On the Trail of the Catalan Sequence

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The Catalan sequence of integers is a long established one which has found wide application since it was first shown to provide the solutions to various cases of a non-trivial problem in the field of geometry. There will, however, be many readers who are unaware that the sequence bears his name as the result of a 19th century paper published by him at the age of twenty four, and even more so that Catalan was by no means the first person to encounter its elements which had actually been known to others working on the problem from at least eighty years previous to this.

INTRODUCTION

In a recent piece of work, a sequence of numbers revealed themselves as being embedded within the methodology one of us (PJL) was using. These numbers were observed to be the Catalan numbers, named after the Belgian author (Eugène Charles Catalan, 1814–1894) of a paper published in 1838 in which he examined a problem receiving fresh attention at the time. This has led us to consider more closely the early occurrences of the numbers and so to present this short, and hopefully informative, exposition which outlines something of their historical development. The trail of the sequence in its geometrical setting, up until the time when Catalan himself became involved, is an interesting one which is deserving of explanation.

THE SEQUENCE AND ITS EARLY DEVELOPMENT

Definition

The Catalan sequence is defined to be the sequence of integers

$$\{c_r\} = \{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, \ldots\} = \{1, 1, 2, 5, 14, 42, 132, 429, \ldots\}, \qquad (1)$$

which can be generated by the closed form for the general $(r+1)$th term

$$c_r = \frac{1}{r+1}\binom{2r}{r}, \quad r = 0, 1, 2, \ldots, \qquad (2)$$

or, given $c_0 = 1$, by the recursion formula

$$c_{m+1} = \sum_{k=0}^{m} c_k c_{m-k}, \quad m = 0, 1, 2, \ldots \qquad (3)$$

The sequence is seen to have the (ordinary) generating function

$$G(x) = \frac{1-\sqrt{1-4x}}{2x}, \qquad (4)$$

being that function whose expansion in infinite series form possesses as its coefficients the sequence terms themselves (that is to say, $G(x) = \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots$).

Discovery in the Context of Geometry

Towards the end of the 1750s, the Hungarian mathematician and physicist Johann Andreas von Segner (Professor at the University of Halle, Germany) presented to the St. Petersburg (Petropolis) Academy a paper concerning the number of ways in which an n-sided convex polygonal region could be dissected into smaller triangular areas by means of non-intersecting diagonals joining its vertices (the answer being c_{n-2}, $n \geq 3$). The original problem lies, however, with Leonhard Euler, who had mentioned it to the Prussian born contemporary mathematician and versatile scholar Christian Goldbach earlier in the decade. In the second half of a letter to him from Berlin dated 4th September, 1751, Euler wrote "Ich bin neulich auf eine Betrachtung gefallen, welche mir nicht wenig merkwürdig vorkam" ["I have recently made another observation, which I found not a little remarkable"] and gave results for polygons of 3, . . ., 10 sides using the explicit formula

$$D_n = \frac{2 \cdot 6 \cdot 10 \cdot 14 \cdot 18 \cdot \cdots \cdot 2(2n-5)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \cdots \cdot (n-1)} = \frac{2}{(n-1)!}\prod_{i=2}^{n-1}(2i-3) \qquad (5)$$

for the number of divisions D_n for an n-sided polygon, a fact which cannot be regarded as in any way common knowledge over the years. Typically, Euler gave no proof, but he did admit that arriving at it had involved some considerable effort and had no doubt that there was an easier method of solution. Von Segner, whom Euler had also informed of the problem, developed a recurrence formula for this number — essentially equation (3), remembering that c_0 has no geometrical analogue D_2 here — and published accordingly in the St. Petersburg Proceedings *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae* (a literal translation is *New Proceedings of the Imperial Academy of Sciences of Petropolis*). The paper appeared in 1761[1] and included a table which quantified the decompositions, unfortunately containing a propagating error which began with what we write as c_{13}.† An article by Euler[2] (unsigned, but evidently by him) accompanied von Segner's, where we find a different recursion enumerating such divisions which is much easier to use than von Segner's for values of n only moderately large and is the basis of (5) in his earlier correspondence with Goldbach. The result is simple, relating D_{n+1} and D_n ($n \geq 3$, $D_3 = 1$ given) as

$$D_{n+1} = \frac{2(2n-3)}{n} D_n. \quad (6)$$

Euler, in this 1761 paper, notes explicitly the Hungarian's failure in his calculations, and himself lists $c_1, \ldots, c_{23}$ without any such mistakes (note that $c_{23} = O(10^{11})$). Over thirty years later Nicolaus Fuss — a Swiss born mathematician and astronomer working at the St. Petersburg Academy — generalised von Segner's work to tackle the division of an n-gon into m-sided sub-regions,[3] of which triangular dissection ($m = 3$) is just a special case. This more difficult problem was brought to his attention in letters from the German mathematician Pfaff (Johann Friedrich, Professor of Mathematics at Helmstedt and in his time mentor to Gauss), as the opening paragraph of Fuss' paper states. According to Fuss, Pfaff had attained a solution and had written to him asking if he had seen any others. Fuss could not refrain from probing the problem himself and duly published the results of his deliberations; we see that the terms $c_1, \ldots, c_9$ appear at the conclusion of the paper, along with other decompositions enumerated for values of $m = 4, \ldots, 8$.

Enter Catalan et al.

It was not until 1838, the year he secured a lectureship in descriptive geometry at the École Polytechnique in Paris, that Catalan produced the article[4] in Volume 3 of Joseph Liouville's *Journal de Mathématiques Pures et Appliquées* (or *Journal de Liouville*, as it was known soon after its launch in 1836) from which the sequence of numbers has since become a named one — the Catalan sequence. Gabriel Lamé — at the time Professor of Physics also at the École Polytechnique and to whom the problem of polygon division had been communicated by Liouville‡ — improved on von Segner's solution methodology and reproduced independently Euler's linear recursive equation (6). His work — in print as an extract of a letter[5] written by him to Liouville — was published adjacent to Catalan's article in 1838, wherein a footnote by editor Liouville makes it clear that Euler had already stated the same result in his publication of the previous century[2] but without the derivation now given by Lamé some seventy seven or so years later. Making no reference to Euler, Catalan, for his part, now began with Lamé's "très simple" recursion, and proceeded to obtain the closed form for c_r (2) in the combinatorial notation which is in common use today.* He also expressed it as the difference

$$c_r = \binom{2r}{r} - \binom{2r}{r-1}, \quad (7)$$

whereby the integer nature of the sequence becomes immediately evident given $c_0 = 1$ (the rhs holding for $r \geq 1$), although there are, of course, other forms available, such as (given c_0, c_1)

$$c_r = \binom{2r-1}{r-1} - \binom{2r-1}{r+1}, \quad r \geq 2. \quad (8)$$

It suffices to say that the Euler-von Segner problem and the more general Pfaff-Fuss problem — having generated a flurry of activity around 1838 and during the next few years[7–13] — continued to be worked on long after this time, and polygon dissection is but one of the many ways in which the Catalan numbers can be interpreted. A comprehensive discussion of these, and some of the one-to-one correspondences which exist between them, can be found in the 1977 American MSc Thesis of Kuchinski.[14] Both this and the 1976 version of Gould's extensive bibliography[15] (from which Kuchinski's references are taken) are useful sources of material relevant to the origins of what we know to be the Catalan sequence, together, for example, with the article of Brown[16] and the 1970 PhD Dissertation of Growney.[17]

Returning to the problem of polygon division, a paper of some interest is a much later one by Taylor and Rowe in the 1881/2 volume of the *Proceedings of the London Mathematical Society*.[18] Acknowledging the work of Euler, von Segner, Catalan, Lamé, Rodrigues and Binet on the triangular decomposition problem, they state that

> "It does not seem to have occurred to any of these writers that the problem admits of extension to the case of division into polygons other than triangles, and none of their methods is alone sufficient for this purpose, although the solution for the general case is as easy as for the simple case of the triangle."

Their subsequent analysis demonstrated this, but no mention is made of Fuss' paper[3] or those related to it[11–13] — including an instructive (but seemingly little read) 1841 article[11] by the German publisher JA Grunert of the journal *Archiv der Mathematik und Physik* who re-worked Fuss' method of solution and developed it to confirm Euler's result (5) — which must have been an oversight, deliberate or otherwise. On this point, it is probably fair to give them the benefit of the doubt since mathematicians of that period did not enjoy the same level of access to disseminated work as we do today. This is evident also from a paper written by Gelin who, in the journal *Mathesis* a short time afterwards,[19] re-derived both Euler's and von Segner's recursions for triangular decomposition of a general polygon, citing neither them nor any relevant prior work at all; it took a follow up piece by one of the journal's editors, P Mansion,[20] to put the work in proper chronological context after Catalan himself (by then aged about seventy years old) had pointed out the extent of such an omission. There are lots of other instances where people appear to have worked on this type of problem 'blind', so to speak, falsely believing that it was a new one. Kirkman, for instance, conducted considerable solo work on polygon partitioning, beginning in earnest with a paper published in 1857[21] which gives no historical context to the analysis.

† Von Segner's table gave c_{13} as 751900 (instead of the correct value of 742900) through an arithmetic slip, causing errors in all subsequent results which accommodated polygons of 16 up to 20 sides. We learn from the paper that Euler had already indicated to von Segner the first few results for such a table, which made the defective entries all the more regrettable.

‡ Who in turn had it previously suggested to him by a Monsieur O Terquem, a much respected geometrist who in his time co-founded the French journal *Nouvelles Annales de Mathématiques*. It is established that he had obtained both Euler's and von Segner's recursions using properties of factorials, and contacted Liouville with a view to seeing a new derivation. It is not known whether he actually published anything on the problem.

* This being just one of "quelques [mathematical] conséquences" he arrived at based on the findings of Lamé. A section was later added to the paper, published separately by Catalan in the 1839 volume of the journal.[6]

Applications: Some General Remarks

From the literature available it is clear that the Catalan numbers were cemented into the mainstream body of mathematical knowledge handed down to us from work inspired by a purely artificial geometrical problem dreamt up by Euler — the key figure of 18th century mathematics — and reported on about two hundred and fifty years ago in Berlin by private letter. Whilst simple to state, its solution proved to be far from easy. According to Binet, the aforementioned Terquem was said to have remarked of it[8] "que ce problème d'analyse a vainement exercé des hommes habiles" ["that this analytical problem has fruitlessly exercised the talents of clever men"]. As already stated, the application of the numbers is diverse, and by way of illustration below are given just three well known combinatorial questions which are resolved with immediate reference to them. Using the given definition (2), then for $n \geq 1$ c_n describes, for example,

- the number of dissections of a convex $(n + 2)$-gon into n triangles by $n - 1$ non-intersecting diagonals (Figure 1(a));
- the number of ways to associate $n + 1$ non-commuting elements by a non-associative binary product (Figure 1(b));
- the number of lattice paths in the plane that begin and end at respective points $(0,0)$ and (n, n) and do not, at any point, cross the main diagonal between the two (Figure 1(c));

three cases (for $n = 1,2,3$) are shown in each figure.

In addition to the polygon problem — which attracted considerable attention in the 1830s — we note the simultaneous formulation of the Catalan numbers by him as being the solution to a question regarding product parenthesisation, as seen in Figure 1(b). Catalan, in the final section of the 1838 paper,[4] first highlighted the connection between the two, writing "Je terminerai cette note par la solution d'un problème qui a une liaison remarquable avec la question de Géométrie traitée M. Lamé" ["I shall finish this note with the solution of a problem which has a remarkable link with the question of Geometry dealt with by Mr. Lamé"]; a short paper appears dated a month later by Rodrigues on the same idea.[22] One finds that this period of the 19th century brought together both past ideas and current ones of the time, causing the Catalan numbers to come to light in the form to which we are accustomed and marking the start of much fruitful research into their properties and so the identification of many problems in which they play an integral role. A Cayley demonstrated the relevance of Catalan numbers to binary tree representations of operator theory in an 1859 paper,[23] and they likewise came into play in Whitworth's 1878 consideration of lattice paths of the type in Figure 1(c).[24] Their surprising ubiquity was noted by Lucas in his 1891 book *Théorie des Nombres*.[25] He recognised the isomorphic nature of seemingly disparate problems which were in fact very much connected through the numbers, and remarked rather aptly that "souvent des vérités qui paraissent distinctes et éloignées l'une de l'autre ne diffèrent que par la forme extérieure et, en quelque sorte, par le vêtement qui les couvre" ["often truths which seem distinct and far removed from each other are only different in their exterior form and, in some way, in the manner they are dressed up"]; it is the magnitude of this phenomenon which has resulted in the sequence becoming one of the most prevalent ones along with the Fibonacci sequence. It has to be said, however, that it has perhaps not enjoyed the recognition it deserves within the mathematical community outside of those working in the field of combinatorics. Many modern textbooks, having defined the sequence, pay only lip service to its range of application, a matter addressed directly in the 1997 London Mathematical Society Popular Lecture on the Catalan numbers (delivered by Professor MD Atkinson, University of St. Andrews) which we were fortunate to attend in London.

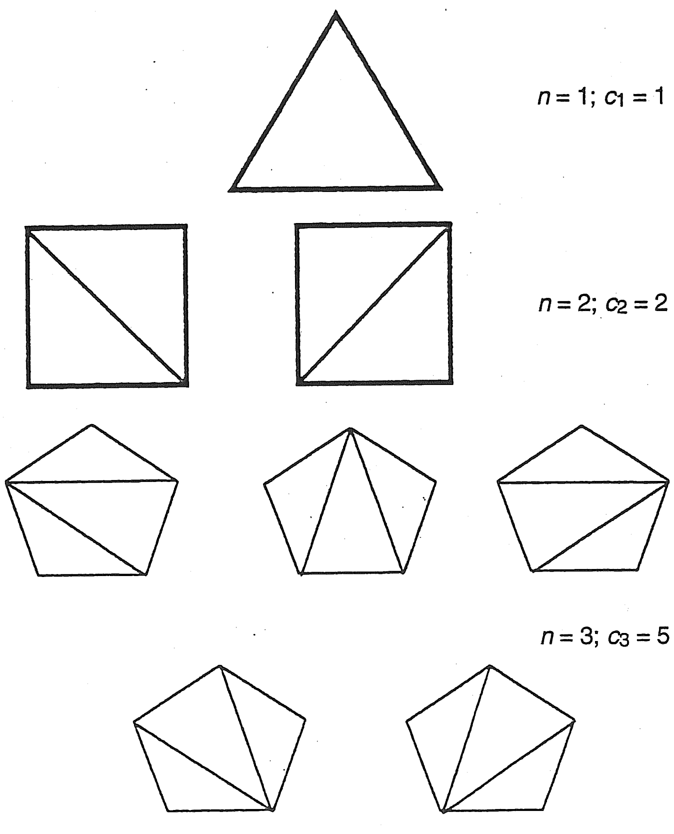

Fig. 1(a). Polygon dissection

ab	$n = 1;\ c_1 = 1$
$(ab)c$ $a(bc)$	$n = 2;\ c_2 = 2$
$(ab)(cd)$ $a(b(cd))$ $a((bc)d)$ $(a(bc))d$ $((ab)c)d$	$n = 3;\ c_3 = 5$

Fig. 1(b). Parenthesising products

SUMMARY

The facts surrounding the emergence of the Catalan numbers from the middle of the 18th century through Euler are certainly interesting, providing a glimpse into some of the working practices of academics in past times.† They remind us that the incremental addition of knowledge to the solution of a problem can, in total, have as great an impact as a more dramatic insight. Equally importantly, in our opinion, they serve to demonstrate the value of communication and dialogue between like-minded

† During the completion of this article it has come to our attention that there may be other past figures of mathematics and science, in addition to those mentioned here, who had encountered the sequence before Catalan. We are currently looking into this.

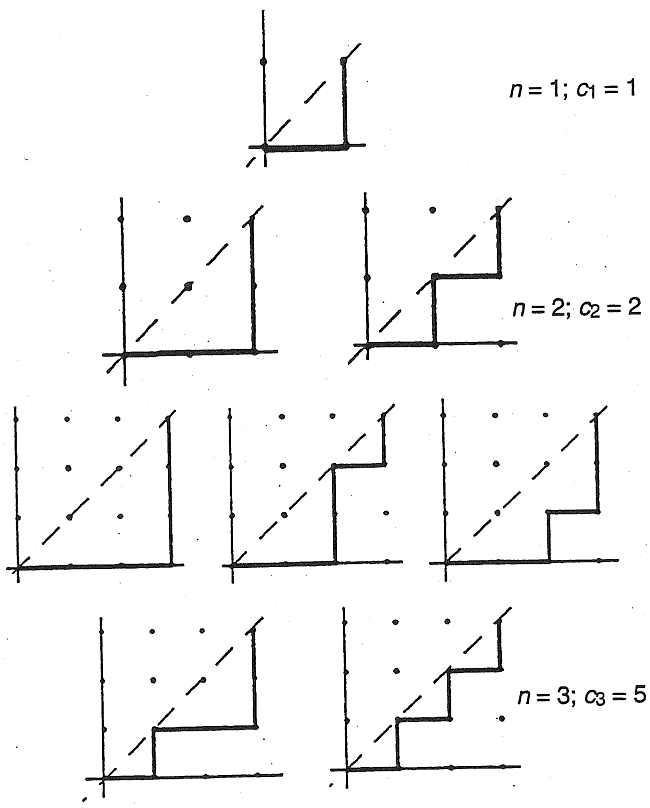

Fig. 1(c). Lattice paths

people. Purely individual intellectual pursuit can at times lead to frustration and isolation, but if one is willing to share one's mathematical ideas (along with some of the potential credit, of course), the benefit to the wider community as a whole can be increased. We consider Euler's personal discovery of the Catalan numbers and the subsequent crystallisation of the work of his and others to the point where, many years later, they became formally recognised as constituting an important sequence to be associated with Catalan, to be a particularly good example of this. In this regard the fact that they took on his name, rather than that of, say, Euler, von Segner or Fuss before him, is largely an irrelevance.

Acknowledgements

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Remark: A cursory glance at References 4–10,12,13 emphasises the influence of Liouville in the events surrounding the publication of Catalan's paper. Aged only twenty two he was selected as *répétiteur* in C L Mathieu's course on analysis and mechanics at the École Polytechnique, where, primarily because of the almost simultaneous demise of two other important review journals (including the only French one), he set about establishing his own. Liouville realised that the vigorous activity of talented French-speaking mathematicians required a suitable outlet for dissemination of their research, and should be awarded mention as a genuine catalyst in encouraging some of his contemporaries to work on the problem of polygon division. The mathematical journal he founded survives today, with the same title, as the world's second oldest.

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