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The 18th Century Chinese Discovery of the Catalan Numbers

P. J. LARCOMBE

A Chinese academic got there first in spotting the significance of the Catalan numbers.

Consider the Catalan sequence $\{c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7, \ldots\} = \{1, 1, 2, 5, 14, 42, 132, 429, \ldots\}$ generated most easily by the expression for the general $(r+1)$th term

$$c_r = \frac{1}{r+1}\binom{2r}{r}, \quad r = 0, 1, 2, \ldots, \qquad (1)$$

with which most of us are conversant. One might well suppose that the sequence originated with Eugène Catalan, for it is in an 1838 paper on mathematical aspects of (triangulated) polygon division that (1) is first given in this form and from which the sequence was to take its name. Readers of *Mathematical Spectrum* may have seen the equation

$$c_{n-1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{n!} 2^{n-1}, \quad n > 1, \qquad (2)$$

in an article by Vun and Belcher (reference 1) in which the Catalan numbers are shown to arise in three different ways, one of these being the polygon dissection problem. In fact Leonhard Euler had already identified the numbers $c_1, c_2, c_3, \ldots$, in this geometrical setting in the middle of the previous century (see references 2 and 3). A readable account of the problem is available in H. Dörrie's book *100 Great Problems of Elementary Mathematics: Their History and Solution* (Dover Publications, New York, 1965) as Problem No. 7. Many people will, however, not have access to a 1988 paper by Luo (reference 4) who asserts that the full sequence was known prior to this by Antu Ming (c.1692–1763), a Chinese scholar with a wide variety of scientific and mathematical interests. Since Luo's article is published in Chinese this fact is not common knowledge in the Western World. I wish here, therefore, to set down some of the historical details he provides and to call attention to a novel formulation of the numbers therein which is due to Ming.

At the beginning of the 18th century a French Jesuit, Pierre Jartoux, brought with him to China the three expansions

$$\begin{aligned}
\pi &= 3\left\{1 + \frac{1^2}{4 \cdot 3!} + \frac{(1 \cdot 3)^2}{4^2 \cdot 5!} + \frac{(1 \cdot 3 \cdot 5)^2}{4^3 \cdot 7!} + \cdots\right\}, \\
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \\
\operatorname{versin}(x) &= 1 - \cos(x) \\
&= \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \cdots,
\end{aligned}$$

all of them without proof. The second and third (valid for all values of x) are immediately recognisable as Maclaurin series and are credited to James Gregory (1638–1675), while the first is attributed to Isaac Newton (1642–1726). Ming saw these results, but, suspecting that Western mathematicians would be unwilling to share their derivations, he set about obtaining them for himself. In this he succeeded, using a mixture of arithmetic and (imported) Euclidean geometry as his basic tools, and he went on to derive six further expansions of other trigonometric functions. Applying recurrence methods systematically, his algorithms have the striking feature of program and calculation, and he unquestionably laid a foundation for the operation of infinite series in China which constitutes an important contribution to the country's historical development in mathematics. His methods were, of course, not rigorous by today's standards, and the question of what was actually meant by a proof there at that time is discussed by Jami (reference 5) in relation to Ming's work. Luo states that Ming discovered the Catalan numbers through his geometric models, and he highlights one or two associated representations of the function $\sin(m\alpha)$ as power series in $\sin(\alpha)$ in which they appear. Ming dealt with the values $m = 2, 3, 4, 5, 10, 10^2, 10^3, 10^4$, and found that the series for $\sin(3\alpha)$ and $\sin(5\alpha)$ terminated whilst his other expansions were infinite. We see, for example, how the full Catalan sequence occurs in each of the following two results:

$$\begin{aligned}
\sin(2\alpha) &= 2\left\{\sin(\alpha) - \sum_{n=1}^{\infty}\left[\frac{c_{n-1}}{2^{2n-1}}\right]\sin^{2n+1}(\alpha)\right\} \\
&= 2\sin(\alpha) - \sin^3(\alpha) - \tfrac{1}{4}\sin^5(\alpha) \\
&\quad -\tfrac{1}{8}\sin^7(\alpha) - \cdots, \qquad (3)
\end{aligned}$$

and

$$\begin{aligned}
\sin(4\alpha) &= 2\left\{2\sin(\alpha) - 5\sin^3(\alpha) + \sum_{n=1}^{\infty}\left[\frac{8c_{n-1} - c_n}{4^n}\right]\sin^{2n+3}(\alpha)\right\} \\
&= 4\sin(\alpha) - 10\sin^3(\alpha) + \tfrac{7}{2}\sin^5(\alpha) \\
&\quad +\tfrac{3}{4}\sin^7(\alpha) + \cdots. \qquad (4)
\end{aligned}$$

We do not concern ourselves with the convergence properties of these series. It is readily seen from (1) that (3) and (4) agree with an alternative formulation of $\sin(m\alpha)$ (m integer) which is said to be Euler's, namely,

$$
\begin{aligned}
\sin(m\alpha) &= m\sin(\alpha) - \left[\frac{m(m^2-1^2)}{3!}\right]\sin^3(\alpha) \\
&\quad + \left[\frac{m(m^2-1^2)(m^2-3^2)}{5!}\right]\sin^5(\alpha) \\
&\quad - \left[\frac{m(m^2-1^2)(m^2-3^2)(m^2-5^2)}{7!}\right] \\
&\quad \times \sin^7(\alpha) + \cdots,
\end{aligned}
$$

from which it is immediately obvious why the series form of $\sin(m\alpha)$ is only infinite for m even; for odd m the sum contains $\frac{1}{2}(m+1)$ terms in odd powers of $\sin(\alpha)$ up to and including the $\sin^m(\alpha)$ term.

Among other findings of Ming are well known recurrence formulae for Catalan numbers which are often assumed in the West to have first been established later in the 18th century or even more recently. Also of interest is a method devised by Ming for generating the sequence which seems to have escaped the notice of the vast majority of combinatorialists and historians of mathematics. In the present brief discussion it is described, for convenience, with reference to polynomial multiplication which is equivalent to the vector based method developed by Luo when interpreting Ming's work.

Let polynomials $M_n(x)$ $(n \geq 1)$ be defined by $M_1(x) = x$, $M_2(x) = x^2$, and thereafter according to the equation

$$M_{p+1}(x) = \left\{2\sum_{k=1}^{p-1} M_k(x) + M_p(x)\right\} M_p(x), \quad p \geq 2.$$

Thus, for instance,

$$
\begin{aligned}
M_3(x) &= 2x^3 + x^4, \\
M_4(x) &= 4x^4 + 6x^5 + 6x^6 + 4x^7 + x^8, \\
M_5(x) &= 8x^5 + 20x^6 + \cdots + 8x^{15} + x^{16}.
\end{aligned}
$$

It is easily proved by induction that

$$M_p(x) = \sum_{k=p}^{2^{p-1}} \alpha_k^{(p)} x^k, \quad p \geq 1,$$

where $\alpha_p^{(p)}, \ldots, \alpha_{2^{p-1}}^{(p)} > 0$. In other words, the degree of $M_p(x)$ is 2^{p-1} and the lowest power of x it contains is x^p. Ming's surprising result is that for $p \geq 3$,

$$\sum_{k=1}^{p} M_k(x) = \sum_{k=1}^{p} c_{k-1}x^k + R_p(x),$$

where the remainder $R_p(x)$ is a polynomial whose lowest power of x is x^{p+1} and highest power is $x^{2^{p-1}}$. Verification of this can be made by performing the necessary mathematical operations on any of the mainstream computer algebra systems which lend themselves to such a task. A proof lies beyond the remit of this article. The above compares with the standard identity

$$\tfrac{1}{2}\left[1 - \sqrt{1-4x}\right] = \sum_{k=1}^{\infty} c_{k-1}x^k,$$

which follows easily from the binomial series

$$(1-t)^{1/2} = \sum_{k=0}^{\infty} \binom{\frac{1}{2}}{k}(-t)^k$$

with $4x$ replacing t, since in view of (2), for $k \geq 1$,

$$
\begin{aligned}
\binom{\frac{1}{2}}{k} &= \frac{\frac{1}{2}(\frac{1}{2}-1)\cdots(\frac{1}{2}-k+1)}{k!} \\
&= \frac{(-1)^{k-1}}{2^{2k-1}} c_{k-1}.
\end{aligned}
$$

Around 1730, Ming started to write a book which included his analysis containing the Catalan numbers. It was completed by his student, Chen Jixin, in 1774, but not actually published until more than sixty years later, over seventy years after Ming's death. As Luo emphasises, therefore, Ming's achievements in connection with the Catalan sequence should not be determined by the date that his work appeared formally in print but by the period during which he worked on the results and formally wrote them up. Since the latter predates Euler's comments on the numbers as being the enumerative solutions to triangular decompositions of polygons (as a 1751 letter of his to Christian Goldbach shows; see reference 2), it is Ming who deserves recognition as the true discoverer of the sequence. During Ming's lifetime Chinese mathematicians had not yet begun to use symbolic mathematical notation, and his calculations, expressed in words, have been re-written in the syntax of contemporary mathematics by Luo who has performed a valuable service to the scientific community.

It is stressed once again that Ming's mathematical work was to a large extent underpinned by particular aspects of Western scientific knowledge introduced into China by Jesuit missionaries during the 17th and 18th centuries. Throughout this period the most significant branch of mathematics to which China was exposed was Euclidean geometry, since until then there was no comparable axiomatic deductive system; conventional Chinese geometry had instead been based on the right-angled triangle. In arithmetic and algebra, areas usually considered more familiar to Chinese custom, the Jesuits' innovations had more to do with mathematical methods than with concepts. By the beginning of the 17th century an important part of China's internal mathematical heritage had been lost or had become incomprehensible. Thus, in learning Western science, the Chinese not only rejuvenated their mathematics but also rediscovered their own history, identifying some of the ancient methods with those they took from the Jesuits. In this context Ming's book, *Ge Yuan Mi Lu Jie Fa*, reflects the manner in which mathematical information was absorbed and assimilated in China, being characteristic of the era in terms of both its contents and the history of its composition. Jami (reference 5), whilst not mentioning explicitly Ming's awareness of the Catalan numbers, provides us with an informative general commentary on his ground-breaking work.

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Sums of Powers

ROGER COOK

Fermat's Last Theorem tells us that when $k > 2$ a kth power cannot be expressed as a sum of two kth powers; so when can a kth power be expressed as a sum of kth powers?

The Pythagorean equation

$$x^2 + y^2 = z^2$$

is probably the most familiar Diophantine equation. If x and y have a common factor d then d also divides z, so the solution is a multiple of a *primitive* solution where the highest common factor of x and y is 1. In a primitive solution x and y cannot both be even; if they were both odd we would have $z^2 \equiv 2 \mod 4$, which is not possible. Thus in a primitive solution, one of x and y, x say, is even and the other is odd. The general primitive solution is then given by

$$x = 2ab, \quad y = a^2 - b^2, \quad z = a^2 + b^2,$$

where a and b are integers satisfying $a > b > 0$, not both odd and having highest common factor 1.

Since the Fermat equation

$$x^k + y^k = z^k$$

has no solution in positive integers when $k > 2$, a natural question to ask next is: 'When can a kth power be expressed as a sum of kth powers?' From work on Waring's problem we know that all positive integers can be expressed as a sum of s kth powers provided that s is large enough. If we just want to express all large integers in this way then it is known that $s = Ck \log k$ terms are sufficient, with a positive constant C. It is probably true that $s = 4k$ terms are sufficient but that is far from what can be proved at present. Here we will look at what is known about expressing a kth power as the sum of a relatively small number of kth powers. We begin with the question: 'What do we know about equations

$$x^k + y^k + z^k = t^k\,?\text{'} \qquad (1)$$

Cubic equations

Two simple solutions of

$$x^3 + y^3 + z^3 = t^3 \qquad (2)$$

in positive integers are

$$x = 3, \quad y = 4, \quad z = 5, \quad t = 6$$

and

$$x = 1, \quad y = 6, \quad z = 8, \quad t = 9.$$

According to Dickson (reference 4, p. 550), the first of these solutions had already been noted by P. Bungus in 1591. In 1753 L. Euler obtained a family of solutions by means of the substitution

$$x = p + q, \quad y = p - q, \quad z = r - s, \quad t = r + s,$$

which reduces the equation to

$$p(p^2 + 3q^2) = s(s^2 + 3r^2).$$

He was then able to go on to find the complete solution to the equation in rationals; see reference 8 (Theorem 235):

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