
[Read March 9th, 1882.]

1. This note contains the solution of a generalized form of a problem interesting, if only from the names of the mathematicians whose attention it attracted half a century ago. The problem is,—“To find the number of independent ways in which a polygon of \( m \) sides can be divided into triangles by means of non-intersecting diagonals.”

The solution seems to have been first discovered by Euler, who gives, but without proof, the number required for polygons up to the enneagon. The proof is supplied by J. A. de Segner, in a paper in the Petersburg Transactions [Nov. Comm., t. vii., pp. 203–210].

The question was next taken up in a series of four papers in Liouville’s Journal for 1838–9. Proposed by Terquem, it was answered first by Lamé in a letter to Liouville (t. iii., pp. 505–7), and further discussed by O. Rodrigues (t. iii., pp. 547–8), M. J. Binet (t. iv., pp. 79–90), and E. Catalan (t. iv., pp. 91–94).

2. The result is most conveniently expressed in terms, not of the number \( m \) of angles of the polygon, but of the number \( n \) of triangles into which it can be divided. This number will be called the ‘order’ of the polygon; and it is clear that \( m = n + 2 \). We write \( P_n \) for the required number of partitions in the case of a polygon of the \( n \)th order.

* [See infra, pp. 106–109, and p. 70.]

10. If we put \( w = 0 \), the second set reduce to the corresponding expressions for \( u + v \) multiplied by \( c_1c_2 \); and the third set reduce to the same quantities multiplied by \( d_1d_2 \). When \( u=v=w \), the formulae reduce directly to the known expressions for \( 3u \).

It is noteworthy that the expressions for the functions of \( u + v + w \), recently communicated to the Society by Mr. M. M. U. Wilkinson, reduce, when \( w=0 \), to the forms \( S_n \), \( O_n \), \( D_n \), and \( N_n \); and that, when rendered homogeneous by the introduction of the letter \( n \), they are symmetrical with respect to the four letters \( s, c, d, \) and \( n \), and therefore constitute of themselves a complete system.


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* [See infra, pp. 106–109, and p. 70.]
3. Lamé, in the paper referred to above, arrives by geometrical reasoning at the two functional equations

\[(\alpha) \ldots P_{n+1} = P_n + P_{n-1} + P_{n-2} + \ldots + P_1 + P_0,\]

\[(\beta) \ldots \quad P_n = \frac{n+2}{2(n-1)} (P_{n-1} + P_{n-2} + \ldots + P_1);\]

whence, immediately, it follows that

\[(\gamma) \ldots P_{n+1} = \frac{4n+2}{n+2} P_n.\]

Rodrigues gives an elegant direct demonstration of (\gamma). Binet's paper contains a solution by the method of generating functions with various 'reflections,' while Catalan, using a somewhat longer process, adds to the above equations a fourth,

\[(\delta) \ldots P_{n-1} - \frac{n}{P_{n-1}} + \frac{n-1}{2} \cdot \frac{n-2}{P_{n-2}} - \ldots = 0.\]

From (\gamma), together with the obvious relation \(P_1 = 1\), we obtain at once

\[P_n = \frac{2n!}{n! (n+1)!}.\]

4. It does not seem to have occurred to any of these writers that the problem admits of extension to the case of division into polygons other than triangles, and none of their methods is alone sufficient for this purpose, though the solution for the general case is as easy as for the simple case of the triangle.

5. We notice first that there is a necessary relation between the number of sides of the polygon to be resolved and that of the polygons into which it is to be resolved. If the latter number be \(p\), we find the number \((m)\) of sides of the polygon which can be divided by diagonals into \(n\) \(p\)-gons (i.e., the polygon which is "of the \(n\)th order for division into \(p\)-gons") thus. There are clearly \(n-1\) diagonals, and each of them is a side of two component polygons. All the sides of the \(m\)-gon, together with \(n-1\) diagonals each counted twice, are therefore all the sides of \(n\) \(p\)-gons, so that

\[m + 2 (n-1) = np,\]

or

\[m = (p-2) n + 2.\]

6. To find the number, \(P_n\), of ways in which the \(m\)-gon, whose vertices we will call \(A_1, A_2, \ldots A_m\), can be divided into \(n\) \(p\)-gons, we fix our attention on one side, for instance \(A_1A_2\). This must be a side of one of the component polygons: suppose that, in a partition chosen at random, the other vertices of the \(p\)-gon which has \(A_1A_2\) for a side are \(A_a, A_b, \&c.,\) the order being \(A_1 A_a A_b \ldots A_2\). It is obviously a necessary
condition that the polygon which has the angular points from \(A_1\) to \(A_n\) inclusive for its vertices should be capable of division into \(p\)-gons; and let it be of the order \(a\). Similarly, the polygon with the angular points from \(A_n\) to \(A_1\) for its vertices must be divisible into \(p\)-gons,—let its order be \(b\), and so on. Then the first of these polygons is divisible into \(a\) \(p\)-gons in \(P_a\) ways; the second into \(b\) \(p\)-gons in \(P_b\) ways, and so on. It follows that the number of independent ways in which the \(m\)-gon can be divided into \(p\)-gons, with the condition that the \(p\)-gon which has \(A_1A_2\) for a side is \(A_1A_2A_3\ldots A_n\), is \(P_aP_bP_c\ldots\); for the central polygon \(A_1A_2A_3\ldots A_n\) serves as a neutral zone to prevent mutual interference among the partitions of the polygons \(A_1\ldots A_n\), \(A_n\ldots A_1\), \&c.

Both the number and the magnitude of the quantities \(a, b, c, \ldots\) are subject to a condition; for, since there is one of these quantities corresponding to a polygon cut off by each side of the central polygon save \(A_1A_n\), their number is \(p-1\); also, the total number of resulting \(p\)-gons is obtained by adding unity for the central polygon to \(a, b, \&c.,\) for the polygons \(A_1\ldots A_n, A_n\ldots A_1\), \&c.; we have, therefore, the further condition
\[a+b+c+\ldots = n-1.\]

It is to be observed that, if any of the vertices \(A_1, A_2, A_3\ldots\) are consecutive, there will be a diminution in the number of the quantities \(a, b, c, \&c.;\) for instance, if \(A_2\) were consecutive to \(A_1\), we should have \(c = 0\). We can, however, allow for this by the introduction of a symbol \(P_0\) equal in fact to unity. We have, then, the final result,
\[P_n = 2P_aP_bP_c\ldots, \quad \text{where} \quad a+b+c+\ldots = n-1,
\]
the number of the quantities \(a, b, c, \ldots\) being \(p-1\), and zero being admissible as a value for any of them.

7. This result may also be obtained by a method which will assign a meaning to the symbol
\[\Sigma P_aP_bP_c\ldots (a+b+c+\ldots = n-1),
\]
where the number of factors in each term is less than \(p-1\).

For the sake of clearness we will consider the case of division into pentagons, so that \(p = 5\), although the method is quite general. Let the polygon to be divided be \(ABCD\ldots XYZA\) and be of the order \(n\) (so that it has \(3n+2\) sides).

Let \(S_n\) be the number of ways of division, subject to the condition that no diagonal shall pass through either of three assigned consecutive angular points, say \(A, Z, Y\). Then, since \(B\) must be joined to \(X\) to form the pentagon \(ABXYZ\), while the polygon cut off by this diagonal (which is of the order \(n-1\)) can then be divided in all possible independent ways, we have
\[S_n = P_{n-1}.\]
Next, let \( R_n \) be the number of ways of division subject to the condition that no diagonal shall pass through either of two assigned consecutive angular points, say \( Z, Y \). Then this number = the number of partitions in which, as a further condition, no diagonal passes through \( A + \) the number in which \( A \) is joined to no vertex further round the polygon (in the direction \( ABC... \)) than \( E + \) the number in which it is joined to no vertex further round than \( H + \ldots \)

\[
= S_n + P_1 S_{n-1} + P_2 S_{n-2} + \ldots + P_{n-1} S_1
\]

\[
= P_0 P_{n-1} + P_1 P_{n-2} + P_2 P_{n-3} + \ldots + P_{n-1} P_0
\]

\[= 2 P_0 P_n \ (a + b = n - 1).\]

Again, let \( Q_n \) be the number of ways of division subject to the condition that no diagonal passes through an assigned angular point, say \( Y \). Then this number = the number of partitions in which no diagonal passes through \( Z + \) the number in which \( Z \) is joined to no vertex further round than \( D + \) the number in which it is joined to no vertex further round than \( G + \ldots \)

\[
= R_n + P_1 P_{n-1} + \ldots + P_{n-1} R_1
\]

\[= \sum P_x P_y P_z \ (a + b + c = n - 1).\]

And, in the same way, we shall obtain

\[
P_n = Q_n + P_1 Q_{n-1} + \ldots + P_{n-1} Q_1
\]

\[= \sum P_x P_y P_z P_r \ (a + b + c + \ldots = n - 1),\]

Hence, in general, if \( P_n \) be the number of ways of dividing a polygon of order \( n \) into \( p \)-gons, the expression

\[
\sum P_x P_y P_z \ldots (a + b + c + \ldots = n - 1),
\]

there being \( r \) factors in each term, is the number of ways of division under the condition that no diagonal is to pass through either of \( p - r - 1 \) assigned consecutive angular points.

8. It only remains to solve the functional equation at the end of §6; which amounts to the statement that, if

\[
f(x) = 1 + P_1 x + P_2 x^2 + \ldots \text{ ad. inf.},
\]

then \( f \) satisfies the equation

\[
f^{r-1} = \frac{f - 1}{x}
\]

or

\[f = 1 + x f^{r-1};\]

whence, by Lagrange's Theorem,

\[
f = 1 + x [f^{r-1}] + \frac{x^3}{3!} \left[ \frac{d}{dx} f^2 (x^{-1}) \right] + \&c.,
\]
where, in the square brackets, \( f \) is to be put equal to unity after differentiation.

Hence

\[
P_n = \frac{1}{n!} \left[ \frac{d^{n-1}}{dy^{n-1}} f'(x) \right]
\]

\[
= \frac{(np-n)(np-n-1) \ldots (np-2n+2) + n!}{n! \{(p-2)n+1\}!}
\]

which is therefore the number of independent ways in which a polygon of the order \( n \) (i.e., a polygon of \((p-2)n+2\) sides) can be divided into \( p \)-gons.

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**Some Elliptic Function Formulae. By the Rev. M. M. U. Wilkinson.**

[Read March 9th, 1882.]

Since

\[
k^2 \text{sn} a \text{sn} \beta \text{sn} (a-\beta) = Z(a-\beta) - Za + Z\beta,
\]

\[
k^2 \text{sn} \beta \text{sn} \gamma \text{sn} (\beta-\gamma) = Z(\beta-\gamma) - Z\beta + Z\gamma,
\]

\[
k^2 \text{sn} \gamma \text{sn} a \text{sn} (\gamma-a) = Z(\gamma-a) - Z\gamma + Za
\]

\[
k^2 \text{sn} (a-\beta) \text{sn} (\beta-\gamma) \text{sn} (\gamma-a) = -Z(a-\beta) - Z(\beta-\gamma) - Z(\gamma-a)
\]

we have

\[
\text{sn} a \text{sn} \beta \text{sn} (a-\beta) + \text{sn} \beta \text{sn} \gamma \text{sn} (\beta-\gamma) + \text{sn} \gamma \text{sn} a \text{sn} (\gamma-a)
\]

\[
= \frac{1}{k^2} \text{sn} (a-\beta) \text{sn} (\beta-\gamma) \text{sn} (\gamma-a) \ldots \ldots \ldots \ldots (1);
\]

a known formula. Substituting \( a+iK \) for \( a, \&c. \), we have

\[
\text{sn} a \text{sn} (\beta-\gamma) + \text{sn} \beta \text{sn} (\gamma-a) + \text{sn} \gamma \text{sn} (a-\beta)
\]

\[
= -k^2 \text{sn} a \text{sn} \beta \text{sn} \gamma \text{sn} (\beta-\gamma) \text{sn} (\gamma-a) \text{sn} (a-\beta) \ldots \ldots (2),
\]

since

\[
k \text{sn} a \text{sn} (a+iK) = 1.
\]

If we examine the way in which \( (1) \) can be obtained by means of the Addition Formulae, it will appear that, *mutatis mutandis*, \( \text{sn}, \text{cn}, \text{dn} \) can be permuted, and corresponding results obtained as follows:

\[
\text{dn} a \text{dn} \beta (\text{dn} a - \text{dn} \beta)(\text{sn} \gamma \text{cn} \gamma \text{dn} \beta + \text{sn} \beta \text{cn} \beta \text{dn} \gamma) (\text{sn} \gamma \text{cn} \gamma \text{dn} a)
\]

\[
+ \text{dn} \beta \text{dn} \gamma (\text{dn} \beta - \text{dn} \gamma)(\text{sn} \gamma \text{cn} \gamma \text{dn} a + \text{sn} \gamma \text{cn} \gamma \text{dn} \beta) (\text{sn} \gamma \text{cn} \gamma \text{dn} \beta + \text{sn} \beta \text{cn} \beta \text{dn} \gamma)
\]

\[
= \text{sn} a \text{cn} a \text{sn} \beta \text{cn} a \text{dn} a \text{dn} \beta \text{dn} a \text{dn} \gamma (\text{dn} a - \text{dn} \gamma - \text{dn} \beta + \text{dn} \beta - \text{dn} \gamma - \text{dn} \gamma - \text{dn} a) + \ldots
\]

\[
+ \text{dn} a \text{dn} \beta (\text{dn} a - \text{dn} \beta) \text{sn} \gamma \text{cn} \gamma + \text{dn} \beta \text{dn} a (\text{dn} \beta - \text{dn} \gamma) \text{dn} \gamma \text{cn} a \text{cn} a
\]

\[
+ \text{dn} \gamma \text{dn} a (\text{dn} \gamma - \text{dn} a) \text{sn} \beta \text{cn} \beta.
\]
Again, \[ (dn^3 \beta - dn^3 \alpha) (dn^3 \alpha - dn^3 \gamma) (dn^3 \gamma - dn^3 \beta) \]
\[ = - \frac{k^4}{k^3} \left[ \frac{dn^2 \alpha \ dn^2 \beta (dn^3 \alpha - dn^3 \beta)}{k^3} - \frac{dn^2 \gamma - k^3}{k^3} + \ldots \right] \]
\[ = - \frac{k^4}{k^3} [dn^2 \alpha \ dn^2 \beta (dn^3 \alpha - dn^3 \beta) \ sn^3 \gamma \ cn^3 \gamma + \ldots] ; \]
so that we have
\[ \frac{k^4}{k^3} \cdot \frac{dn^3 \beta - dn^3 \gamma}{\sn \cn \sn \cn \dn} + \frac{dn^3 \gamma - dn^3 \alpha}{\sn \cn \sn \cn \dn} \]
\[ = \frac{dn \beta \ dn \gamma (dn^3 \beta - dn^3 \gamma)}{\sn \cn \dn} + \ldots ; \]
and, since \[ \frac{\sn (\alpha - \beta)}{\cn (\alpha - \beta)} = \frac{\sn^3 \alpha - \sn^3 \beta}{\cn \cn \dn + \sn \cn \dn} \]
we have
\[ \frac{dn \beta \ dn \gamma \ sn (\beta - \gamma)}{\cn (\beta - \gamma)} + \ldots = \frac{k^3 \ sn (\beta - \gamma) \ sn (\gamma - \alpha) \ sn (\alpha - \beta)}{\cn (\beta - \gamma) \ cn (\gamma - \alpha) \ cn (\alpha - \beta)} \ldots (3) ; \]
and, in like manner, or from (3) by changing
\[ \sn, \cn, \dn, k^3, k^3, \text{ into } k \sn, \dn, \cn, \frac{1}{k^3}, - \frac{k^3}{k^3} \]
we have
\[ \frac{\cn \dn \sn (\beta - \gamma) \ cn (\beta - \gamma) \ sn (\beta - \gamma)}{\cn (\beta - \gamma) \ cn (\gamma - \alpha) \ cn (\alpha - \beta)} + \ldots = - \frac{k^3 \ sn (\beta - \gamma) \ sn (\gamma - \alpha) \ sn (\alpha - \beta)}{\cn (\beta - \gamma) \ cn (\gamma - \alpha) \ cn (\alpha - \beta)} \ldots (4) ; \]
and, increasing in (3) \( a, \ldots, \) by \( K, \) we get
\[ \frac{dn \ a \ sn (\beta - \gamma)}{\cn (\beta - \gamma)} + \frac{dn \ beta \ sn (\gamma - \alpha)}{\cn (\gamma - \alpha)} + \frac{dn \ gamma \ sn (\alpha - \beta)}{\cn (\alpha - \beta)} \]
\[ = dn \ a \ dn \ beta \ dn \ gamma \ sn (\beta - \gamma) \ sn (\gamma - \alpha) \ sn (\alpha - \beta) \ldots (5) ; \]
increasing in (4) \( a, \ldots, \) by \( K + k \), we have
\[ \frac{cn \ a \ sn (\beta - \gamma)}{\cn (\beta - \gamma)} + \frac{cn \ beta \ sn (\gamma - \alpha)}{\cn (\gamma - \alpha)} + \frac{cn \ gamma \ sn (\alpha - \beta)}{\cn (\alpha - \beta)} \]
\[ = k^4 \ cn \ a \ cn \ beta \ cn \ gamma \ sn (\beta - \gamma) \ sn (\gamma - \alpha) \ sn (\alpha - \beta) \ldots (6) ; \]
It can be easily seen from (1) that

\[ \text{sn}(a + \beta + \gamma) = \frac{\text{sn}(\alpha + \beta) \text{sn}(\gamma) \text{sn}(a + \gamma)}{\text{sn}(\alpha + \beta + \gamma)} \]

Again, since \( \gamma > 3 \)

\[ \text{sn}(\alpha + \beta + \gamma) = \frac{\text{sn}(\alpha + \beta) \text{sn}(\gamma) \text{sn}(a + \gamma)}{\text{sn}(\alpha + \beta + \gamma)} \]

we have, then,

\[ \text{sn}(\alpha + \beta + \gamma) = \frac{\text{sn}(\alpha + \beta) \text{sn}(\gamma) \text{sn}(a + \gamma)}{\text{sn}(\alpha + \beta + \gamma)} \]

and

\[ \text{sn}(\alpha + \beta + \gamma) = \frac{\text{sn}(\alpha + \beta) \text{sn}(\gamma) \text{sn}(a + \gamma)}{\text{sn}(\alpha + \beta + \gamma)} \]

or by permuting \(\alpha, \beta, \gamma\), we have

\[ \text{sn}(\alpha + \beta + \gamma) = \frac{\text{sn}(\alpha + \beta) \text{sn}(\gamma) \text{sn}(a + \gamma)}{\text{sn}(\alpha + \beta + \gamma)} \]
We may remark that we consider the formulae

\[
\begin{align*}
\text{sn} (\alpha + \beta) &= \text{sn}^2 \alpha - \text{sn}^2 \beta, \\
\text{cn} (\alpha + \beta) &= \text{sn} \alpha \text{cn} \beta - \text{sn} \beta \text{cn} \alpha, \\
\text{dn} (\alpha + \beta) &= \text{sn} \alpha \text{dn} \beta - \text{sn} \beta \text{dn} \alpha,
\end{align*}
\]

where denominator = \(\text{sn} \alpha \text{cn} \beta \text{dn} \alpha - \text{sn} \beta \text{cn} \alpha \text{dn} \alpha\),

simpler than the elementary formulae usually given. These do not involve \(k\) explicitly, and have numerators and denominators one degree lower than those have.

They become vanishing fractions when

\[\alpha - \beta = 2mK' + 2nK.\]

**Thursday, April 6th, 1882.**

S. ROBERTS, Esq., F.R.S., President, in the Chair.

Messrs. Buchheim, Muir, and Charles Smith were admitted into the Society.

The following communications were made:


"Note on the Condensation of Skew Determinants which are partially zero-axial, and on a Symmetric Determinant connected with Lagrange's Interpolation Problem:" T. Muir.


"On certain Loci and Envelopes belonging to triangles of given form inscribed and circumscribed to a given triangle:" Prof. Wolstenholme.

"On Binomial Biordinals:" Sir J. Cockle.


"On Polygons circumscribed about a confocal Cubic:" R. A. Roberts.

The following presents were received:

"Carte-de-Visite Likeness," from F. Scott Haydon, B.A.

"Educational Times" for April, 1882.


"The Mathematical Visitor," Vol ii., No. 1: from the Editor, Artemas Martin, M.A.


[Read April 6th, 1882.]

We define as follows:

\[ V(a, \beta) = \frac{\Theta_0 (a - \beta)}{\Theta \alpha \Theta \beta}; \]

\[ V(a, \beta, \gamma) = \frac{\Theta_0 (a - \beta) \Theta_1 (a - \gamma) \Theta_1 (\beta - \gamma)}{\Theta \alpha \Theta \beta \Theta \gamma}; \]

or

\[ V(a, \beta, \gamma) = \frac{\Theta_0 \Theta_1 (a - \beta) \Theta_1 (a - \gamma)}{\Theta \alpha \Theta \beta \Theta \gamma} V(\beta, \gamma); \]

\[ \ldots = \ldots \ldots \ldots \]

\[ V(a_1, a_2, a_3, \ldots a_n) = \frac{\Theta \alpha \Theta \beta \Theta \gamma \cdots \Theta \alpha \Theta \beta \Theta \gamma}{\Theta \alpha \Theta \beta \Theta \gamma \cdots \Theta \alpha \Theta \beta \Theta \gamma} V(a_1, a_2, a_3, \ldots a_n); \]

The definitions which we shall now give are by determinants. One line of the determinant only will be written down. The other lines are in every case got by substituting the letters in order.

\[ S(a, \beta) = \mid \sin a, 1 \mid; \]

\[ C(a, \beta) = \mid \sin a \cos a, \cos a \mid; \]

\[ D(a, \beta) = \mid \sin a \cos a, \sin a \mid; \]

\[ N(a, \beta) = - \mid \cos a \sin a, \sin a \mid; \]