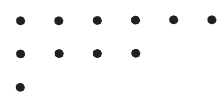
CHAPTER V

LATTICE PERMUTATIONS

93. Consider an assemblage of letters $\alpha^p\beta^q\gamma^r \ldots$ in which the numbers $p, q, r, \ldots$ are in descending order of magnitude. This particular permutation of the assemblage can be denoted by a regular graph consisting of rows of nodes. The successive rows will have $p, q, r, \ldots$ nodes respectively and the graph is the same as serves to denote the partition $(pqr\ldots)$ of the number $p + q + r + \ldots$.

Such a graph may be

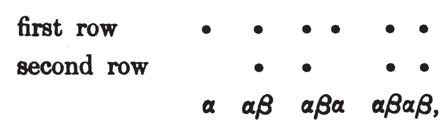


for $p = 6$, $q = 4$, $r = 1$.

The successive rows correspond to the letters $\alpha, \beta, \gamma, \ldots$ respectively.

If we take *any* permutation of $\alpha^p\beta^q\gamma^r \ldots$ we shall arrive finally at the same graph by proceeding from left to right of the permutation and placing a node in the first row, or in the second, or in the third according as we reach a letter α or β or γ, etc.

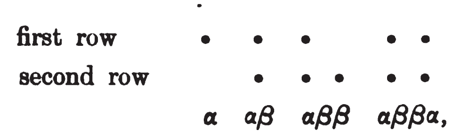
Thus if we take the permutation $\alpha\beta\alpha\beta$ of the assemblage we obtain successively in this manner



and it will be observed that each of the four graphs thus reached is regular and is in fact the graph of a partition of a number.

Since the permutation possesses this property of yielding a succession of regular graphs it is termed a "lattice permutation."

On the other hand if we treat the permutation $\alpha\beta\beta\alpha$ in the same way we reach the graphs



and since the third of these graphs is irregular we have not had before us a "lattice permutation."

In general for a permutation of $\alpha^p\beta^q\gamma^r \ldots$ if the successive graphs are all regular it is a "lattice permutation."

In other words if a dividing line be drawn between *any two* letters of the permutation and the assemblage of letters to the left of the line is found to be $\alpha^{p_1}\beta^{q_1}\gamma^{r_1} \ldots$, where $p_1 \geqslant q_1 \geqslant r_1 \geqslant \ldots$, the permutation is said to be a lattice permutation.

Ex. gr. of the assemblage $\alpha^2\beta^2$, there are only two lattice permutations, viz. $\alpha\alpha\beta\beta$ and $\alpha\beta\alpha\beta$.

These special permutations are of much use in the Theory of Partitions taken up in Volume II of this work, but they also have a special interest of their own. For instance in the Theory of Probabilities:

Suppose that there are $p + q + r + \ldots$ electors at an election and that $p, q, r, \ldots$ electors vote for candidates $\alpha, \beta, \gamma, \ldots$ by handing in tickets marked $\alpha, \beta, \gamma, \ldots$ respectively. The electors may present themselves in any order and if such gives a lattice permutation it is clear that if the flow of electors be stopped at any time and the votes be counted, the count will give a result which is not inconsistent with the final result. The enumeration of the lattice permutations leads therefore to the probability of such non-inconsistency obtaining.

94. We will first set forth certain properties possessed by the permutations.

Every permutation necessarily commences with α.

Consider any lattice permutation, say $\alpha\beta\alpha\gamma\beta$, of the assemblage $\alpha^2\beta^2\gamma$.

Write underneath the letters the first five numbers in descending order, viz.:

$$\begin{matrix} \alpha & \beta & \alpha & \gamma & \beta \\ 5 & 4 & 3 & 2 & 1 \end{matrix}$$

Starting from the left, place each number in the first, second or third row of a graph according as it stands beneath an α, a β or a γ. Thus:

$$\begin{matrix} 5 & 3 \\ 4 & 1 \\ 2 & \end{matrix}$$

This graph, since it has been formed from a lattice permutation, has the property that the numbers are in descending order of magnitude in each row read from left to right and in each column read from top to bottom. There is a one-to-one correspondence between the two-dimensional array of numbers, formed upon the graph of the partition (221) of the number 5, which possess this property and the lattice permutations of the assemblage $\alpha^2\beta^2\gamma$. For we can pass uniquely from any such array to the corresponding permutation. Thus from the array

$$\begin{matrix} 5 & 4 \\ 3 & 1 \\ 2 & \end{matrix}$$

we consider the numbers in descending order of magnitude and write down from left to right an α, a β or a γ according as the number considered is in the first, second or third row. We thus reach the permutation

$$\alpha\alpha\beta\gamma\beta.$$

In general we see that there exists a one-to-one correspondence between the lattice permutations of the assemblage $\alpha^p\beta^q\gamma^r\ldots$ and the two-dimensional arrays of the first $p+q+r+\ldots$ numbers at the nodes of the graph of the partition $(pqr\ldots)$ of the number $p+q+r+\ldots$, which are such that there is a descending order of magnitude alike in each row and in each column of the graph.

95. We can now transform these arrays so as to establish and exhibit an important property of lattice permutations. For suppose that we take the array

$$\begin{matrix} 5 & 3 \\ 4 & 1 \\ 2 & \end{matrix}$$

and write the rows as columns thus:

$$\begin{matrix} 5 & 4 & 2 \\ 3 & 1 & \end{matrix}$$

we obtain an array at the nodes of the graph of the partition (32) which is the partition conjugate to the partition (221) appertaining to the former graph. The transformed array leads to the lattice permutation

$$\alpha\alpha\beta\alpha\beta \text{ of the assemblage } \alpha^3\beta^2,$$

and in consequence there must be a one-to-one correspondence between the lattice permutations of the two assemblages

$$\alpha^2\beta^2\gamma, \quad \alpha^3\beta^2.$$

The lattice permutations of these two assemblages are therefore equinumerous. In general we may say that there is a one-to-one correspondence between the lattice permutations of the two assemblages

$$\alpha^p\beta^q\gamma^r\ldots, \quad \alpha^{p'}\beta^{q'}\gamma^{r'}\ldots,$$

if $(pqr\ldots)$, $(p'q'r'\ldots)$ are conjugate partitions.

96. There is another correspondence which it is useful to note.

If we take the graph

$$\begin{matrix} 5 & 3 \\ 4 & 1 \\ 2 & \end{matrix}$$

we observe descending orders of magnitude five times, viz. in the three rows and in the two columns. The descending orders are 53, 41, 2, 542, 31.

We can arrange these five numbers in a row so that the five descending orders are in evidence. To do this we proceed from the lattice permutations

$\alpha\alpha\beta\beta\gamma$	$\alpha\alpha\beta\gamma\beta$	$\alpha\beta\alpha\gamma\beta$	$\alpha\beta\alpha\beta\gamma$	$\alpha\beta\gamma\alpha\beta$
5 3 4 1 2	5 3 4 2 1	5 4 3 2 1	5 4 3 1 2	5 4 2 3 1

and write 5, 3 in order underneath the letters α; 4, 1 in order underneath the letters β and 2 underneath the γ.

We have thus a line-arrangement exhibiting the five descending orders in correspondence with each lattice permutation, and the descending orders have been derived from one (any one) of the associated two-dimensional arrays.

In general we determine a one-to-one correspondence between the lattice permutations of the assemblage $\alpha^p\beta^q\gamma^r\ldots$ and the permutations of the first $p+q+r\ldots$ natural numbers which exhibit the descending orders which are derived from the rows and columns of *any one* of the two-dimensional arrays associated with the lattice permutations.

97. We will now be concerned with the enumeration of the lattice permutations. First let us take the assemblage $\alpha^p\beta^q$ and denote by $(pq;)$ the number of the lattice permutations. If $q=p$, the last letter of a lattice permutation must be β, and if we delete this β we shall get every lattice permutation of the assemblage $\alpha^p\beta^{p-1}$. Hence

$$(pp;) = (p, p-1;).$$

If $p>q$, the last letter may be α or β; if this last letter be deleted we obtain all the lattice permutations of $\alpha^{p-1}\beta^q$ and of $\alpha^p\beta^{q-1}$; hence

$$(pq;) = (p-1, q;) + (p, q-1;).$$

Of this difference equation $\dfrac{(p+q)!}{(p+s)!\,(q-s)!}$ is a solution, and the particular solution required is

$$(pq;) = \frac{(p+q)!}{p!\,q!} - \frac{(p+q)!}{(p+1)!\,(q-1)!} = \frac{(p+q)!}{(p+1)!\,q!}(p-q+1).$$

Also
$$(pp;) = \frac{(2p)!}{(p+1)!\,p!}.$$

It follows from the expression obtained for $(pq;)$ that if two candidates A, B at an election can command p, q voters respectively $(p > q)$, the probability that at no time during the balloting A will have fewer votes than B is

$$\frac{p-q+1}{p+1}.$$

It is seen without difficulty that $(pq;)$ is equal to the coefficient of $x^p y^q$ in the expansion of the function

$$\frac{1-\frac{y}{x}}{1-x-y},$$

for $(pq;)$ has been shewn to be the difference of $\frac{(p+q)!}{p!\,q!}$ and $\frac{(p+q)!}{(p+1)!(q-1)!}$.

This is a redundant generating function because it contains many terms which are not applicable to the question under examination.

98. The exact generating function is obtained in the following manner. Consider

$$\sum_p (pp;)(xy)^p = 1 + xy + 2x^2y^2 + 5x^3y^3 + 14x^4y^4 + \ldots = u_{xy}.$$

The general term in u_{xy} is

$$\frac{(2p)!}{(p+1)!\,p!} x^p y^p,$$

and since

$$\sqrt{(1-4xy)} = 1 - 2xy - 2x^2y^2 - \ldots - \frac{2\,(2p)!}{(p+1)!\,p!}(xy)^{p+1} - \ldots,$$

we find that
$$2xy\,u_{xy} = 1 - \sqrt{(1-4xy)}.$$

Thus
$$u_{xy} = \frac{1}{2xy}\{1 - \sqrt{(1-4xy)}\},$$

and thence a relation that will be useful

$$u_{xy} = \frac{1}{1 - xy\,u_{xy}}.$$

99. There is another way of establishing this result which is valuable for the purpose in hand. If we examine the various lattice permutations of the assemblage $\alpha^p\beta^p$ we find that they are of two kinds, viz. prime and composite. Composite arrangements are those which are decomposable into shorter lattice permutations appertaining to assemblages $\alpha^q\beta^q$, where $q < p$. The prime lattice permutations are those which are not so decomposable.

Thus the assemblage $\alpha^2\beta^2$ has the two lattice permutations

$$\alpha\alpha\beta\beta, \quad \alpha\beta \mid \alpha\beta;$$

the first is prime; the second is composite, because it is decomposable into two shorter lattice permutations, each of which is $\alpha\beta$.

Similarly the assemblage $\alpha^3\beta^3$ has two prime arrangements, viz. $\alpha\alpha\alpha\beta\beta\beta$, $\alpha\alpha\beta\alpha\beta\beta$, and three which are composite, viz.

$$\alpha\alpha\beta\beta \mid \alpha\beta, \quad \alpha\beta \mid \alpha\alpha\beta\beta, \quad \alpha\beta \mid \alpha\beta \mid \alpha\beta.$$

The theory of prime lattice permutations becomes very simple directly the observation is made, that from *every* lattice permutation of the assemblage $\alpha^{p-1}\beta^{p-1}$ a prime lattice permutation of the assemblage $\alpha^p\beta^p$ is derivable by simply prefixing the letter α and affixing the letter β, and that in this way the whole of the lattice permutations are obtained.

Thus we have

Lattice Permutations	Prime Lattice Permutations
1	$\alpha\beta$
$\alpha\beta$	$\alpha\alpha\beta\beta$
$\alpha\alpha\beta\beta$	$\alpha\alpha\alpha\beta\beta\beta$
$\alpha\beta\alpha\beta$	$\alpha\alpha\beta\alpha\beta\beta$
$\alpha\alpha\alpha\beta\beta\beta$	$\alpha\alpha\alpha\alpha\beta\beta\beta\beta$
$\alpha\alpha\beta\alpha\beta\beta$	$\alpha\alpha\alpha\beta\alpha\beta\beta\beta$
$\alpha\alpha\beta\beta\alpha\beta$	$\alpha\alpha\alpha\beta\beta\alpha\beta\beta$
$\alpha\beta\alpha\alpha\beta\beta$	$\alpha\alpha\beta\alpha\alpha\beta\beta\beta$
$\alpha\beta\alpha\beta\alpha\beta$	$\alpha\alpha\beta\alpha\beta\alpha\beta\beta$
$\vdots$	$\vdots$

Lattice permutations are either prime or decomposable into primes.

This fact will lead us to the generating functions. For it is clear that the enumerating generating function of the prime lattice permutations is

$$xy\,u_{xy}.$$

Moreover these permutations may be combined *in all ways* to produce lattice permutations. Hence the relation

$$u_{xy} = \frac{1}{1 - xy\,u_{xy}}.$$

The relation shews simply the derivation of all lattice permutations from prime lattice permutations.

The same principle will now be applied to determine the enumerating generating function of lattice permutations of the assemblage $\alpha^p\beta^q$ where $p \geqslant q$. When $p > q$ there is no prime lattice permutation except in the particular case $p = 1$, $q = 0$, when the form is

$$\alpha.$$

Ex. gr. for the assemblage $\alpha^3\beta^2$ the arrangements are all composite, viz.

$$\alpha \mid \alpha\alpha\beta\beta, \quad \alpha \mid \alpha\beta \mid \alpha\beta, \quad \alpha\alpha\beta\beta \mid \alpha,$$
$$\alpha\beta \mid \alpha \mid \alpha\beta, \quad \alpha\beta \mid \alpha\beta \mid \alpha.$$

The generating function for the prime lattice permutations is now

$$x + xy\, u_{xy},$$

and if we write $\sum\limits_{p,q} (pq\,;)\, x^p y^q = v_{x,y}$, the fact that the arrangements are either prime or composed of primes leads at once to the relation

$$v_{x,y} = \frac{1}{1 - x - xy\, u_{xy}}\,;$$

and this, using the relation above satisfied by u_{xy}, may be written

$$v_{x,y} = \frac{u_{xy}}{1 - x u_{xy}},$$

where

$$u_{xy} = \frac{1}{2xy}\{1 - \surd(1 - 4xy)\}.$$

This is the exact form of generating function to which we have been led by the notion of prime lattice permutations.

If s be a given integer we deduce from the above result that

$$\sum_q (q + s,\ q\,;)\,(xy)^q = u_{xy}^{s+1},$$

a relation which leads to the expression of $(q + s,\ q\,;)$ in terms of $(11\,;)$, $(22\,;)$, $(33\,;)$, etc.

Thus

$$(s + 1,\ 1\,;) = \binom{s+1}{1}(11\,;),$$

$$(s + 2,\ 2\,;) = \binom{s+1}{1}(22\,;) + \binom{s+1}{2}(11\,;)^2,$$

etc.

100. Reverting to the difference equation, we find

$$\begin{aligned}(pp\,;) &= (p,\ p - 1\,;)\\ &= (p,\ p - 2\,;) + (p - 1,\ p - 1\,;)\\ &= (p,\ p - 3\,;) + 2\,(p - 1,\ p - 2\,;)\\ &= (p,\ p - 4\,;) + 3\,(p - 1,\ p - 3\,;) + 2\,(p - 2,\ p - 2\,;),\end{aligned}$$

and we notice that the last result may be written

$$(pp\,;) = (40\,;)(p,\ p - 4\,;) + (31\,;)(p - 1,\ p - 3) + (22\,;)(p - 2,\ p - 2\,;).$$

In view of this it is natural to suspect the law

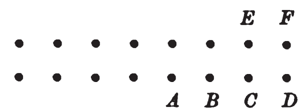
$$(pp\,;) = \Sigma\,(st\,;)(p - t,\ p - s\,;),$$

where

$$s + t = \text{constant}.$$

This may be established by utilizing the correspondence between lattice permutations and the arrangements of different numbers at the nodes of the graph of the partition (pp) of the number $2p$.

For consider the graph



The four lowest numbers may be placed

(i) at the points A, B, C, D,

(ii) " B, C, D, F,

(iii) " C, D, E, F.

Taking the case (ii) the numbers may be

$$\begin{matrix} 2 & 3 & 4 \\ 431, & 421, & 321. \end{matrix}$$

Subtracting each of these numbers from the number 5, the arrangements become

$$\begin{matrix} 3 & 2 & 1 \\ 124, & 134, & 234, \end{matrix}$$

and it is clear that they are enumerated by the number $(31\,;)$.

Similarly the arrangements at the nodes A, B, C, D and C, D, E, F are enumerated by the numbers $(40\,;)$, $(22\,;)$ respectively, and we are led to the relation

$$(pp\,;) = (40\,;)(p, p-4\,;) + (31\,;)(p-1, p-3\,;) + (22\,;)(p-2, p-2\,;).$$

Similarly it is shewn that

$$(pp\,;) = \Sigma\,(st\,;)(p-t, p-s\,;), \text{ where } s+t = \text{constant}.$$

101. Putting herein $s+t=p$, we find

$$(pp\,;) = (p, 0\,;)^2 + (p-1, 1\,;)^2 + (p-2, 2\,;)^2 + \ldots \text{ to } \tfrac{1}{2}(p+1) \text{ or } \tfrac{1}{2}(p+2) \text{ terms},$$

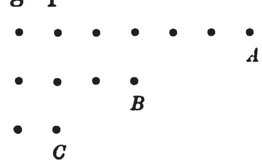
according as p is uneven or even.

Hence the identity

$$\frac{(2p)!}{(p+1)!\,p!} = 1^2 + (p-1)^2 + \{\tfrac{1}{2}p(p-3)\}^2 + \left\{\frac{1}{3!}p(p-1)(p-5)\right\}^2 + \ldots$$

$$\text{to } \tfrac{1}{2}(p+1) \text{ or } \tfrac{1}{2}(p+2) \text{ terms}.$$

102. Taking up the lattice permutations of the assemblage $\alpha^p\beta^q\gamma^r$ and considering the associated graph



with different numbers placed at the nodes in such wise that descending order of magnitude is visible alike in each row and in each column, we find that we may detach a node without destroying the regularity of the graph in three ways. The nodes are marked A, B, C, ...

But if q were equal to p we could not detach A; if r were equal to q we could not detach B; if q and r were both equal to p we could not detach B or A.

Hence the difference equations

$$(pqr;) = (p-1, q, r;) + (p, q-1, r;) + (p, q, r-1;),$$
$$(ppq;) = (p, p-1, q;) + (p, p, q-1;),$$
$$(pqq;) = (p-1, q, q;) + (p, q, q-1;),$$
$$(ppp;) = (p, p, p-1;).$$

A solution of the first of these equations is

$$\frac{(p+q+r)!}{(p+p')!(q+q')!(r+r')!},$$

where p', q', r' are positive or negative (including zero) integers, such that

$$p' + q' + r' = 0.$$

This may be at once established by applying to it the difference equation.

The particular solution which corresponds to the present problem is

$$\frac{(p+q+r)!}{p!q!r!} - \frac{(p+q+r)!}{(p+1)!(q-1)!r!} - \frac{(p+q+r)!}{p!(q+1)!(r-1)!}$$
$$+ \frac{(p+q+r)!}{(p+2)!(q-1)!(r-1)!} + \frac{(p+q+r)!}{(p+1)!(q+1)!(r-2)!} - \frac{(p+q+r)!}{(p+2)!q!(r-2)!}.$$

This may be simplified so as to exhibit the result

$$(pqr;) = \frac{(p+q+r)!}{p!q!r!}\left(1 - \frac{q}{p+1}\right)\left(1 - \frac{r}{q+1}\right)\left(1 - \frac{r}{p+2}\right),$$

which is true whatever equalities subsist between p, q, and r.

The unsimplified form shews that $(pqr;)$ is equal to the coefficient of $x^p y^q z^r$ in the expansion of the redundant generating function

$$\frac{\left(1 - \frac{y}{x}\right)\left(1 - \frac{z}{y}\right)\left(1 - \frac{z}{x}\right)}{1 - x - y - z},$$

for the expanded numerator is

$$1 - \frac{y}{x} - \frac{z}{y} + \frac{yz}{x^2} + \frac{z^2}{xy} - \frac{z^2}{x^2},$$

and its six terms yield respectively the six terms of the unsimplified expression for $(pqr;)$.

The reader should also make note of another form of $(pqr;)$, viz.

$$\frac{(p+q+r)!}{(p+2)!(q+1)!r!}(p-q+1)(q-r+1)(p-r+2).$$

The Theory of the Prime Lattice Permutations of the assemblage $\alpha^p\beta^q\gamma^r$ awaits investigation. It seems to present a certain amount of difficulty. Until this has been surmounted we cannot pass to the real generating function from the redundant form above given.

103. We now pass to the general case, viz. the number

$$(p_1 p_2 p_3 \dots p_n;).$$

The difference equation to be satisfied is

$$(p_1 p_2 p_3 \dots p_n;)$$
$$=(p_1-1, p_2 p_3 \dots p_n;)+(p_1, p_2-1, p_3 \dots p_n;)+\dots+(p_1 p_2 p_3 \dots p_n-1;).$$

We are led to the redundant generating function

$$\frac{\prod_{s=t+1}^{s=n}\prod_{t=1}^{t=n-1}\left(1-\frac{x_s}{x_t}\right)}{1-(x_1+x_2+x_3+\dots+x_n)},$$

and to the two forms of result

$$(p_1 p_2 p_3 \dots p_n;)=\frac{(p_1+p_2+p_3+\dots+p_n)!}{p_1!p_2!p_3!\dots p_n!}\prod_{s=t+1}^{s=n}\prod_{t=1}^{t=n-1}\left(1-\frac{p_s}{p_t+s-t}\right),$$

$$(p_1 p_2 p_3 \dots p_n;)$$
$$=\frac{(p_1+p_2+p_3+\dots+p_n)!}{(p_1+n-1)!(p_2+n-2)!(p_3+n-3)!\dots p_n!}\prod_{s=t+1}^{s=n}\prod_{t=1}^{t=n-1}(p_t-p_s+s-t).$$

The question in probability that is here solved may be stated as follows:

If n candidates at an election have

$$p_1, p_2, p_3, \dots p_n$$

voters in their favour respectively, where

$$p_1 \geqslant p_2 \geqslant p_3 \dots \geqslant p_n,$$

and if at any instant $P_1, P_2, P_3, \dots P_n$ voters have recorded their votes in favour of the several candidates respectively, the probability that

$$P_1 \geqslant P_2 \geqslant P_3 \dots \geqslant P_n$$

always is

$$\prod_{s=t+1}^{s=n}\prod_{t=1}^{t=n-1}\left(1-\frac{p_s}{p_t+s-t}\right).$$

It will be remarked also that the number $(p_1 p_2 p_3 \dots p_n;)$ enumerates the arrangements of $p_1+p_2+p_3+\dots+p_n$ different numbers at the nodes of a lattice which has $p_1, p_2, p_3, \dots p_n$ nodes (respectively) in the successive rows where the arrangements are such that there is in evidence a descending order of magnitude alike in each row and in each column.

The circumstance that this enumeration is not altered by interchanging all the rows and all the columns establishes the fact that if

$$(p_1 p_2 p_3 \dots),\ (q_1 q_2 q_3 \dots)$$

be conjugate partitions

$$(p_1 p_2 p_3 \dots ;) = (q_1 q_2 q_3 \dots ;).$$

In view of the results above set forth this involves a remarkable property of numbers.