 1 0 1 0 0 1 1 1. If we add a 0 in front (corresponding to the root) we have a sequence as in (3.1.2), (3.1.3) with $n = 5$. That (3.1.3) is satisfied is obvious and (3.1.2) follows from the fact that $(x_1,x_2,\dots,x_j)$ is a sequence describing a partial tree corresponding to a lower part of figure 11. To finish the sequence a number of ones would have to be added. E.g. 0 0 1 0 1... and 0 0 1 0 1 1 correspond to figure 12:

Fig. 12

(iii) *Problems 2 and 3*: Consider a planted plane tree as in figure 13. Again the vertices are in levels and numbered in the obvious way.

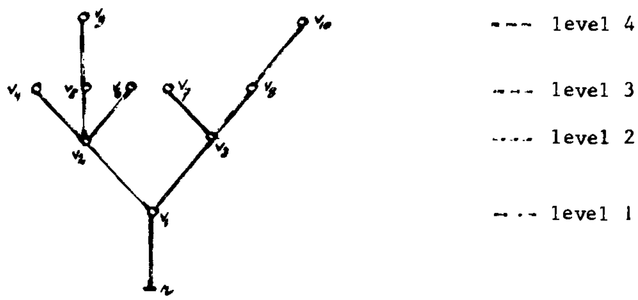

Fig. 13

We describe this tree by a sequence of e's (for edge) and the vertices v_i, each followed by as many e's as there are edges going up from v_i:

$$e\,v_1\,e\,e\,v_2\,e\,e\,e\,v_3\,e\,e\,v_4\,v_5\,e\,v_6\,v_7\,v_8\,e\,v_9\,v_{10}\ .$$

If we now replace each e by 0, each v_i by 1 we have a sequence $x_1, x_2, \ldots, x_{20}$ with $x_1 + x_2 + \ldots + x_{20} = 10$ and $x_1 + x_2 + \ldots + x_j \leq \frac{1}{2}j$ for $j = 1,\ldots,20$. This corresponds to the first question in Problem 2. Another mapping giving a correspondence with the second question in Problem 2 is given by the "up-down" code. This code is discussed in [4]. The idea is shown in figure 14.

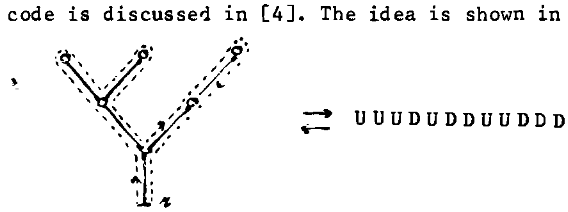

Fig. 14

The dotted line describes a path around the tree. For each edge, U (up) or D (down) gives the direction of the path. Clearly the number of U's exceeds the number of D's at every stage except when the path is complete. This corresponds to (3.1.2), (3.1.3) by taking U = 0, D = 1.

(iv) *Problems 3 and 4*: We take this correspondence from [4]. As the authors of [4] say: "The principle is so simple that it seems to be a pity to obscure it by giving a formal description ...". See figure 15.

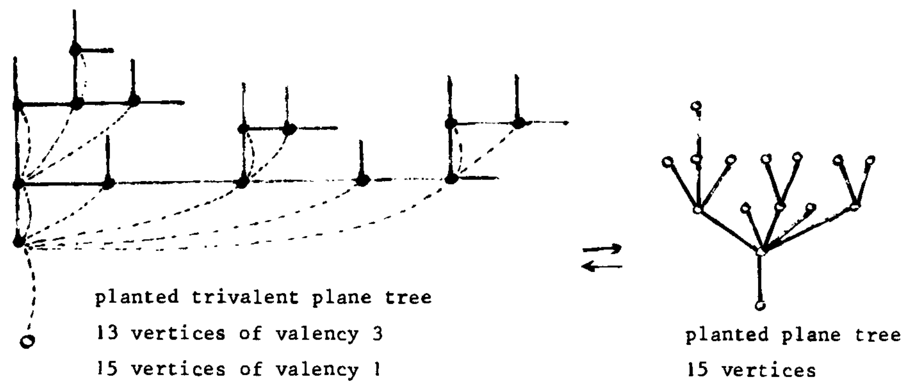

Fig. 15

(v) *Problems 1, 4 and 5*: We distinguish an edge of an (n+1)-gon, then consider the (n+1)-gon, decomposed into triangles as a planar graph and draw a modified "dual" graph of this graph. The rule is demonstrated in figure 16a. Subsequent application of the mapping discussed in (i) yields figure 16b.

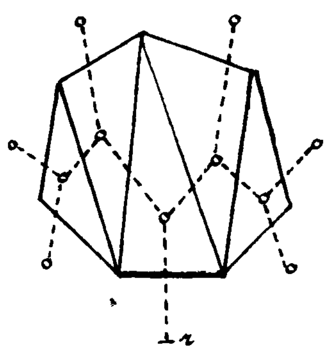

Fig. 16a

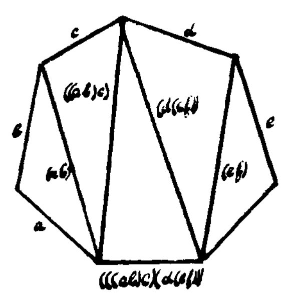

Fig. 16b

Clearly the "dual" graph in figure 16a is a trivalent plane tree which becomes planted if we consider the edge crossing the distinguished edge of the (n+1)-gon as coming from the root. (Note that the usual concept of dual graph is the same as ours if we identify all vertices of valency 1.) In this case a decomposition of an (n+1)-gon corresponds to a trivalent planted plane tree with n + 1 vertices of valency 1. Hence the solution to problem 5 is u_n.

(vi) *Problems 3 and 6*: Again a figure (figure 17) illustrates the equivalence. We leave a formal proof to those readers who are not convinced by figures.

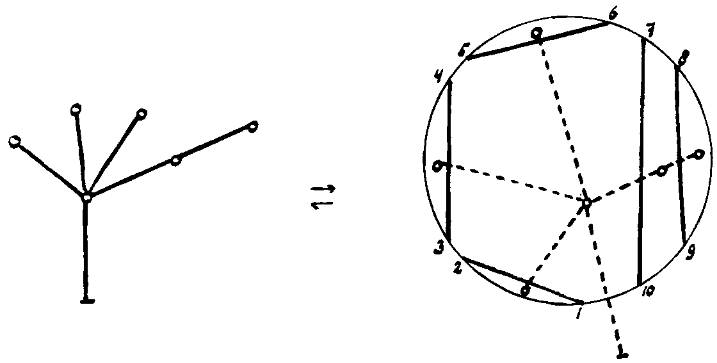

Fig. 17

The chords divide the circle into n parts. For each we have a vertex. The root is outside the circle and the edge from the root crosses the circle between 1 and 2n. The tree has n + 1 edges. Hence the solution to problem 6 is u_{n+1}.

Problems 7 and 2: To show the correspondence with the previous combinatorial problems we analyze a sequence $1,a_1,a_2,\dots,a_n,1$ as described in Problem 7. If, for any i, we have $a_i = a_{i+1}$ then the divisibility condition for the integers a_j implies that all of them are divisible by a_i and hence all the a_j's are 1. Otherwise there is at least one a_i such that $a_{i-1} < a_i$ and $a_i > a_{i+1}$. Then $a_i \mid (a_{i-1} + a_{i+1})$ implies that $a_{i-1} + a_{i+1} = a_i$ (we take $a_0 = a_{n+1} = 1$). It is easily checked that we can now remove a_i from the sequence, thus obtaining a sequence with one element less which still satisfies the divisibility condition. Conversely, any sequence can be lengthened by adding the term $a_i + a_{i+1}$ between a_i and a_{i+1}. For example

$$(1,1) \to (1,2,1) \to (1,2,3,1) \to (1,2,5,3,1) \to (1,2,5,3,4,1)$$

or

$$(1,1) \to (1,2,1) \to (1,2,3,1) \to (1,2,3,4,1) \to (1,2,5,3,4,1) .$$

We repeat this example, but now when a term $a_i + a_{i+1}$ is added between a_i and a_{i+1} we insert a mark before a_i and are allowed to make subsequent changes after the mark only. The second of the sequences does not satisfy this condition. The first example becomes

$$(1,1) \to (|1,2,1) \to (|1|2,3,1) \to (|1||2,5,3,1) \to (|1||2,5|3,4,1) .$$

The places of the 4 marks completely determine the sequence a_1,a_2,a_3,a_4 (in this case 2,5,3,4) and obviously the marks precede the corresponding numbers. The sequence starts with a mark. If we replace the sequence of marks and a_i's by 0's and 1's respectively and then omit the last two zeros, we have shown the correspondence with the first question in Problem 2. To show that this is indeed a 1-1 correspondence we note that a given sequence $a_1,a_2,\dots,a_n$ can be reduced inductively by subsequently removing the term a_i with $a_{i-1} + a_{i+1} = a_i$, i maximal. This reverses the procedure described above.

The sequences forming the set U_n are treated in the same way, starting from (0,1,0). In this case we have three more integers than marks. Our rule says these are at the end. An example illustrates the procedure. Start from a sequence of 0's and 1's as in Problem 2, say 0 0 1 0 1 1. This corresponds to

$$||0|\,?\,?\,?\,1\,0\,,$$

where we have added three more integers at the end, of which 1 and 0 are known. This describes the sequence generated as follows:

$$(0,1,0) \to (|0,1,1,0) \to (||0,1,1,1,0) \to (||0|1,2,1,1,0)\ .$$

We have shown that

(3.1.4) $$|U_n| = |A_{n+1}| = u_{n+2}\ .$$

3.2. *Stirling numbers.*

We recall the definitions of the Stirling numbers as given in H.Ch.3, Problem 2. We have

(3.2.1) $$(x)_0 := 1\ ,$$

(3.2.2) $$(x)_n := x(x-1)\ \dots\ (x-n+1)\ , \qquad (n \in \mathbb{N})\ .$$

Then the Stirling numbers of the first kind s(n,r) are defined by

(3.2.3) $$(x)_n =: \sum_{r=0}^{n} s(n,r)x^r \qquad (n \ge 0)\ .$$

The Stirling numbers of the second kind S(n,r) are defined by

(3.2.4) $$x^n =: \sum_{r=0}^{n} S(n,r)(x)_r \qquad (n \ge 0)\ .$$

It is often useful to extend these definitions by defining s(n,r) = S(n,r) = 0 if r < 0 or r > n (e.g. in H.p.27, Problem 3).

Generating functions. From (3.2.3) we find, for $|z| < 1$,

$$(1+z)^x = \sum_{n=0}^{\infty} \binom{x}{n} z^n = \sum_{n=0}^{\infty} \frac{1}{n!}(x)_n z^n =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^n \sum_{r=0}^{n} s(n,r)x^r = \sum_{r=0}^{\infty} x^r \sum_{n=r}^{\infty} s(n,r)\frac{z^n}{n!}\ .$$

On the other hand, we have

$$(1 + z)^x = e^{x \log(1+z)} = \sum_{r=0}^{\infty} \frac{1}{r!} (\log(1 + z))^r x^r .$$

Hence it follows that

$$(3.2.5) \qquad \sum_{n=r}^{\infty} s(n,r) \frac{z^n}{n!} = \frac{1}{r!} (\log(1 + z))^r .$$

In the same way we find from (3.2.4)

$$e^{xz} = \sum_{n=0}^{\infty} \frac{x^n z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{r=0}^{n} S(n,r)(x)_r = \sum_{r=0}^{\infty} (x)_r \sum_{n=r}^{\infty} S(n,r) \frac{z^n}{n!} .$$

Since we also have

$$e^{xz} = (1 + (e^z - 1))^x = \sum_{r=0}^{\infty} (x)_r \frac{(e^z - 1)^r}{r!} ,$$

we find that

$$(3.2.6) \qquad \sum_{n=r}^{\infty} S(n,r) \frac{z^n}{n!} = \frac{1}{r!} (e^z - 1)^r .$$

For different proofs of (3.2.5) and (3.2.6) see [7].

Relations. The Stirling numbers of the first and second kind are connected by the relation

$$(3.2.7) \qquad \sum_r S(n,r)s(r,m) = \delta_{nm} .$$

This immediately follows from (3.2.4) by substituting (3.2.3). Now we interpret this using the terminology of H. § 2.2. Let $P := \mathbb{N} \cup \{0\}$ with the usual ordering reversed. Then the functions s and S are elements of the incidence algebra A(P) of P. Since $S(n,n) = 1$ for $n \geq 0$, we find from H. Lemma 2.2.1 that S has an inverse $S^{\leftarrow}$, i.e.

$$\sum_{n \geq r \geq m} S(n,r)S^{\leftarrow}(r,m) = \delta_{nm} .$$

Apparently s is the inverse of S in A(P). If $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are sequences, we define the functions a and b in A(P) by

$$a(x,y) := a_{x-y} \qquad (x \geq y) ,$$

$$b(x,y) := b_{x-y} \qquad (x \geq y) .$$

By (H.2.2.1) we can then interpret a relation

$$a_n = \sum_r s(n,r)b_r \qquad (n = 1,2,\dots)$$

as

$$a = sb \ .$$

This implies

$$b = s^{\leftarrow} a = Sa \ ,$$

i.e.

$$b_n = \sum_r S(n,r) a_r \ .$$

This explains the relations (a), (b) of H.Ch.2 Problem 2 in terms of incidence algebras.

We now return to the formula (3.2.6). Expand the right-hand side and then expand e^{kz} in a power series and change the order of summation. This yields

$$r! \sum_{n=r}^{\infty} S(n,r) \frac{z^n}{n!} = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} e^{kz} =$$

$$= \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \sum_{n=0}^{\infty} k^n \frac{z^n}{n!} =$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} k^n \ .$$

It follows that

$$(3.2.8) \qquad \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} k^n = \begin{cases} r! \ S(n,r) & (n \geq r) \ , \\ 0 & (n < r) \ . \end{cases}$$

The special case $r = n$ was treated in (2.1.6). We take a second look at (2.1.7). Apply (3.2.4):

$$f_n(x) = \sum_{k=0}^{\infty} k^n x^k = \sum_{k=0}^{\infty} x^k \sum_{r=0}^{n} S(n,r)(k)_r =$$

$$= \sum_{r=0}^{n} S(n,r) \sum_{k=0}^{\infty} (k)_r x^k = \sum_{r=0}^{n} S(n,r) r! \ x^r (1-x)^{-r-1} \ .$$

Hence $f_n(x) = (1-x)^{-n-1} P_n(x)$ where

$$(3.2.9) \qquad P_n(x) = \sum_{r=0}^{n} S(n,r) r! \ x^r (1-x)^{n-r}$$

and again we find $P_n(1) = n!$.

Combinatorial interpretations. There are a number of combinatorial interpretations of the Stirling numbers of the second kind. We shall consider these below. For the sake of completeness we remark that $s(n,r)$ is the number of permutations of n symbols which have exactly r cycles (cf. [7] Ch.4.3).

Consider all the permutations of b_1 1's, b_2 2's, ..., b_r r's. Their number is the multinomial coefficient $\frac{n!}{b_1! \dots b_r!}$ (cf. (H.2.1.20)). Now, we wish to count the number of permutations (with repetition) of n symbols chosen from $x_1, x_2, \dots, x_r$ with the property that each symbol occurs at least once. Clearly this is the coefficient of $\frac{t^n}{n!}$ in the expansion of $(\frac{t}{1!} + \frac{t^2}{2!} + \dots)^r$. Hence, by (3.2.6) we have

THEOREM 3.2.1. *The number of permutations of* r *things taken* n *at a time, repeats permitted, such that each of the* r *things occurs at least once, is* $r!\,S(n,r)$.

This can also be formulated as follows.

THEOREM 3.2.2. *The number of ways* n *distinct objects can be divided over* r *distinct boxes, with no box empty, is* $r!\,S(n,r)$.

Proof. Let $o_1, o_2, \dots, o_n$ be the objects and number the boxes $x_1, x_2, \dots, x_r$. Consider one of the permutations $a_1, a_2, \dots, a_n$ counted by Theorem 3.2.1. This permutation corresponds to a division of the objects over the boxes in which o_i is in box x_j if $a_i = x_j$ $(i = 1,2,\dots,n)$. This is clearly a 1-1 correspondence.

COROLLARY. $S(n,r)$ *is the number of ways of partitioning a set of* n *elements into* r *nonempty subsets.*

Proof. This follows from Theorem 3.2.2 by disregarding the order of the boxes.

Remark. If we also no longer consider the n elements of the set as distinguishable, then the number of partitions is $p_r(n)$ (cf. H.Ch.4). (For further results see [7] Ch.5.)

Two recent problems. We apply the results of this section to two interesting problems which appeared in Elemente der Mathematik.

PROBLEM 1 (El. d. Math. 27 (1972), Aufgabe 654, p.110). Show that

$$S(n+r,n) = \sum_{1 \le k_1 \le \dots \le k_r \le n} k_1 k_2 \dots k_r \qquad (k_i \in \mathbb{N}) . \tag{3.2.10}$$

(This is a different formulation from the one which originally appeared.)

First solution. Apply the corollary of Theorem 3.2.2. Let the elements be $x_1, x_2, \dots, x_{n+r}$. If we divide $\{x_1, x_2, \dots, x_{n+r-1}\}$ into n subsets then there are n choices for the place of x_{n+r}. We can also let $\{x_{n+r}\}$ be one of the subsets and then divide $\{x_1,x_2,\dots,x_{n+r-1}\}$ into $n-1$ subsets. It follows that

$$(3.2.11)\qquad S(n+r,n) = n\,S(n+r-1,n) + S(n+r-1,n-1)\ .$$

Let $F(r,n)$ be the right-hand side of (3.2.10). Divide the sum into two parts: (i) the terms with $k_r = n$ and (ii) the terms with $k_r \le n-1$. It follows that

$$(3.2.12)\qquad F(r,n) = n\,F(r-1,n) + F(r,n-1)\ .$$

Now, (3.2.10) follows by induction from (3.2.11) and (3.2.12) since the two sides are obviously equal for $r = 1$. (We could also define $F(0,n) := 1$, in which case (3.2.12) remains correct for $r = 1$.)

Second solution. Instead of using recursion we can also prove (3.2.10) by dividing the partitions counted by $S(n+r,n)$ into classes, each of which is counted by one term on the right-hand side of (3.2.10). Let the set $\{x_1,x_2,\dots,x_{n+r}\}$ be partitioned into n nonempty subsets. We label a subset by the minimal i such that x_i is in the subset. Then order the labels: $1 = a_1 < a_2 < \dots < a_n \le n+r$. Let $b_1 < b_2 < \dots < b_r$ be the remaining x_i's. Define $k_i := |\{j \in \mathbb{N} \mid a_j < b_i\}| = b_i - i$. Then the numbers $b_1,b_2,\dots,b_r$ can be divided over the subsets in $k_1\,k_2 \dots k_r$ ways in accordance with the labeling. It is easily seen that the sequence $k_1,k_2,\dots,k_r$ satisfies $1 \le k_1 \le k_2 \le$ $\le \dots \le k_r \le n$ and that any such a sequence uniquely determines a sequence of labels $1 = a_1 < a_2 < \dots < a_n \le n+r$. This proves (3.2.10).

PROBLEM 2 (El. d. Math. <u>27</u>, 1972, Aufgabe 673, p.95). Let Φ denote a permutation of $1,2,\dots,n$ and let $F(\Phi)$ denote the number of fixed points of Φ. Show that

$$(3.2.13)\qquad A(n,k) := \frac{1}{n!}\sum_{\Phi} (F(\Phi))^k = A_k\ ,$$

where A_k is the number of partitions of $\{1,2,\dots,k\}$ and the summation is over all permutations of $\{1,2,\dots,n\}$.

Solution. Let D_m be the number of derangements of m symbols. Then

$$A(n,k) = \frac{1}{n!}\sum_{m=0}^{n} \binom{n}{m} D_{n-m}\, m^k$$

and by (2.2.5) this is the coefficient of x^n in the product

$$(3.2.14)\qquad e^{-x}(1-x)^{-1}\sum_{m=0}^{\infty} m^k \frac{x^m}{m!}\ .$$

An immediate consequence of (3.2.11) is

$$(3.2.15)\qquad e^x \sum_{\ell=1}^{k} S(k,\ell)x^\ell = \left(x\frac{d}{dx}\right)^k e^x = \sum_{m=0}^{\infty} m^k \frac{x^m}{m!}\ .$$

(3.2.14) and (3.2.15) imply that $A(n,k)$ is the coefficient of x^n in

$(1-x)^{-1} \sum_{\ell=1}^{k} S(k,\ell)x^{\ell}$ and for $n \geq k$ this coefficient is $\sum_{\ell=1}^{k} S(k,\ell)$. By the corollary to Theorem 3.2.2 this is equal to A_k.

The material of this section gives an impression of the many connections between the Stirling numbers and several of the topics treated in H.Ch.1,2,3. For much more on Stirling numbers we refer to [7].

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