On the K-Partitions of the R-Gon and R-Ace

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Source: *Philosophical Transactions of the Royal Society of London*, 1857, Vol. 147 (1857), pp. 217-272

Published by: Royal Society

Stable URL: http://www.jstor.com/stable/108618

XII. *On the* K-*partitions of the* R-*gon and* R-*ace.*

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Communicated by ARTHUR CAYLEY, *Esq.*, *F.R.S.*

Received November 13,—Read December 11, 1856.

I. BY the k-*partitions of an* r-*gon*, I mean the number of ways in which it can be divided by $k-1$ diagonals, of which none crosses another; two ways being different only when no cyclical permutation or reversion of the numbers $1\,2\,3..r$ at the angles can make them alike: and by the k-*partitions of an* r-*ace* (a pencil of r rays in space or a plane), I mean the number of ways in which it can be divided into k smaller pencils, by the introduction of $k-1$ connecting lines, of which none enclose a space; two ways being different only when by no cyclical permutation or reversion of the numbers $1\,2\,3..r$ in the angular spaces of the r-ace they can be made identical. The polygon here considered is the section of a pyramid, and its discussion includes that of the polyace.

The enumeration of the partitions of the polygon and polyace is indispensable in the theory of the polyedra. In a memoir "On the x-edra which have an $(x-1)$-gonal base, and all their Summits Triedral," in the Transactions of the Royal Society, 1856, page 399, I have investigated the $(r-2)$-partitions of the r-ace, or the r-gon; for the number of x-edra there determined is exactly that of these $(r-2)$-partitions. What follows may be considered as a completion of the investigation in that memoir begun; yet not properly a continuation, inasmuch as the results there obtained are here deduced by a different and more general method.

II. A partition of an r-gon is *reversible* or *irreversible*: reversible, when it is symmetrical about a diameter or bisector of the figure, so that the configuration is unaltered by a semirevolution about that line, which is called *an axis of reversion*, of which axes there may be one or many; and *irreversible*, when it is reversible about no axis. An irreversible is about no axis its own reflexion.

An axis of reversion is always a bisector of the r-gon, and is *agonal*, *monogonal*, or *diagonal*, according as it passes through no angle, one angle only, or two angles of the r-gon; and the polygon is said to be about that axis, *agonally*, *monogonally*, or *diagonally reversible.*

A diagonal axis may be drawn or undrawn; a monogonal or agonal axis is always undrawn.

III. A partition is said to be m-ly reversible when it has m axes of reversion. The simple $2n$-gon ($k=1$) is $2n$-ly reversible, having n agonal and n diagonal axes; and its sides may be so loaded with polygons, that this number of axes shall be either retained or diminished. The simple $(2n+1)$-gon has $2n+1$ monogonal axes, on which an r-gon may be built to have the same or a smaller number. To help our conceptions, we may always suppose our r-gon regularly inscribed in a circle; but it is evident that the *syntypy* of two identically-partitioned r-gons in no wise depends on such *symmetry*, but may remain after any distortion of either r-gon which does not change the angles on any diagonal. So that, if we wish to build an r'-gon on an inscribed r-gon, we need not fear exceeding the limits of the circle by our additions, while we may *suppose* these all contained within it.

A partition is said to be m-ly irreversible, when it has an irreversible sequence of configuration m times repeated in the circuit of the r-gon. This sequence will occupy $\frac{r}{m}$ angles; and, from whatever angle we begin to read, we shall see a sequence of $\frac{r}{m}$ sides irreversible, such that through the mid-point of it no axis of reversion can be drawn.

Obs. 1. Hence a $2m$-ly irreversible has an irreversible sequence, simple if $m=1$, and m-ple if $m>1$, occupying half the circuit of the r-gon; but a $(2m+1)$-ly irreversible has no repeated sequence occupying half its circuit.

IV. THEOREM A. *Every reversible* (1+k)-*partition of an* r-*gon has two reversible sequences of configuration which are bisected by alternate and equidistant axes of reversion, and has not more than two, whatever be the number of these axes.*

For, first, let there be only one axis of reversion in the r-gon: there must be two aspects of configuration observable from opposite ends of that axis, otherwise the figure would be reversible about a perpendicular to that axis, *i. e.* there would be two axes, contrary to hypothesis.

Secondly, let there be more axes of reversion than one; any axis a bisects an aspect A, because the figure is unchanged by a semirevolution about that axis; and the axis b next in order to a along the circumference bisects an aspect B. This B is different from A; for if not, the series of configurations read from a to b will be that read from b to a, and there will be either a vertex or a side centrally placed between a and b, having on both sides the same aspect, or an axis of reversion can be drawn between a and b; but b is the next in order to a, which is absurd; therefore B is not A. Now b bisecting the aspect B must have the axial termination a at the same distance on either side of it, and for the same reasons a must have the axial termination b at the same distance on either side of it; so that the terminations of the axes must recur at equal distances in the order $..ababa...$ bisecting the aspects ..ABAB.. And this series of aspects has as many terms as there are axial terminations, viz. $4m$ terms, if the number of axes is even, and $4m+2$ if it be odd. Wherefore no aspect different from A and B can be bisected by any axis, and A and B, different reversible aspects or sequences, are bisected by alternate equidistant axes. Q. E. D.

We may call A and B the two axial configurations.

V. *Obs.* 2. A reversible partition of the r-gon having more than one axis of reversion, has both reversible and irreversible sequences repeated in the circuit of the r-gon, which occupy an interval equal to that between alternate axes; those being reversible sequences which begin and end with a side or angle carrying an axial termination, and those being irreversible which begin and end at any other side or angle.

Obs. 3. But a singly reversible partition has no sequence repeated in the circuit of the r-gon; for if it had a repeated sequence reversible as read from no point, the r-gon would not be reversible; and if it had a reversible sequence repeated in the circuit of the r-gon, it would not be singly reversible.

Obs. 4. A $2m$-ly reversible r-gon, if $r>2m$, has beginning at any angle of the r-gon which is not the termination of an axis, an irreversible sequence, simple, if $m=1$, and m-ple, if $m>1$, occupying half the circuit of the r-gon, and repeated in the other half.

VI. THEOREM B. *When the number of axes of reversion is odd in any partitioned* r-*gon, none is perpendicular to another; and when that number is even, every one is perpendicular to some other.*

For when the number is odd, there is on each side of any one a an equal number of terminations of other axes, all equidistant from a and from each other. And when that number is even, there is an odd number there of such terminations. Whence the truth of the proposition is evident.

VII. THEOREM C. *When the axes of reversion are odd in any partitioned* r-*gon, each one bisects both axial configurations; and when they are even in number, each bisects but one, read on it alike at either end, and half the axes carry one, and half the other, axial configuration.*

This is very evident from what is proved in Theorem A, that the axial configurations present themselves alternately upon the axes in order.

Cor. 1. If there be both agonal and diagonal axes, there is an equal number of each kind; and, as this number is even or odd, so is each axis perpendicular to one of its own or of the other kind.

Obs. 5. A $(2m+1)$-ly reversible partition never has an irreversible sequence occupying half the circuit of the r-gon and repeated in the other half; for this would require that every axis of reversion should carry the same configuration at both ends, which are points in those two sequences.

VIII. THEOREM D. *If a diagonal be perpendicular to an axis of reversion in any* h-*ly reversible* r-*gon, it is one of a system of not fewer than* h *diagonals symmetrically placed about the centre. And all diagonals not perpendicular to that axis form pairs making each an angle bisected by that axis, or that produced.*

For if h be odd, every axis carries the same perpendicular; and if h be even, at least $\frac{1}{2}h$ axes carry that perpendicular on opposite sides of the centre; and these axes are equidistant from each other: whence the first part of the theorem is evident. The second part follows from the definition of an axis of reversion.

Cor. The intersection of two produced diagonals equidistant from the centre, in any reversibly partitioned r-gon, is upon an axis of reversion.

IX. THEOREM E. *No* (2m+1)-*gon has an agonal or diagonal, and no* (2m)-*gon has a monogonal, axis of reversion.*

For a monogonal axis must have an equal number of vertices of the r-gon on each side of it, besides the vertex through which it passes; hence r is odd. And a diagonal axis must have an equal number of vertices on either side, besides the two through which it passes. And an agonal axis, which passes through no angles, must have an equal number of vertices on either side. Hence, in these two latter cases, r is even.

X. THEOREM F. *If one axis of reversion is monogonal in a partitioned* r-*gon, all its axes of reversion are monogonal and odd in number.*

For r is odd, and the r-gon cannot have either a diagonal or an agonal axis; and as each axis bisects two aspects, A and B, A must be opposite to B in the circle ..ABABAB.. of axial configurations; *i. e.* the number of its terms is $2(2\dot{m}+1)$, wherefore the axes are odd in number.

XI. THEOREM G. *If there be a drawn axis of reversion,* a, *in a* $(1+\mathrm{k})$-*partitioned* r-*gon, there cannot be more than one other axis. If there be another,* b, *it is undrawn, and perpendicular to the former,* a, *and is either agonal or diagonal, as* $\mathrm{r}=4\mathrm{m}+2$, *or* $\mathrm{r}=4\mathrm{m}$.

For, if there be a second axis, b, it cannot meet the drawn one, and must be undrawn. And all the k diagonals are symmetrically placed about or upon this b; therefore a, meeting it and bisecting it in the centre of the r-gon, meets it at right angles; otherwise (Theorem D) two diagonals would meet b in the centre, which is impossible. And no line besides b can so meet a; wherefore a and b are the only axes. As a is not a monogonal axis, neither is b (Theorem F). If $r=4m$, a has on either side an even number of sides of the r-gon, and b, bisecting that system, is diagonal; if $r=4m+2$, a has on either side an odd number of sides of the r-gon bisected by b, which is therefore an agonal axis. Q. E. D.

XII. THEOREM H. *If there be more than one undrawn axis of reversion in a partitioned* r-*gon, the* r-*gon is built regularly on a polygonal nucleus* (Q), *which is reversible about all the axes of reversion of the* r-*gon, and has no drawn diagonal.*

For consider the symmetry of the r-gon about any one of its axes, a, which are all undrawn (Theorem G). We see, on each side of a, f *marginal faces*, limited each by one diagonal d and certain sides of the r-gon, the $2f$ diagonals d forming pairs making angles bisected by a. Let these $2f$ faces be erased: the $2f$ lines d are now sides of an r'-gon ($r'<r$), which is still reversible about a, for the symmetry about a is not disturbed by the erasures. This r'-gon has also $2f'$ marginal faces which can be erased, leaving an r''-gon ($r''<r'$) still reversible about a; and thus by erasure of all the pairs of faces about a, the r-gon will be finally reduced to a polygon P, having no diagonals but what are bisected at right angles by a (Theorem D). Let Q be that portion of P which contains the centre of the r-gon, and Q_1, Q_2, &c. the remaining portions of P limited by perpendiculars to a. Then Q has evidently no diagonals.

Next consider the symmetry of the r-gon about any other axis b. We can reduce it by erasure of pairs of faces about b to a polygon P′, consisting of a portion Q′ about the centre and having no diagonals, and of portions limited by diagonals perpendicular to b. Q and Q′ are polygons about the centre having no diagonals; they are therefore one polygon Q. The polygons Q_1, Q_2, the remaining portions of P, have disappeared in the erasure of pairs of faces about b and Q; that is, every one of them, Q_m, is one of a set of two or more faces standing symmetrically about Q. Therefore Q is a nucleus, reversible about all the axes of the r-gon, free from diagonals, and having everything arranged symmetrically about it in the r-gon. Q. E. D.

XIII. THEOREM K. *If there be no axis of reversion in a partitioned* r-*gon except one undrawn*, a, *the figure is built symmetrically about* a *and about a polygonal nucleus* P, *which has either no diagonals, or only what are at right angles to* a.

This is established by the former part of the preceding demonstration.

But the polygon P, in this singly reversible r-gon, if it has diagonals perpendicular to the axis, is not properly a nucleus; nor is there any reason why one portion of it, Q, should be called a nucleus rather than Q_1 or Q_2; for P can be constructed by loading opposite sides of any of them with proper polygons. But if P is a simple polygon, it is properly the nucleus of the figure, which is made by loading opposite sides of it, on different sides of the axis, with the same polygons, thus preserving the reversibility about the axis.

XIV. THEOREM L. *If a diagonal axis of the nucleus is an axis of reversion of the* r-*gon, it is a diagonal axis thereof.*

For it passes through two angles of the r-gon.

THEOREM M. *An agonal axis of the* r-*gon is an agonal axis of the nucleus.*

For a diagonal or monogonal axis of the nucleus passes through a vertex of the r-gon, and cannot be an agonal axis of it.

THEOREM N. *If an agonal axis of the nucleus is an axis of the* r-*gon, it may be an agonal, or a diagonal, or a monogonal axis of it.*

For the sides of the nucleus through which that agonal axis passes may be sides of the r-gon, or they may be loaded with agonally reversible polygons having the same agonal axis: this will then be an agonal axis of the r-gon.

Or those sides may be loaded both with monogonally reversible polygons, having their axes in prolongation of that agonal axis: this makes it a diagonal axis of the r-gon.

Or one of those sides may be loaded with a monogonally reversible polygon, and the other either not at all, or with an agonally reversible: this, if the axes of the imposed polygon are in direction with that agonal axis, makes it a monogonal axis of the r-gon.

XV. THEOREM O. *A monogonal axis of the nucleus may be either a monogonal or a diagonal axis of the* r-*gon.*

For the side of the nucleus which that axis bisects may be unloaded, or loaded with an agonally reversible polygon, so as to make it a monogonal axis of the r-gon. Or that side can be loaded with a monogonally reversible polygon, so as to make the axis a diagonal one of the r-gon.

It is easily seen, and unnecessary to be formally propounded, how and how far the converses of these latter propositions are to be laid down.

Def. A *clear axis of reversion* meets no diagonal at right angles.

A *scored or loaded axis* of reversion meets one or more diagonals at right angles.

An *e-scored axis* of reversion meets e diagonals at right angles, of course bisecting them.

The clear and the loaded axis are both supposed undrawn.

XVI. THEOREM P. *If a clear axis of a* $(2m+3)$*-ly reversible* r*-gon be scored in any way by* e *diagonals at right angles to it, the* r*-gon becomes singly reversible about that scored axis.*

For, let ABC..be the axes of reversion of the r-gon N. The scored axis A remains an axis of reversion, because the symmetry about is not disturbed by the perpendicular scores; but none of the other axes BC...is perpendicular to A (Theorem B); wherefore each meets singly all the scores upon A, and is no axis of reversion (Theorem D) of the scored r-gon N′. Let then, M, any other diameter of N′ not amongst ABC... be an axis of reversion; it meets all the diagonals to which it is not perpendicular in pairs whose angles it bisects (Theorem D); wherefore these are pairs of equidistants from the centre; now the intersections of all these, except the newly-added scores upon A, are on the axes BC... (*Cor.* Theorem D); wherefore M, passing through the centre and one of these intersections, is one of the lines ABC..., contrary to hypothesis, which is absurd. Therefore A is the only axis of reversion of N′. Q. E. D.

XVII. THEOREM Q. *If any loaded axis* (A) *of a* $(2m+3)$*-ly reversible* r*-gon* (N) *be cleared by erasure of the diagonals perpendicular to* A, *the cleared figure* (N′) *is singly reversible about that axis* A.

For, let ABC...be the axes of reversion of N, on which are the diagonals at right angles to them *abc*...forming a system of lines symmetrically placed about the centre (Theorem D). No one of these B, after the erasure of a from A, is an axis of reversion of N′, because the diagonal b at right angles to B is not one of a system of lines symmetrically placed about the centre (Theorem D). Let then, M, any other diameter of N′ be an axis of reversion of N′; this line meets all the diagonals of N′ not perpendicular to it in pairs, whose angles it bisects, wherefore these pairs are equidistants from the centre of N′; but all the intersections of these pairs lie on the lines ABC...(*Cor.* Theorem D); wherefore M is one of these lines ABC... Q. E. A. Therefore A is the only axis of reversion of N′. Q. E. D.

XVIII. THEOREM R. *If a clear axis* (A) *of reversion of a* $2m$*-ly reversible partitioned* r*-gon* (N) *be scored by perpendiculars to it symmetrically about the centre, the figure is not made singly reversible about* (A); *but if it be so scored unsymmetrically about the centre, the scored figure* (N′) *is singly reversible about that axis* (A).

For there is an axis perpendicular to A (Theorem B) about which the symmetry is not disturbed by symmetric scores, *i. e.* pairs of parallels to it equidistant from the centre, or a diameter parallel with such pairs.

But when the scores are not such pairs, or a diameter and such pairs, the axis perpendicular to A is no axis of reversion evidently; nor is any other diameter of N′, for it meets all those scores singly. Wherefore A is the only axis of reversion of N′. Q. E. D.

XIX. THEOREM S. *If a clear axis* A *of a singly reversible partitioned* r-*gon* N′ *be scored by diagonals perpendicular to it, the scored figure* N *becomes sometimes singly, and at other times* $(2m+3)$-*ly reversible about the scored axis.*

For the r-gon N′ being singly reversible has no axis of reversion perpendicular to A; and no addition of diagonals parallel to the perpendicular diameter can make it an axis of reversion; for that addition cannot alter its intersections with the diagonals of N′; wherefore N is not $2m$-ly reversible (Theorem D and B). If N′ should be the N′ of Theorem Q, and the scores upon A should be those erased in that theorem, N will be the $(2m+3)$-ly reversible of that theorem: but if this is not the case in both these conditions, N will remain, like N′ unscored, singly reversible about A, since the scores do not disturb the symmetry about that axis. Q. E. D.

What precedes about singly reversibles with loaded axes is sufficient for our present purpose, which is to show that before we can determine the number of singly reversibles, with clear and loaded axes, it is necessary that we should know the number of $(2m+1)$-ly reversible $(1+k-e)$-partitions of the r-gon which have a clear axis, *i. e.* which have clear axes; for here the configurations about all the axes are alike (Theorem C); and also that of the $2m$-ly reversible $(1+k-e)$-partitioned r-gons which have one configuration about clear axes, and also of those which have both their configurations about clear axes. This matter will be more evident as we proceed.

XX. Let $R^{2hagdi}(r, k)_n$, $R^{hag}(r, k)_n$, $R^{hdi}(r, k)_n$, $R^{hmo}(r, k)_n$ denote the whole number of $(1+k)$-partitioned r-gons built on the n-gonal nucleus $(n>2)$, which have (Theorem C)

$R^{2h.agdi}(r, k)_n$, h agonal and h diagonal,

$R^{h.ag}(r, k)_n$, h agonal only,

$R^{h.di}(r, k)_n$, h diagonal only, and

$R^{h.mo}(r, k)_n$, h monogonal axes only, of reversion.

We shall denote those having all their axes clear by c subscript to R; those having no clear axes by zero subscript to R; those of the second and third classes which have half their axes, bearing one configuration, clear, by $\frac{1}{2}c$ subscript, and those of the first, which have half their axes clear, the agonal or the diagonal ones, by ac or dc subscript to R. We write

$$R^{2h.agdi}(r, k)_n = R_c^{2h.agdi}(r, k)_n + R_{ac}^{2h.agdi}(r, k)_n + R_{dc}^{2h.agdi}(r, k)_n + R_0^{2h.agdi}(r, k)_n,$$
$$R^{h.ag}(r, k)_n = R_c^{h.ag}(r, k)_n + R_{\frac{1}{2}c}^{h.ag}(r, k)_n + R_0^{h.ag}(r, k)_n,$$
$$R^{h.di}(r, k)_n = R_c^{h.di}(r, k)_n + R_{\frac{1}{2}c}^{h.di}(r, k)_n + R_0^{h.di}(r, k)_n,$$
$$R^{h.mo}(r, k)_n = R_c^{h.mo}(r, k)_n + R_0^{h.mo}(r, k)_n.$$

In the second and third lines the second subclass is of course nothing when h is odd. In the fourth class the number of axes of reversion is always odd. And in all those equations we suppose $n \not< 3$.

XXI. Before we can proceed to investigate formulæ for the determination of these numbers, it is necessary,—

Problem a. *To find the* k-*divisions of the* r-*gon* or r-*ace.*

By the $(1+k)$-divisions of an r-gon I mean the entire number of ways in which k diagonals can be drawn in it, none crossing another, all ways being different which occupy different angles $1\,2\,..\,r$ of the r-gon. Thus there are five 3-divisions, but only one 3-partition of the pentagon made by drawing a pair of diagonals. And there are five 3-divisions of a pentace made by breaking it into three triaces, but these are all the same 3-partition.

If we call the number sought of $(1+k)$-divisions of the r-gon $D(r, k)$, we can express it in terms of $D(r', k')$, $r'<r$, and $k'<k$.

For consider any diagonal b, drawn from any angle β of the r-gon, dividing it into a $(3+h)$-gon and a $(r-h-1)$-gon. This line b will be drawn in the $(1+k)$-divisions along with every $(1+\varepsilon)$-division of the $(3+h)$-gon, combined with every $(k-\varepsilon)$-division of the $(r-h-1)$-gon, ε diagonals being drawn on one side, and $k-\varepsilon-1$ diagonals on the other side of b.

That is, $$\Sigma_\epsilon D(3+h, \varepsilon)\times D(r-h-1, k-\varepsilon-1)$$

taken from $\varepsilon=0$ to $\varepsilon=k-1$, is the number of $(1+k)$-divisions in which that line b will be seen.

If, now, we give to h every value from $h=0$ to $h=r-4$, we shall have counted every $(1+k)$-division in which any line b appears that can be drawn from that angle β. If we put for β each of the r angles in succession, that is, if we multiply by r, we shall have enumerated every $(1+k)$-division in which any line b appears, that is drawn from any angle β. But we have thus handled twice, once from either extremity, every line in every set of k diagonals; that is, we have counted every $(1+k)$-division $2k$ times. Wherefore the correct result is

$$2k.D(r, k)=r.\Sigma_h\Sigma_\epsilon\{D(3+h, \varepsilon).D(r-h-1, k-\varepsilon-1)\},$$

where every value of h from $h=0$ to $h=r-4$ is to be combined with every value of ε from $\varepsilon=0$ to $\varepsilon=k-1$. If, then, we know these divisions for all values of r and k up to $D(r-1, k-1)$, we obtain $D(r, k)$ by addition. And as $D(r+0)=1$, and $D(r, 1)=\frac{1}{2}r.(r-3)$, $D(r, 2)$ is given, and thence $D(r, 3)$, and so on, up to $D(r, k)$, for any given values of r and k.

XXII. To find $D(r, k)$ in terms of r, for the general value of r, and for a given value of k, we write

$$D(r, k)=Ar^{2k}+Br^{2k-1}+..+Lr+M;$$

for we know that this number is not greater than $\{\frac{1}{2}(r.(r-3))\}^{k|-1}\times\overline{|k+1}^{-1}$, that of all possible sets of k diagonals that cross or not. Then, from the $2k+1$ equations,

$$\begin{aligned}
D(\ k+3, k)&=A(\ k+3)^{2k}+B(\ k+3)^{2k-1}+..+M,\\
D(\ k+4, k)&=A(\ k+4)^{2k}+B(\ k+4)^{2k+1}+..+M,\\
\vdots\quad &= \quad\vdots\\
D(3k+3, k)&=A(3k+3)^{2k}+B(3k+3)^{2k+1}+..+M,
\end{aligned}$$

the left members of which are to be calculated by the preceding article, we can determine the coefficients A, B, .. LM.

So far as I have pursued the inquiry, I find always one factorial form of the function $D(r, k)$; but in order to prove that this is always the form, it is necessary to show that, if

$$D(3+h, \varepsilon) \qquad =\frac{(3+h)^{\varepsilon|1}}{\underline{|\varepsilon+1}}\times\frac{(3+h-\varepsilon-2)^{\varepsilon|1}}{\underline{|\varepsilon+2}}, \text{ and}$$

$$D(r-h-1, k-\varepsilon-1)=\frac{(r-h-1)^{k-\varepsilon-1|1}}{\underline{|k-\varepsilon}}\times\frac{(r-h-k+\varepsilon-2)^{k-\varepsilon-1|1}}{\underline{|k-\varepsilon+1}},$$

$$D(r, k) \qquad =\frac{r}{2k}\Sigma_h\Sigma_\varepsilon\{D(3+h, \varepsilon).D(r-h-1, k-\varepsilon-1)\}=\frac{r^{k|1}}{\underline{|k+1}}\times\frac{(r-k-2)^{k|1}}{\underline{|k+2}},$$

from $h=0$ to $h=r-4$, and from $\varepsilon=0$ to $\varepsilon=k-1$; which is the expression that I continually find.

This summation I must leave to the learned and industrious reader; but, meanwhile, I shall venture to enunciate with the best demonstration, such as it is, that occurs to me, the following

THEOREM T. *The number of* (1+k)-*divisions of an* r-*gon*, i. e. *of all the ways in which* k *diagonals can be drawn in it, none crossing another, is*

$$D(r, k)=\frac{r^{k|1}}{\underline{|k+2}}\times\frac{(r-k-2)^{k|1}}{\underline{|k+1}}.$$

For first, let the r-gon be divided into triangles, *i. e.* let $k=r-3$. Here the $(3+h)$-gon of the preceding article can have only h diagonals, or $\varepsilon=h$; and the final result, the last equation of that article, becomes

$$D(r, r-3)=\frac{r}{2(r-3)}.\Sigma_h\{D(3+h, h).D(r-h-1, r-h-4)\},$$

from $h=0$ to $h=r-4$.

Hence we obtain for $r=4$, $r=5$, $r=6$, &c.,

$$D(4, 1)=\tfrac{4}{2}.D(3, 0).D(3, 0)=2=\frac{4^{4-3|1}}{\underline{|4-1}},$$

$$D(5, 2)=\tfrac{5}{4}.\{D(3, 0).D(4, 1)+D(4, 1).D(3, 0)\}$$

$$=5=\frac{5^{5-3|1}}{\underline{|5-1}},$$

$$D(6, 3)=\tfrac{6}{6}\{D(3, 0)D(5, 2)+D(4, 1).D(4, 1)+D(5, 2)(D3, 0)\}$$

$$=14=\frac{6^{6-3|1}}{\underline{|6-1}},$$

$$D(7, 4)=\tfrac{7}{8}\{D(3, 0)D(6, 3)+D(4, 1)D(5, 2)+D(5, 2)D(4, 1)+D(6, 3).D(3, 0)\}$$

$$=\tfrac{7}{8}\{14+10+10+14\}=42$$

$$=\frac{7^{7-3|1}}{\underline{|7-1}};$$

and as we have demonstrated that $D(8, 5)$ is a sum of products of these continuous functions, it is a continuous function, and has a permanent form: wherefore it is sufficiently evident that

$$D(r,\ r-3)=\frac{r^{r-3|1}}{\lceil r-1}.$$

We know that $D(r, k)=0$, if $k>0$, for every value of r from $r=3$ to $r=k-2$; for no $(k-2)$-gon has k diagonals none crossing another. Hence $(r-k-2)^{k|1}$ is a factor of $D(r, k)$. Therefore

$$D(r,\ k)=(r-k-2)^{k|1}(A'r^k+B'r^{k-1}+\ldots+L'r+M'),$$

$$D(k+3,\ k)=1.2\ldots k\big(A'(k+3)^k+B'(k+3)^{k-1}+\ldots+L'(k+3)+0\big),$$

$$\frac{(k+3)^{k|1}}{\lceil k+2\lceil k+1}=A'(k+3)^k+B'(k+3)^{k-1}+\ldots+L'(k+3),$$

$$\frac{(k+3+1)(k+3+2)\ldots(k+3+k-1)}{\lceil k+2\lceil k+1}=A'(k+3)^{k-1}+B'(k+3)^{k-2}+\ldots+L'.$$

Hence
$$A'=\frac{1}{\lceil k+2\lceil k+1}$$

$$L'=\frac{1.2.3.(k-1)}{\lceil k+2\lceil k+1};$$

consequently

$$\frac{D(r,\ k)}{r}=(r-k-2)^{k|1}\times\frac{(r+1)(r+2)\ldots(r+k-1)}{\lceil k+2\lceil k+1},\ \text{and } D(r,\ k)=\frac{r^{k|1}}{\lceil k+2}\cdot\frac{(r-k-2)^{k|1}}{\lceil k+1};\ \text{Q. E. D.}$$

XXIII. Theorem U. *Every* m*-ly irreversible* k*-partition of an* r*-gon occurs* 2r : m *times among the* k*-divisions, and every* m*-ly reversible* k*-partition of it is found* r : m *times among the* k*-divisions.*

For any m-ly irreversible k-partition has a different configuration about $r:m$ successive angles (III.), and is nowhere its own reflexion; so that $r:m$ more configurations the reflexions of the former are read on the reversed face of the r-gon. And as all these configurations are found among the k-divisions separately enumerated about the same angle, this partition is counted $2r:m$ times among them.

And any m-ly reversible k-partition of the r-gon has $r:m$ different configurations in the interval between the termination of any axis A and its repetition, that is, about $r:m$ angles if A be an agonal axis, and about $r:m$ sides if it be diagonal. The central configuration of these $r:m$ is about the axis alternate with A; the others form pairs of configurations reflecting each other. All these are constructed and counted separately among the k-divisions about the same point of the r-gon; *i. e.* this partition is counted $r:m$ times among the k-divisions. Q. E. D.

Let $R^m(r, k)$ denote the entire number of m-ly reversible $(1+k)$-partitions of the r-gon about all nuclei, and all kinds of axes, and let $I^m(r, k)$ stand for the complete number of m-ly irreversible $(1+k)$-partitions.

The theorem U may be thus expressed:

$$D(r,\ k)=\Sigma_m\Big\{\frac{r}{m}\cdot R^m(r,\ k)+\frac{2r}{m}\cdot I^m(r,\ k)\Big\},\ \text{or, } (m>0),$$

$$2r.\mathrm{I}(r, k)=\mathrm{D}(r, k)-\Sigma_m\left\{\frac{r}{m}\mathrm{R}^m(r, k)+\frac{2r}{m+1}\cdot\mathrm{I}^{m+1}(r, k)\right\};$$

which shows that, if we can find $\mathrm{R}^m(r, k)$ and $\mathrm{I}^{m+1}(r, k)$ for all values of $m>0$, we can obtain the most numerous of all the classes, $\mathrm{I}(r, k)$, by a simple subtraction and division.

XXIV. Our first step towards the actual solution of our problem of partitions, which, after all these tedious prolegomena, we may now think of taking, is to find the numbers in (XX.). We begin with

$$\mathrm{R}^{2hagdi}(r, k)_n=\mathrm{R}_c^{2h.agdi}(r, k)_n+\mathrm{R}_{ac}^{2hag.di}(r, k)_n+\mathrm{R}_{dc}^{2h.agdi}(r, k)_n+\mathrm{R}_0^{2h.agdi}(r, k)_n.$$

Problem b. *To find* $\mathrm{R}_c^{2h.agdi}(\mathrm{r, k})_n$, *the number of* (1+k)*-partitioned* r*-gons, built on the* n*-gonal nucleus, and having* h *agonal and* h *diagonal axes of reversion, all clear.*

There are, in every one of these, $2h$ equidistant terminations of diagonal axes, between which intervene $\frac{n}{2h}$ sides of the n-gon; the central side of these $\frac{n}{2h}$ is bisected by an agonal axis, wherefore $\frac{n}{2h}$ is an odd number.

If we mark this central side as the nth of the n-gon, we see on either side of it $\frac{1}{2}\left(\frac{n}{2h}-1\right)$ sides between it and a termination of a diagonal axis. This nth side is one of $2h$ equidistant sides of the n-gon, bisected by terminations of agonal axes, namely the

$$\frac{n}{2h}, \frac{2n}{2h}, \frac{3n}{2h}, \ldots n\text{th sides},$$

which are all unloaded, being sides both of the r-gon and the n-gon. The $\frac{1}{2}\left(\frac{n}{2h}-1\right)$ sides on either side of the nth are loaded, the first and $(n-1)$th with a certain e_1-partitioned $(2+a_1)$-gon, so as to form a configuration bisected by the agonal axis between them; the second and $(n-2)$th by a e_2-partitioned $(2+a_2)$-gon placed in like reversible manner, and so on; in such a way as to satisfy the equations following,

$$\left.\begin{aligned} r&=n+4h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{4h}}\right),\\ k&=n-2h+4h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{4h}}\right);\end{aligned}\right\}\quad\ldots\ldots\ldots\quad (\mathrm{A.})$$

where $a_m \overline{\gtrless} 0$, and $e_m \not> a_m-1$, since no $(2+a_m)$-gon can have more than a_m-1 diagonals drawn none crossing another; wherefore if $a_m=0$, $e_m=-1$.

These $\frac{n-2h}{4h}$ polygons a_1a_2, &c. are laid on in the same manner to correspond about the $\frac{n}{2h}$th $\frac{2n}{2h}$th, &c. sides; *i. e.* they fill up the $4h$ intervals between agonal and diagonal axes, making a configuration reversible about any one of them.

XXV. The equations (A.) must be satisfied, because the n summits of the n-gon, with the a_1 summits added $4h$ times in the $(2+a_1)$-gon, and the a_m summits added $4h$ times in the $(2+a_m)$-gon, must make up the r summits of the r-gon. And, supposing that a_m everywhere >0, the $n-2h$ loaded sides of the n-gon, with the e_m diagonals of every $(2+a_m)$-gon $4h$ times laid on, must make up the k diagonals of the r-gon. And if $a_m=0$, in which case the mth and $(n-m)$th sides of the n-gon are unloaded, whereby $4h$ sides besides the $2h$ above-mentioned remain sides common to the r-gon and the n-gon, the appearance of $e_m=-1$ corrects the error made by counting those $4h$ sides for diagonals of the r-gon, and the second equation still remains true, however many of the numbers a_1, a_2, &c. $=0$. But these equations are impossible, and $R_e^{2hagdi}(r, k)_n=0$, unless $r-n$, $n-2h$, and $k-n+2h$ are all multiples of $4h$.

Consider any single solution of the equation (A.), and let N be an r-gon bearing, on the mth side everywhere reckoned in both directions from the one bisected by an agonal axis, a $(2+a_m)$-gon having e_m diagonals. Whatever be the posture of those e_m diagonals, and whatever be the character of the $(2+a_m)$-gon, the figure is reversible at least about h agonal and h diagonal axes. And let C be the configuration read in N about the agonal axis through the nth side of the n-gon.

XXVI. Let, next, any disturbance be made in the arrangement of the e_m diagonals in one or more of these imposed $\frac{n-2h}{4h}$ polygons, so that the reversible character about the $2h$ axes be preserved, and the diagonals of the $(2+a_m)$-gon on both sides of any axis shall form pairs making angles bisected by the intervening axis of reversion. The solution before us of (A.) being undisturbed in both values and order of the numbers $a_1\,e_1$, $a_2\,e_2$, &c., we can combine every $(1+e_m)$-division of the $(2+a_m)$-gon in the interval with every $(1+e_p)$-division of the $(2+a_p)$-gon, &c.; that is, we can make C take

$$D(2+a_1, e_1).D(2+a_2, e_2).D(2+a_3, e_3)\ldots D(2+a_{\frac{n-2h}{4h}}, e_{\frac{n-2h}{4h}})=\Delta_{\frac{n-2h}{4h}}$$

different forms, by mere variation of arrangement of diagonals, the same variation being made about every agonal axis, and consequently the same variation occurring about every diagonal axis. Of these $\Delta_{\frac{n-2h}{4h}}$ configurations no two can be alike; for the intervals of $\frac{n-2h}{4h}$ sides between two adjoining axes are irreversible configurations, if we look at the axes which limit them, because one is an agonal and the other a diagonal axis; so that, in fact, no disturbance of one or more diagonals of N can produce N$'$ a repetition of N. If the limiting axes were alike, it might possibly occur that N$'$ should be a reflexion of N, the configuration of one read from right to left in the interval being that of the other read from left to right.

It is to be observed here, and in all products that we may handle of the form of $\Delta_{\frac{n-2h}{4h}}$, that when $a_m=0$ and consequently $e_m=-1$, $D(2+a_m, e_m)$ is either to be omitted, or counted unity, so that $\Delta 0$ is always $=1$.

XXVII. Next, let us make a variation in our solution of equation (A.), either by altering two or more values of $a_1\,a_2$.. or of $e_1\,e_2$.. or of both; or by exchanging the places of certain values without altering them. It is evident that C$'$, the result of such disturbance of the solution, must differ from every C that we have before regarded about the axis through the nth side of the n-gon; for it has no longer the same polygons in the same places of the interval between the axes. We have either changed the order of the imposed polygons, or we have in certain places put a $(1+e_p)$-partitioned $(2+a_p)$-gon which was not employed in the construction of C, for a $(1+e_m)$-partitioned $(2+a_m)$-gon, which was. Whatever be the solution we work from, we shall be able to produce from it, by mere disturbances of diagonals in the imposed polygons, $\Delta_{\frac{n-2h}{4h}}$ distinct configurations, all different from the $\Delta_{\frac{n-2h}{4h}}$ counted before; and all the r-gons thus produced will have the property of being reversible about h agonal and h diagonal axes. And it is evident, that every configuration about an agonal axis of any $(1+k)$-partitioned r-gon, having h agonal and h diagonal axes of reversion, will be produced from some solution of equations (A.), and some arrangement of diagonals in the imposed polygons.

If, then, we denote by $\Sigma.\Delta_{\frac{n-2h}{4h}}$ the sum of the products $\Delta_{\frac{n-2h}{4h}}$ made from every solution of equations (A.), every change, either in value or order, of $a_1 e_1$, $a_2 e_2$, &c. being counted as a solution, we find that $\Sigma\Delta_{\frac{n-2h}{4h}}$, many terms of which will be equal numbers, is the exact number of configurations about an agonal axis which can be seen on any $(1+k)$-partitioned r-gon having h agonal and h diagonal clear axes of reversion.

Now no ago-diagonally reversible has two configurations about agonal axes; and every $(1+k)$-partitioned r-gon having ih agonal and ih diagonal axes, is reversible about h agonal and h diagonal axes, whatever positive number i may be; for it has h equidistant agonal axes, and between every pair of these $(i-1)$ more agonal axes, and it has h diagonal axes, because ih diagonal axes bisect the angles between the ih agonal ones. Hence it follows that

$$\Sigma\Delta_{\frac{n-2h}{4h}}=\Sigma_i \mathrm{R}_c^{2ihagdi}(r, k)_n,$$

where i is every whole number giving $\frac{n-2ih}{4ih}$, $\frac{r-n}{4ih}$, and $\frac{k-n+2ih}{4ih}$ positive integers; conditions necessary to the existence of equations (A.).

XXVIII. Consequently $(i>0)$,

$$\mathrm{R}_c^{2h.agdi}(r, k)_n=\Sigma\Delta_{\frac{n-2h}{4h}}-\Sigma_i\mathrm{R}_c^{2(i+1)h.agdi}(r, k)_n,$$

for all values of $(i+1)h=h'$, which make $n-2h'$, $r-n$, and all multiples of $4h'$.

As $r>n$, $n-2h>0$ in equations (A.); hence $n\not<6h$, and if $2h>\frac{n}{3}$, $\mathrm{R}_c^{2h}=0$; *i. e.*

$$\mathrm{R}_c^{n.agdi}(r, k)_n=\mathrm{R}_c^{\frac{n}{2}agdi}(r, k)_n=0.$$

If $n=6h$, the equations (A.) become

$$r-n=\frac{2n}{3}a_1,$$

$$k-\frac{2n}{3}=\frac{2n}{3}e_1,$$

$$\therefore \quad 2+a_1=\frac{3r+n}{2n}, \quad e_1=\frac{3k-2n}{2n}$$

$$\Delta_{\frac{n-2h}{4h}}=D(2+a_1,\ e_1)=D\left(\frac{3r+n}{2n},\ \frac{3k-2n}{2n}\right),$$

and this is $\Sigma\Delta_{\frac{n-2h}{4h}}$, as there is no other solution of (A.). Wherefore

$$R_c^{\frac{n}{3}agdi}(r,\ k)_n=D\left(\frac{3r+n}{2n},\ \frac{3k-2n}{2n}\right).$$

Next let $\frac{n-2h}{4h}=2$, or $2h=\frac{n}{5}$; we obtain

$$R_c^{\frac{n}{5}agdi}(r,\ k)_n=\Sigma\Delta_{\frac{n-2h}{4h}}-R_c^{\frac{n}{3}}(r,\ k)_n,$$

which is a given number. Here, as before, $\Sigma\Delta$ has as many terms as there are solutions of

$$r=n+\frac{2n}{5}(a_1+a_2)$$

$$k=.\frac{4n}{5}+\frac{2n}{5}(e_1+e_2).$$

For an example, let $r=78$, $k=48$, $n=30$; $\frac{n}{5}=6$: to find $R_c^{6agdi}(78,\ 48)_{30}$, we form the solutions of

$$4=a_1+a_2 \qquad (a_m \geqq 0, \quad e_m \not> a_m-1)$$
$$2=e_1+e_2,$$

of which there are five, namely,

$$4=4+0=\ \ 0+4=3+1=1+3=2+2$$
$$2=3-1=-1+3=2+0=0+2=1+1;$$

whence, disregarding the negative -1, we obtain

$$2.D(6,\ 3)+2.D(5,\ 2).D(3,\ 0)+D(4,\ 1).D(4,\ 1)=\Sigma\Delta_{\frac{n-2h}{4h}},$$

or
$$\Sigma\Delta_{\frac{n-2h}{4h}}=2.14+2.5+2^2=42;$$

and
$$R_c^{6.agdi}(78,\ 48)_{30}=42-R_c^{10agdi}(78,\ 48)_{30}=42,$$

because $\frac{78-30}{20}$ is not a whole number.

We may then put $\frac{30-2h}{4h}=7$, the next possible integral value after 2 of that subindex; and thus finding $h=1$, proceed to enumerate $R_c^{2agdi}(78,48)_{30}$. The equations (A.) become

$$\frac{78-30}{4}=12=a_1+a_2+\ldots+a_7$$

$$\frac{48-28}{4}=\ 5=e_1+e_2+\ldots+e_7,$$

and the solutions are already numerous. The sum of products Δ_7 being found, we obtain

$$R_c^{2.agdi}(78,\ 48)_{30}=\Sigma\Delta_7-42.$$

Thus it is evident that we can register $R_c^{2h.agdi}(r, k)_n$, from $h=\frac{n}{3}$ to $h=1$, for all values that make $r-n$, $n-2h$, and $k-n+2h$, multiples of $4h$.

For a simple and complete example, to be carried out through all our investigation, we may take the partitions of the octagon. Here we are to seek $R_c^{2hagdi}(8, k)_n$. Evidently $n<8$ (if $k>0$), and $=6$ or $=4$.

When $n=6$, $\frac{6-2h}{4h}=1$, the only value, gives $h=1$, and $R_c^{2agdi}(8, k)_6=0$ whatever be k, because $\frac{8-6}{4}$ is not integer.

When $n=4$, $\frac{4-2h}{4h}$ has no integer value >0; wherefore, since $r>n$, $R_c^{2hagdi}(8, k)_4=0$, for all values of k. Hence

$$R_c^{2hagdi}(8, k)=0,$$

whatever be the nucleus, except that

$$R_e^{8agdi}(8, 0)_8=1 \text{ (art. III.)}.$$

XXIX. Problem c. *To find* $R_{dc}^{2hagdi}(r, k)_n$, *the number of* $(1+k)$*-partitioned* r*-gons, built on an* n*-gonal nucleus, having* h *loaded agonal, and* h *clear diagonal, axes of reversion.*

The investigation to be made differs from that in the five preceding articles in that the $2h$ sides of the n-gon, bisected by the agonal axes, are all loaded with a certain $(1+\varepsilon_0)$-partitioned $(4+2\alpha_0)$-gon A_0, having at least one agonal axis of reversion (Theorem G.). This must be so posited that its agonal axis shall be in a line with that of the n-gon (Theorem N.), and must be placed alike on the $2h$ equidistant sides, since all agonal axes of the r-gon must bisect the same configuration at each end (Theorem C.). The remaining sides of the n-gon may be loaded, the mth from A_0, with any $(1+e_m)$-partitioned (or divisioned) $(2+a_m)$-gon so as to form a figure reversible about the $2h$ axes, and so that the equations following be satisfied:—

$$\left.\begin{aligned} r&=n+2h(2+2\alpha_0)+4h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{4h}}\right)\\ k&=n+2h\varepsilon_0+4h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{4h}}\right); \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (A'.)$$

where $\qquad a_m \gtreqless 0,\ e_m \ngtr a_m-1;$ and $e_m=-1$, if $a_m=0$;

the numbers α_0 and ε_0 being any that we may select from our register of polygons having at least one agonal axis, which is supposed complete for values of $(2+2\alpha_0)$ not greater than $\frac{r-n}{2h}$. All that is said of the loading of the $\frac{n-2h}{4h}$ sides between the axes, and about the import and necessity of equations (A.), applies to equations (A'.).

The sides being thus loaded, we have an r-gon which has only clear diagonal axes, and is reversible about $2h$ equidistant and alternate agonal and diagonal axes. Leaving now the $2h$ agonally reversible $(1+2\alpha_0)$-gons undisturbed, we can vary the positions of the diagonals, exactly as in (XXVI.), so as to produce about the agonal axis through the nth side of the r-gon

$$D(2+a_1, e_1).D(2+a_2, e_2)\ldots D\left(2+a_{\frac{n-2h}{4h}}, e_{\frac{n-2h}{4h}}\right)=\Delta_{\frac{n-2h}{4h}}$$

different configurations, the numbers a_1 e_1, &c. here being those of a solution of equations (A′.). And every one of these may be varied again by changing the posture of the $(4+2\alpha_0)$-gon, if it has a second configuration about an agonal axis, so that this axis shall come into a line with that of the n-gon, or by exchanging the $(4+2\alpha_0)$-gon for any other that is also $(1+\varepsilon_0)$-partitioned and has an agonal axis. The number of variations in our power by disturbance of the $(4+2\alpha_0)$-gons alone, is

$$\Sigma_m\{2\mathrm{R}^{mag}(4+2\alpha_0,\ \varepsilon_0)+\mathrm{R}^{2magdi}(4+2\alpha_0,\ \varepsilon_0)\}=\mathrm{H},$$

because each m-agonally reversible has two configurations about agonal axes and thus admits of two postures, while each ago-diagonally reversible has only one; whatever be the number m of agonal axes (Theorem A.).

Thus we see, that for every solution of equations (A′.) that we can write down, all values and orders of the numbers counting in the solution, we can make

$$\mathrm{H}\,.\,\Delta_{\frac{n-2h}{4h}}$$

different configurations about the axis through the nth side of the n-gon. The proof that there are no two of them alike need not be repeated here from art. XVI.

If then
$$\Sigma\mathrm{H}\,.\,\Delta_{\frac{n-2h}{4h}}$$

denote the sum of those products HΔ made from every solution of equations (A′.), this number is that of the r-gons $(1+k)$-partitioned, and having h agonal and h diagonal axes of reversion, the latter only being clear axes, which are constructible on the n-gonal nucleus. This is shown exactly as in art. XXVII. And by the reasoning of that article we obtain

$$\Sigma\,.\,\mathrm{H}\,.\,\Delta_{\frac{n-2h}{4h}}=\Sigma_i\mathrm{R}_{dc}^{2hi\,.\,agdi}(r,\ k)_n,$$

hi being any number h' such that

$$\frac{n-2h'}{4h'},\ \frac{r-n-2h'.(2+2\alpha_0)}{4h'}\text{ and }\frac{k-n-2h'\varepsilon_0}{4h'}$$

are positive integers, for some values of α_0 and ε_0 that are found in our register of $(1+\varepsilon_0)$-partitioned $(2+2\alpha_0)$-gons, having an agonal axis of reversion.

Hence $(i>0)$,

$$\mathrm{R}_{dc}^{2h\,.\,agdi}(r,\ k)_n=\Sigma\mathrm{H}\,.\,\Delta_{\frac{n-2h}{4h}}-\Sigma_i\mathrm{R}_{dc}^{2h(i+1)\,.\,agdi}(r,\ k)_n.$$

XXX. The highest value of h in equations (A′.) gives $\frac{n-2h}{4h}=0$, or $2h=n$. In this case A_0 is the only imposed polygon, and the equations (A′.) become

$$\frac{r-3n}{n}=2\alpha_0,\ \frac{k-n}{n}=\varepsilon_0,$$

$$\mathrm{H}=\Sigma_m\left\{2\mathrm{R}^{mag}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right)+\mathrm{R}^{magdi}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right)\right\},$$

which $=0$ of course if the fractions are irreducible, or if no such values are in our register of reversible polygons with an agonal axis. $\Delta_{\frac{n-2h}{4h}}=\Delta_0$ is here to be considered unity; wherefore

$$\mathrm{R}_{dc}^{n \cdot agdi}(r,\ k)_n = \Sigma_m\left\{2\mathrm{R}^{m \cdot ag}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right)+\mathrm{R}^{2m \cdot agdi}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right)\right\},$$

which is a number given in our register, or else $=0$.

The next step is to put the subindex $\frac{n-2h}{4h}=1$, or $\frac{n}{3}=2h$, and find

$$\mathrm{R}_{dc}^{\frac{n}{2}agdi}(r,\ k)_n=0,$$

because $2h=\frac{n}{2}$ does not give an integer $\frac{n-2h}{4h}$.

$$\mathrm{R}_{dc}^{\frac{n}{3}\cdot agdi}(r,\ k)_n=\Sigma\mathrm{H}.\Delta_1-\mathrm{R}_{dc}^{n \cdot agdi}(r,\ k)_n,$$

a number readily obtained, and so on through all integer values of $\frac{n-2h}{4h}$, from $2h=\frac{n}{2}$ to $2h=2$.

XXXI. For an example take

$$r=22,\ k=12,\ n=6,\ h=1, \text{ and seek } \mathrm{R}_{dc}^{2 \cdot agdi}(22,\ 12)_6.$$

By equations (A′.) we have

$$22=6+4+4\alpha_0+4a_1$$

or

$$\alpha_0 \not> 2,\ \ 4+2\alpha_0 \not> 8.$$

Our register as far as $r=8$ contains, under R^{agdi},

	$\mathrm{R}^{8agdi}(8,\ 0)=1,$	$\mathrm{R}^{6 \cdot agdi}(6,\ 0)=1,$	$\mathrm{R}^{2 \cdot agdi}(6,\ 1)=1,$	$\mathrm{R}^{2agdi}(6,\ 2)_4=1,$	$\mathrm{R}^{4agdi}(4,\ 0)=1,$
where	$\alpha_0=2$	$\alpha_0=1$	$\alpha_0=1$	$\alpha_0=2$	$\alpha=0$
	$\varepsilon_0=0,$	$\varepsilon_0=0,$	$\varepsilon_0=1$	$\varepsilon_0=2$	$\varepsilon_0=0.$

But as $\frac{12-6-2.\varepsilon_0}{4}$ is integer, $\varepsilon_0=1=\alpha_0$; and the only solution of (A′.) is

$$22=6+2.(2+2.1)+4.2$$
$$12=6+2.1 \qquad +4.1,$$

where

$$a_1=2,\ e_1=1.$$

$$\mathrm{D}(2+2,\ 1)=2=\Sigma\Delta_1,\ \Sigma\mathrm{H}=\mathrm{R}^{2agdi}(6,\ 1)=1$$
$$\mathrm{R}_{dc}^{2agdi}(22,\ 12)_6=\Sigma\mathrm{H}\Delta_1-\mathrm{R}_{dc}^{2 \cdot 3agdi}(22,\ 12)_6$$
$$=2-0,$$

because $\frac{r+n}{n}=\frac{22+6}{6}$, in $\mathrm{R}_{dc}^{n}(r,\ k)_n$, is not integer.

We proceed with our partitions of the octagon, and seek $\mathrm{R}_{dc}^{2hagdi}(8,\ k)_n$.

Now $\mathrm{R}_{dc}^{6agdi}(8,\ k)_6=0$, because $\frac{r+n}{n}$ here is not integer: let $n=4$; then, $\frac{n-2h}{4h}=0$, the only integer value, and $\mathrm{R}_{dc}^{4agdi}(8,\ k)_4=0$, because $\frac{8+4}{4}=3$, a value of r not found in our register under R^{ag} or R^{agdi}. Therefore $\mathrm{R}_{dc}^{2h \cdot agdi}(8,\ k)=0$.

XXXII. Problem d. *To find* $R_{ac}^{2h.agdi}(r, k)_n$, *the number of* $(1+k)$*-partitioned* r*-gons built on an* n*-gonal nucleus, to have* h *loaded diagonal and* h *clear agonal axes of reversion.*

All the axes are agonal axes of the nucleus (Theorems M, N). There are $2h$ unloaded sides, carrying the agonal axes, and midway between each pair of these, a side loaded with a $\left(1+\varepsilon_{\frac{n}{4h}}\right)$-partitioned $\left(3+2\alpha_{\frac{n}{4h}}\right)$-gon, having at least one monogonal axis (Theorem N), which is laid alike on the $\frac{n}{4h}$th, $\frac{3n}{4h}$th, ... $\frac{2h-1}{4h}n$th sides of the nucleus; $\alpha_{\frac{n}{4h}}$ and $\varepsilon_{\frac{n}{4h}}$ being numbers selected from our register of monogonally reversibles. And on the mth side, counted from the unloaded one in both directions, is laid any $(1+e_m)$-partitioned $(2+a_m)$-gon, once in every interval between an agonal and a diagonal axis.

The equations to be satisfied are

$$\left.\begin{aligned} r&=n \qquad +4h\left(a_1+a_2+\ldots+a_{\frac{n-4h}{4h}}\right)+2h\left(2\alpha_{\frac{n}{4h}}+1\right)\\ k&=n-2h+4h\left(e_1+e_2+\ldots+e_{\frac{n-4h}{4h}}\right)+2h\varepsilon_{\frac{n}{4h}},\end{aligned}\right\} \quad \ldots \quad (A''.)$$

$$a_m \overline{>} 0, \quad e_m \not> a_m-1;$$

for reasons sufficiently given in art. XXV. These cannot exist unless

$$r-n-2h\left(2\alpha_{\frac{n}{4h}}+1\right) \text{ and } k-n-2h\left(\varepsilon_{\frac{n}{4h}}-1\right)$$

are divisible by $4h$. Every r-gon constructed by loading the sides of the n-gon as above described will be $(1+k)$-partitioned and reversible about h clear agonal and h loaded diagonal axes.

Leaving now undisturbed the $\left(3+2\alpha_{\frac{n}{4h}}\right)$-gons, we can change the arrangements of diagonals in the $(2+a_1)$-gon, $(2+a_2)$-gon, &c. in

$$D(2+a_1, e_1).D(2+a_2, e_2)..D\left(2+a_{\frac{n-4h}{4h}}, e_{\frac{n-4h}{4h}}\right)=\Delta_{\frac{n-4h}{4h}}$$

different ways, without altering at all our solution of equations (A''.). And we can combine each of these arrangements of diagonals with every $\left(1+\varepsilon_{\frac{n}{4h}}\right)$-partitioned $\left(3+2\alpha_{\frac{n}{4h}}\right)$-gon in our list of monogonally reversibles, of which there are

$$M=\Sigma_m R^{m.mo}\left(3+2\alpha_{\frac{n}{4h}}, \varepsilon_{\frac{n}{4h}}\right);$$

so that the entire number of $(1+k)$-partitioned r-gons, having not fewer than h clear agonal and h loaded diagonal axes, constructible from a single solution of (A''.), is the product $M\Delta_{\frac{n-4h}{4h}}$; and a similar product being formed from every solution of (A''.), counting every change of value or order of $a_1 e_1$, $a_2 e_2$, &c. as a solution, we obtain the result

$$\Sigma.M.\Delta_{\frac{n-4h}{4h}}=\Sigma_i R_{ac}^{2ih.agdi}(r, k)_n,$$

the number of products on the left under Σ being that of those solutions; or

$$R_{ac}^{2h.agdi}(r, k)_n=\Sigma.M.\Delta_{\frac{n-4h}{4h}}-\Sigma_i.R_{ac}^{2h(i+1)agdi}(r, k)_n; \ (i>0).$$

XXXIII. In equations (A″.) $\frac{n-4h}{4h}$ may have any value $\gtreqless 0$. When $n=4h$, we have

$$r-n=\frac{n}{2}\cdot(2\alpha_1+1),$$

$$k-\frac{n}{2}=\frac{n}{2}\varepsilon_1;$$

$$3+2\alpha_1=\frac{2r}{n};\ \varepsilon_1=\frac{2k-n}{n};$$

hence $$R_{ac}^{n.agdi}(r,k)_n=0;$$

and as $\Delta_0=1$,

$$R_{ac}^{\frac{n}{2}agdi}(r,k)_n=\Sigma_m R^{m.mo}\left(\frac{2r}{n},\ \frac{2k-n}{n}\right);$$

which of course $=0$ if the fractions are irreducible, and also (IX.), if r is divisible by n.

From this we obtain, putting the subindex $\frac{n-4h}{4h}=1$, or $2h=\frac{n}{4}$,

$$R_{ac}^{\frac{n}{4}.agdi}(r,k)_n=\Sigma M\Delta_1-\Sigma_m R^{m.mo}\left(\frac{2r}{n},\ \frac{2k-n}{n}\right),$$

and thus $R_{ac}^{2hagdi}(r,k)$ can be found in succession for every whole value of $\frac{n}{4h}$ from $2h=\frac{n}{2}$ to $2h=2$, which gives

$$r-n-2h\left(2\alpha_{\frac{n}{4h}}+1\right)\ \text{and}\ k-n-2h\left(\varepsilon_{\frac{n}{4h}}-1\right)$$

divisible by $4h$.

We proceed with our partitions of the octagon, by finding $R_{ac}^{2hagdi}(8,k)_n$.

$R_{ac}^{2hagdi}(8,k)_6=0$, because the subindex $\frac{6-4h}{4h}$ is not integer.

If $n=4$, $\frac{4-4h}{4h}=0$ and $h=1$; when (A″.) becomes $8=4+2(2\alpha_1+1)$, which is absurd;

$\therefore\ R_{ac}^{2hagdi}(8,k)=0$, for any nucleus.

XXXIV. Problem e. *To find* $R_0^{2hagdi}(r,k)_n$, *the number of* $(1+k)$*-partitioned* r*-gons having an* n*-gonal nucleus and* h *diagonal and* h *agonal axes of reversion, of which none are clear.*

The construction is like the preceding, with the exception that $2h$ equidistant sides, viz. the

$$n\text{th},\ \frac{n}{2h}\text{th},\ \frac{2n}{2h}\text{th},\ \ldots\ \frac{2h-1}{2h}n\text{th},$$

are all loaded with the same $(1+\varepsilon_0)$-partitioned $(4+2\alpha_0)$-gon, having at least one agonal axis of reversion, ε_0 and α_0 being any numbers in our register of agonally reversibles, $\varepsilon_{\frac{n}{4h}}$ and $\alpha_{\frac{n}{4h}}$ being found as in XXXII.

The equations to be satisfied are,

$$\left.\begin{aligned} r&=n+2h.(2+2\alpha_0)+4h\left(a_1+a_2+\ldots+a_{\frac{n-4h}{4h}}\right)+2h\left(2\alpha_{\frac{n}{4h}}+1\right)\\ k&=n+2h.\varepsilon_0\qquad\quad +4h\left(e_1+e_2+\ldots+e_{\frac{n-4h}{4h}}\right)+2h.\varepsilon_{\frac{n}{4h}}\end{aligned}\right\}\quad . \quad . \quad . \quad (\text{A}'''.)$$

where $a_m\gtreqless 0$, and $e_m \ngtr a_m-1$.

An r-gon so constructed from any one solution has k diagonals, and is reversible about at least h agonal and h diagonal axes, all loaded.

We can, as before, make the product

$$D(2+a_1,\ e_1)D(2+a_2,\ e_2)\dots D\left(2+a_{\frac{n-4h}{4h}},\ e_{\frac{n-4h}{4h}}\right)=\Delta_{\frac{n-4h}{4h}}$$

with the numbers a_1, a_2, &c. in our solution of (A‴.), which is the number of variations of diagonals that can be made in the $(2+a_1)$-gon, $(2+a_2)$-gon, &c., while the $4h$ reversible polygons upon the axes are undisturbed. We can also disturb these polygons alone, varying the $2h$ agonally reversibles, in

$$\Sigma_m\left(2R^{mag}(4+2\alpha_0,\ \varepsilon_0)+R^{2magdi}(4+2\alpha_0,\ \varepsilon_0)\right)=H$$

different ways, as in XXIX., and the $2h$ monogonally reversibles in

$$\Sigma_m R^{m.mo}\left(3+\alpha_{\frac{n}{4h}},\ \varepsilon_{\frac{n}{4h}}\right)=M$$

different ways, as in XXXII., wherefore the entire number of r-gons $(1+k)$-partitioned and having h equidistant agonal and as many diagonal axes of reversion all loaded, that can be formed by one solution of (A‴.), is the product

$$H.M.\Delta_{\frac{n-4h}{4h}},$$

all the numbers $\alpha_0\,\varepsilon_0$, $a_1\,e_1$, &c. appearing in the product being those found in that one solution. The sum of these products, one for every possible solution of equation (A‴.), all orders and values of the quantities $\alpha\,\varepsilon$, $a\,e$, &c. counting as solutions, is the total number of $(1+k)$-partitioned r-gons, built on the n-gonal nucleus, to have h agonal and h diagonal axes of reversion none of them clear, *i. e.*

$$\Sigma_i R_0^{2hi.agdi}(r,\ k)_n=\Sigma H.M.\Delta_{\frac{n-4h}{4h}};$$

the number of products under Σ on the right being that of the solutions of (A‴.).

Wherefore $\quad R_0^{2h.agdi}(r,\ k)_n=\Sigma H.M.\Delta_{\frac{n-4h}{4h}}-\Sigma_i R_0^{2h(i+1)agdi}(r,\ k)_n\ (i>0)$;

for whatever be the value of i, there must be h equidistant agonal and as many diagonal axes bisecting the intervals between them.

XXXV. The highest value of h in (A‴.) gives $n-2h=0$, $2h=\frac{n}{2}$. The equations become in this case

$$\frac{2r-5n}{2n}=\alpha_0+\alpha_1,$$

$$\frac{2k-2n}{n}=\varepsilon_0+\varepsilon_1,$$

which have of course no solution if the fractions are irreducible; and $\Sigma\Delta_0=1$: wherefore $R_0^{nagdi}(r,\ k)_n=0$,

$$R_0^{\frac{n}{2}agdi}(r,\ k)_n=\Sigma\left\{\Sigma_m\left(2R^{mag}(4+2\alpha_0,\ \varepsilon_0)+R^{2magdi}(4+2\alpha_0,\ \varepsilon_0)\right)\Sigma_m R^{m.mo}(3+\alpha_1,\ \varepsilon_1)\right\}=\Sigma HM\Delta_0,$$

a sum of products easily obtained from our register of reversibles, a different product for every solution of the two equations just written in α_0, α_1, ε_0, ε_1.

We next put $\frac{n-4h}{4h}=1$, or $2h=\frac{n}{4}$, and find

$$\mathrm{R}_0^{\frac{n}{4}agdi}(r,\, k)_n=\Sigma\mathrm{HM}\Delta_1-\mathrm{R}_0^{\frac{n}{2}agdi}(r,\, k)_n,$$

H, M and Δ_1 being properly determined from the numbers that satisfy

$$r=n+\frac{n}{4}(2+2\alpha_0)+\frac{n}{2}\cdot\alpha_1+\frac{n}{4}(2\alpha_2+1),$$

$$k=n+\frac{n}{4}\cdot\varepsilon_0 \qquad +\frac{n}{2}\cdot e_1+\frac{n}{4}\cdot\varepsilon_2.$$

In like manner we can obtain $\mathrm{R}_0^{2h\,.\,agdi}(r,\, k)_n$ for every value of $2h$ that gives, in equations (A‴.),

$$n,\ r-6h, \text{ and } k-n-2h\left(\varepsilon_0+\varepsilon_{\frac{n}{4h}}\right),$$

each divisible by $4h$; α_0, ε_0, $\alpha_{\frac{n}{4h}}$ and $\varepsilon_{\frac{n}{4h}}$ being numbers in our register of reversibles.

It is easily seen that

$$\mathrm{R}_0^{2hagdi}(8,\, k)_6=0=\mathrm{R}_0^{2hagdi}(8,\, k)_4,$$

whence by what precedes we obtain, for $k>0$,

$$\mathrm{R}^{2hagdi}(8,\, k)=0,$$

or the octagon has no partition ago-diagonally reversible, except itself.

Thus we have determined the four numbers

$$\mathrm{R}_c^{2h\,.\,agdi}(r,\, k)_n+\mathrm{R}_{dc}^{2h\,.\,agdi}(r,\, k)_n+\mathrm{R}_{ac}^{2h\,.\,agdi}(r,\, k)_n+\mathrm{R}_0^{2h\,.\,agdi}(r,\, k)_n=\mathrm{R}^{2h\,.\,agdi}(r,\, k)_n.$$

XXXVI. The next step is to obtain

$$\mathrm{R}^{h\,.\,ag}(r,\, k)_n=\mathrm{R}_c^{h\,.\,ag}(r,\, k)_n+\mathrm{R}_{\frac{1}{2}c}^{h\,.\,ag}(r,\, k)_n+\mathrm{R}_0^{h\,.\,ag}(r,\, k)_n.$$

Problem f. *To find* $\mathrm{R}_c^{\mathrm{h}\,.\,\mathrm{ag}}(\mathrm{r},\, \mathrm{k})_{\mathrm{n}}$, *the number of* (1+k)*-partitioned* r*-gons built on the* n*-gonal nucleus, to be reversible about* h *agonal axes, all clear.*

Whether n be odd or even, $2h$ sides of the n-gon, viz.—

$$\text{the } n\text{th, } \frac{n}{h}\text{th, } \frac{2n}{h}\text{th} \ldots \frac{h-1}{h}n\text{th sides,}$$

$$\text{and the } \frac{n}{2h}\text{th, } \frac{3n}{2h}\text{th} \ldots \frac{2h-1}{2h}n\text{th sides,}$$

which are equidistant from pairs of the preceding h, are unloaded, and the sides between the nth and $\frac{n}{2h}$th are loaded, the mth with a $(1+e_m)$-partitioned $(2+a_m)$-gon, &c., the configuration of this interval being reversed in the two adjoining intervals of $\frac{n-2h}{2h}$ sides each, so as to make a figure reversible about all the axes. And the equations following will be true:—

$$\left.\begin{array}{l} r=n+2h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{2h}}\right),\ a_m \overline{\gtrless} 0, \\ k=n-2h+2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right),\ e_m \not> a_m-1. \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{B.})$$

The above construction from every solution of this equation gives an r-gon k-partitioned end reversible about at least h clear agonal axes bisecting $2h$ equidistant sides, both of the r-gon and n-gon (Theorems B, N). And with this solution, *i. e.* without changing the weight or order of the imposed loads, we can, by disturbing the arrangement of the e_m diagonals of the $(2+a_m)$-gon in any way whatever, produce a new configuration about the axis bisecting the nth side; and the entire number of configurations so brought into view by such disturbances will be

$$\mathrm{D}(2+a_1,\ e_1)\mathrm{D}(2+a_2,\ e_2)\ldots\mathrm{D}\left(2+a_{\frac{n-2h}{2h}},\ e_{\frac{n-2h}{2h}}\right)=\Delta_{\frac{n-2h}{2n}};$$

and a like expression for the number of configurations generable from every solution of equation (B.), *i. e.* for every change in the weight or order of imposed loads, being added to the above, we obtain $\Sigma\Delta_{\frac{n-2h}{2h}}$ different configurations about the axis through the nth side, the number of terms under Σ being that of the solutions. Among these will be found once, and once only, every aspect of any $(1+k)$-partitioned r-gon about a clear agonal axis, which has $2h$ equidistant sides in common with the n-gon, bisected by agonal axes of reversion. Now two of these configurations are found on every one of $\mathrm{R}_c^{hiag}(r,\ k)_n$, for each has $i\times 2h$ such equidistant sides of the n-gon; one will be found on each of $\mathrm{R}_{\frac{1}{2}c}^{2hi\cdot ag}(r,\ k)_n$ for the same reason, and one will be found on each of $\mathrm{R}_{ac}^{2hiagdi}(r,\ k)_n$ and on each of $\mathrm{R}_c^{2hiagdi}(r,\ k)_n$ for the same reason (Theorem C); that is $(i>0)$,

$$\Sigma\Delta_{\frac{n-2h}{2h}}=\Sigma_i\{2\mathrm{R}_c^{hiag}(r,\ k)_n+\mathrm{R}_{\frac{1}{2}c}^{2hiag}(r,\ k)_n+\mathrm{R}_{ac}^{2hiagdi}(r,\ k)_n+\mathrm{R}_c^{2hiagdi}(r,\ k)_n\},\ \text{and}$$

$$\mathrm{R}_c^{h\cdot ag}(r,\ k)_n=\tfrac{1}{2}\left\{\Sigma\Delta_{\frac{n-2h}{2h}}-\Sigma_i\left(2\mathrm{R}_c^{h(i+1)ag}(r,\ k)_n+\mathrm{R}_{\frac{1}{2}c}^{2hiag}(r,\ k)_n+\mathrm{R}_{ac}^{2hiagdi}(r,\ k)_n+\mathrm{R}_c^{2hi\cdot agdi}(r,\ k_n)\right)\right\}.$$

XXXVII. Before we proceed with the discussion of this formula, it is necessary,

Problem g. *To find* $\mathrm{R}_{\frac{1}{2}c}^{\mathrm{hag}}(\mathrm{r},\ \mathrm{k})_{\mathrm{n}}$, *the number of* $(1+\mathrm{k})$-*partitioned* r-*gons having* $\frac{\mathrm{h}}{2}$ *clear and* $\frac{\mathrm{h}}{2}$ *loaded agonal axes, and built on the* n-*gonal nucleus.*

These r-gons have all on h sides of the n-gon, the

$$n\text{th},\ \frac{n}{h}\text{th},\ \frac{2n}{h}\text{th},\ \ldots\frac{h-1}{h}n\text{th sides},$$

a $(1+\varepsilon_0)$-partitioned $(4+2\alpha_0)$-gon having an agonal axis, and nothing on the

$$\frac{n}{2h}\text{th},\ \frac{3n}{2h}\text{th}\ldots\frac{2h-1}{2h}n\text{th sides}.$$

The equations to be satisfied are

$$\left.\begin{array}{l} r=n+h(2+2\alpha_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{2h}}\right), \\ k=n-h+h\varepsilon_0 \qquad +2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right); \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{B}'.)$$

α_0 and ε_0 being any number in our list of reversibles about an agonal axis.

The configurations producible about the loaded agonal axis through the nth side of the n-gon, from any one solution of (B′.), are the product $\mathrm{H}\Delta_{\frac{n-2h}{2h}}$, of

$$\mathrm{H}=\Sigma_m\{2\mathrm{R}^{mag}(4+2\alpha_0, \varepsilon_0)+\mathrm{R}^{2magdi}(4+2\alpha_0, \varepsilon_0)\},$$

and
$$\Delta_{\frac{n-2h}{2h}}=\mathrm{D}(2+a_1, e_1).\mathrm{D}(2+a_2, e_2)\ldots.\mathrm{D}\left(2+a_{\frac{n-2h}{2h}}, e_{\frac{n-2h}{2h}}\right);$$

and the sum of these products, one for every solution of (B′.), all values and orders of the quantities counting among the solutions, is the sum of the configurations bisected by the loaded agonal axes of all $(1+k)$-partitioned r-gons built on the n-gonal nucleus, to have $\frac{1}{2}h$ loaded agonal axes, and $\frac{1}{2}h$ clear agonal axes bisecting the equal intervals between the loaded ones.

Now the whole number of these r-gons is that of these configurations, since no such r-gon has more than one; and these r-gons are in number exactly

$$\Sigma_i \mathrm{R}_{\frac{1}{2}c}^{(2i+1)h.ag}(r, k)_n=\Sigma\mathrm{H}.\Delta_{\frac{n-2h}{2h}}, (i\gtreqless 0);$$

for all these have $(2i+1)\times\frac{1}{2}h$ loaded agonal axes, and among these are $\frac{h}{2}$ equidistant ones, in the intervals of which lie an odd number $(2i+1)$ of clear agonal axes. But $\mathrm{R}_{\frac{1}{2}c}^{2ihag}(r, k)_n$ is not in the number of these r-gons, because in the intervals of any $\frac{1}{2}h$ equidistant loaded axes of these, lies an even number of clear axes. Wherefore, $(i>0)$

$$\mathrm{R}_{\frac{1}{2}c}^{hag}(r, k)_n=\Sigma\left(\mathrm{H}\Delta_{\frac{n-2h}{2h}}\right)-\Sigma_i\mathrm{R}_{\frac{1}{2}c}^{(2i+1)hag}(r, k)_n.$$

XXXVIII. The highest value of h in (B′.) gives $n-2h=0$; which reduces them to

$$\frac{2r-2n}{n}=2+2\alpha_0, \frac{2k-n}{n}=\varepsilon_0, \Delta_0=1;$$

wherefore
$$\mathrm{R}_{\frac{1}{2}c}^{nag}(r, k)_n=0$$
$$\mathrm{R}_{\frac{1}{2}c}^{\frac{n}{2}ag}(r, k)_n=\Sigma_m\left\{2\mathrm{R}^{mag}\left(\frac{2r}{n}, \frac{2k-n}{n}\right)+\mathrm{R}^{2magdi}\left(\frac{2r}{n}, \frac{2k-n}{n}\right)\right\}.$$

We can next put $\frac{n-2h}{2h}=1$ in equations (B′.) and find $\mathrm{R}_{\frac{1}{2}c}^{\frac{n}{4}ag}(r, k)_n$, and thus every number $\mathrm{R}_{\frac{1}{2}c}^{hag}(r, k)_n$, for any even value of h, so that

$$n, r, \text{ and } k-h(\varepsilon_0-1)$$

shall be multiples of $2h$, ε_0 being some number of diagonals on our list of agonally reversibles.

In XXXVI., equations (B.), the highest value of h gives $\frac{n-2h}{2h}=1$, or $h=\frac{n}{4}$, when those equations become

$$\frac{2r-2n}{n}=a_1, \quad \frac{2k-n}{n}=e_1,$$

so that r is divisible by n: $\mathrm{D}(2+a_1, e_1)=\Sigma\Delta_1$.

Hence
$$\mathrm{R}_c^{\frac{n}{2}agdi}(r, k)_n=0;$$

and since $\qquad \mathrm{R}_c^{\frac{n}{2}agdi}(r,\ k)_n=0$, by XXVIII.,

and $\qquad \mathrm{R}_{ac}^{\frac{n}{2}agdi}(r,\ k)_n=0$, by XXXIII.,

when r is divisible by n,

$$\mathrm{R}_c^{\frac{n}{4}ag}(r,\ k)_n=\tfrac{1}{2}\{\mathrm{D}(2+a_1,\ e_1)-\mathrm{R}_{\frac{1}{2}c}^{\frac{n}{2}ag}(r,\ k)_n\}$$
$$=\tfrac{1}{2}\Big\{\mathrm{D}\Big(\frac{2r}{n},\ \frac{2k-n}{n}\Big)-\Sigma_m\Big\{2\mathrm{R}^{mag}\Big(\frac{2r}{n},\ \frac{2k-n}{n}\Big)+\mathrm{R}^{2magdi}\Big(\frac{2r}{n},\ \frac{2k-n}{n}\Big)\Big\}\Big\}.$$

If now we put in (B.) $\frac{n-2h}{2h}=2$, we can obtain $\mathrm{R}_c^{\frac{n}{6}ag}(r,k)_n$, and every number $\mathrm{R}_c^{hag}(r,k)_n$ in which n, r, and k are multiples of $2h$, including the case of $h=1$.

We shall now find $\mathrm{R}_c^{hag}(8,\ k)_n$ and $\mathrm{R}_{\frac{1}{2}c}^{hag}(8,\ k)_n$.

For $n=6$ in equations (B.), we can only put $\frac{6-2h}{2h}=2$, $h=1$, and $k=2$; then

$$r=8=6\quad+2.1(1+0)=6+2.1(0+1)$$
$$k=2=6-2+2.1(0-1)=6+2.1(-1+0)$$
$$\Delta_2=2\mathrm{D}(3,\ 0)=2$$
$$\mathrm{R}_c^{ag}(8,\ 2)_6=\tfrac{1}{2}2=1.$$

If $n=4$, $\frac{4-2h}{2n}=$ integer gives only $(r>n)$ $h=\frac{n}{4}$, and $\frac{2k-4}{4}$ being integer in the formula for $\mathrm{R}_c^{\frac{n}{4}ag}$, gives $k=2$, or else $k=4$. Also, by the same formula,

$$\mathrm{R}_c^{ag}(8,\ 2)_4=\tfrac{1}{2}\{\mathrm{D}(4,\ 0)-\mathrm{R}^{4agdi}(4,\ 0)\}=\tfrac{1}{2}(1-1)=0$$
$$\mathrm{R}_c^{ag}(8,\ 4)_4=\tfrac{1}{2}\Big\{\mathrm{D}(4,\ 1)-2\big(\mathrm{R}^{ag}(4,\ 1)+\mathrm{R}^{agdi}(4,\ 1)\big)\Big\}=\tfrac{1}{2}(2-0)=1;$$

for the only entries in our register for $r=4$ are

$$\mathrm{R}^{4agdi}(4,\ 0)=1 \text{ and } \mathrm{R}^{2di}(4,\ 1)=1.$$

When $n=6$ in equations (B′.), $\frac{n-2h}{2h}$ is not integer for an even value of h; but when $n=4$, $h=2=\frac{n}{2}$, and by the formula for $\mathrm{R}_{\frac{1}{2}c}^{\frac{n}{2}agdi}(r,\ k)_n$,

$$\mathrm{R}_{\frac{1}{2}c}^{2ag}(8,\ 2)_4=\mathrm{R}^{4agdi}(4,\ 0)=1;$$

for $k=2$ is the only value that finds $\frac{2k-n}{n}$ integer in our register.

XXXIX. Problem h. *To find* $\mathrm{R}_0^{hag}(\mathrm{r},\ \mathrm{k})_n$, *the number of* $(1+\mathrm{k})$*-partitioned* r*-gons having* h *agonal axes only of reversion, all loaded axes.*

In any one of these there are $2h$ equidistant sides, the nth, the $\frac{n}{2h}$th, the $\frac{2n}{2h}$th, &c. of the n-gon loaded, half with a $(1+\varepsilon_0)$-partitioned $(4+2\alpha_0)$-gon, and the rest, alternating with these, loaded with a $\big(1+\varepsilon_{\frac{n}{2h}}\big)$-partitioned $\big(4+2\alpha_{\frac{n}{2h}}\big)$-gon, all having at least one agonal axis of reversion, which is also one in the n-gon. The numbers α_0, ε_0, as well as $\alpha_{\frac{n}{2h}}$, $\varepsilon_{\frac{n}{2h}}$, which may be the same or not with the former, are any that we choose from our register of agonally reversibles. The intervals are loaded as before, so as to satisfy

$$\left.\begin{aligned} r &= n+h(2+2\alpha_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{2h}}\right)+h\left(2+2\alpha_{\frac{n}{2h}}\right) \\ k &= n+h\varepsilon_0 \qquad\quad +2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right)+h\varepsilon_{\frac{n}{2h}}\,; \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (\mathrm{B}''.)$$

where $\qquad a_m \overline{>} 0, \quad e_m \not> a_m - 1.$

Having chosen any solution of these equations, and posited the polygons, we can, without disturbing the first-named $2h$ polygons on the axes, make

$$\mathrm{D}(2+a_1, e_1).\mathrm{D}(2+a_2, e_2)..\mathrm{D}\left(2+a_{\frac{n-2h}{2h}}, e_{\frac{n-2h}{2h}}\right)=\Delta_{\frac{n-2h}{2h}}$$

variations of the e_1, e_2 ... diagonals, and produce about the axis through the nth side as many configurations. And with each of these variations we can combine any $(4+2\alpha_0)$-gon in our register which has ε_0 diagonals and an agonal axis, and also any $\left(4+2\alpha_{\frac{n}{2h}}\right)$-gon having $\varepsilon_{\frac{n}{2h}}$ diagonals and an agonal axis, and in any posture giving h agonal axes of the r-gon. That is, we can, without altering our solution of (B''.), produce about the axis through the nth side $\mathrm{H}.\mathrm{H}'.\Delta_{\frac{n-2h}{2h}}$ different configurations reversible about h agonal axes all loaded as described, where

$$\mathrm{H} = \Sigma_m\{2.\mathrm{R}^{mag}(4+2\alpha_0, \varepsilon_0)+\mathrm{R}^{2magdi}(4+2\alpha_0, \varepsilon_0)\}$$
$$\mathrm{H}'=\Sigma_m\left\{2.\mathrm{R}^{mag}\left(4+2\alpha_{\frac{n}{2h}}, \varepsilon_{\frac{n}{2h}}\right)+\mathrm{R}^{2magdi}\left(4+2\alpha_{\frac{n}{2h}}, \varepsilon_{\frac{n}{2h}}\right)\right\}.$$

If then $\Sigma\mathrm{HH}'\Delta$ denote the sum of these products made from every solution of (B''.), obtained by every change of value or order in the numbers αa εe, this will denote the entire number of such configurations.

Now two of these are found in each of $\qquad \mathrm{R}_0^{hi.ag}(r, k)_n$;

one of them, namely, about the loaded axes, in each of $\qquad \mathrm{R}_{\frac{1}{2}c}^{2hi.ag}(r, k)_n$,

and these are among the configurations obtained with $\alpha_0=\alpha_{\frac{n}{2h}}$ and $\varepsilon_0=\varepsilon_{\frac{n}{2h}}$;

also one of them is in each of $\qquad \mathrm{R}_0^{2hi.agdi}(r, k)_n$,

and one of them in each of $\qquad \mathrm{R}_{dc}^{2hi.agdi}(r, k)_n$;

for both these have h equidistant agonal axes loaded, of course alike $\left(\alpha_0=\alpha_{\frac{n}{2h}}\text{ and }\varepsilon_0=\varepsilon_{\frac{n}{2h}}\right)$. Nor is there anything to prevent the $2hi$ terminations of axes in some of $\mathrm{R}_0^{hiag}(r, k)_n$ being loaded all alike; for this cannot hinder the configurations about alternate axes from being different. Therefore

$$\Sigma\mathrm{H}.\mathrm{H}'.\Delta_{\frac{n-2h}{2h}}=\Sigma_i\left(2\mathrm{R}_0^{hi.ag}(r, k)_n+\mathrm{R}_{\frac{1}{2}c}^{2hi.ag}(r, k)_n+\mathrm{R}_0^{2hi.agdi}(r, k)_n+\mathrm{R}_{dc}^{2hi.agdi}(r, k)_n\right), \text{ or}$$

$$\mathrm{R}_0^{hag}(r, k)_n=\tfrac{1}{2}\left\{\Sigma\mathrm{H}.\mathrm{H}'.\Delta_{\frac{n-2h}{2h}}-\Sigma_i\left(2\mathrm{R}_0^{h(i+1)ag}(r, k)_n+\mathrm{R}_{\frac{1}{2}c}^{2hiag}(r, k)_n+\mathrm{R}_0^{2hi.agdi}(r, k)_n+\mathrm{R}_{dc}^{2hi.agdi}(r, k)_n\right)\right\}.$$

XL. The highest value of h in (B''.) comes from $n-2h=0$, the case when there are no intervals to load. Then,

$$\frac{r-3n}{n}=\alpha_0+\alpha_1$$

$$\frac{k-n}{n}=\varepsilon_0+\varepsilon_1$$

$$\Delta_0=1=\Sigma\Delta_0;\text{ so that}$$

$$R_0^{nag}(r,k)_n=0,$$

and since $\qquad R_0^{nagdi}(r,k)_n=0$ (by XXIV.) $=R_{\frac{1}{2}c}^{nag}(r,k)_n$ by XXVII.,

$$R_0^{\frac{n}{2}ag}(r,k)_n=\tfrac{1}{2}\{\Sigma H.H'-R_{dc}^{nagdi}(r,k)_n\},$$

H and H' being given functions of $\alpha_0\varepsilon_0$ and $\alpha_1\varepsilon_1$ expressed in the preceding article, of course both $=0$, if r and k are not multiples of n.

We next put $n-2h=2h$ in (B''.), and readily deduce $R_0^{\frac{n}{4}ag}(r,k)_n$; and thus in order we obtain $R_0^{hag}(r,k)_n$ for every value of $h>1$, which makes

$$n,\ r,\text{ and } k-h\left(\varepsilon_0+\varepsilon_{\frac{n}{2h}}\right)\text{ divisible by } 2h;$$

conditions necessary for the solution of equations (B''.). We cannot find $R_0^{ag}(r,k)_n$ by this formula ($h=1$), for there is no such division reducible to a nucleus (K, XIII.). We are to proceed with our partitions of the octagon. In equation (B''.) $r-n\not<4$, $\therefore n\not>4$. When $n=4$, $\frac{4-2h}{2h}=1$ or $=0$, giving $h=1$ or $h=2$, the latter of which is inadmissible, since $8-4\not<2h.2$. Therefore

$$r=8=4+1.2+2.0+1.2;\text{ or }\alpha_0=\alpha_{\frac{n}{2h}}=a_1=0,$$
$$k=2=4+2.-1,$$

and D(2, -1) is to be considered unity $=\Delta_1$;

$$R_0^{ag}(8,2)_4=\tfrac{1}{2}\{R^{4agdi}(4,0).R^{4agdi}(4,0)-R_{\frac{1}{2}c}^{2ag}(8,2)_4\}$$
$$=\tfrac{1}{2}(1.1-1)=0.$$

Thus we have found (XXXVI. XXXVII. XXXIX.),

$$R^{h.ag}(r,k)_n=R_c^{hag}(r,k)_n+R_{\frac{1}{2}c}^{hag}(r,k)_n+R_0^{h.ag}(r,k)_n.$$

XLI. We have to investigate next, formulæ for

$$R_0^{hdi}(r,k)_n+R_{\frac{1}{2}c}^{hdi}(r,k)_n+R_c^{hdi}(r,k)_n=R^{hdi}(r,k)_n.$$

Problem i. *To find* $R_0^{hdi}(r,k)_n$, *the number of* $(1+k)$*-partitioned* r*-gons having diagonal axes only of reversion, all loaded, and built on an* n*-gonal nucleus.*

Each r-gon of this number has a $(1+\varepsilon_0)$-partitioned $(3+2\alpha_0)$-gon laid on h equidistant sides, the nth, the $\frac{n}{h}$th, the $\frac{2n}{h}$th, &c. of the r-gon, and a $\left(1+\varepsilon_{\frac{n}{2h}}\right)$-partitioned $\left(3+2\alpha_{\frac{n}{2h}}\right)$-gon laid on h sides each equidistant from two of the other h, all being polygons reversible about at least one monogonal axis, which is an agonal axis of the n-gon (Theorem N), bisecting the side on which the polygon is placed. The intervals of $\frac{n-2h}{2h}$ sides being loaded as in the previous constructions, the equations of condition are

$$\left.\begin{array}{l} r=n+h(1+2\alpha_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{2h}}\right)+h\left(1+2\alpha_{\frac{n}{2h}}\right) \\ \mathrm{R}=n+h\,.\,\varepsilon_0 \qquad +2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right)+h\varepsilon_{\frac{n}{2h}} \end{array}\right\}, \quad . \; . \; . \quad \text{(C.)}$$

$$a_m \gtreqless 0, \quad e_m \ngtr a_m-1,$$

and both α, ε are chosen from our register of monogonally reversibles. Leaving undisturbed the $2h$ polygons on the axes, we have in our power, from any one solution of equations (C.),

$$\mathrm{D}(2+a_1,\,e_1)\mathrm{D}(2+a_2,\,e_2)\ldots\mathrm{D}\left(2+a_{\frac{n-2h}{2h}},\,e_{\frac{n-2h}{2h}}\right)=\Delta_{\frac{n-2h}{2h}}$$

variations of arrangement of the e_1, e_2, &c. diagonals in the intervals, producing as many configurations about the axis through the nth side of the n-gon, and all reversible about at least h monogonal axes. With any one of these variations we can combine any polygon of the number

$$\mathrm{M}=\Sigma_m\mathrm{R}^{m\,.\,mo}(3+2\alpha_0,\,\varepsilon_0),$$

and any one of

$$\mathrm{M}'=\Sigma_m\mathrm{R}^{m\,.\,mo}\left(3+2\alpha_{\frac{n}{2h}},\,\varepsilon_{\frac{n}{2h}}\right);$$

so that we obtain $\mathrm{MM}'\Delta_{\frac{n-2h}{2h}}$ configurations from this one solution of (C.).

If we form this product for every possible solution, in every order of values of $\alpha\,a$, $\varepsilon\,e$, &c., we shall produce about the axis through the nth side of the n-gon,

$$\Sigma\mathrm{MM}'\Delta_{\frac{n-2h}{2h}}$$

different configurations.

Now two of these are seen in each of $\mathrm{R}_0^{hi\,.\,di}(r,\,k)_n$,

one of them in each of $\mathrm{R}_{\frac{1}{2}c}^{2hi\,.\,di}(r,\,k)_n$,

one in each of $\mathrm{R}_0^{2hi\,.\,agdi}(r,\,k)_n$,

and one in each of $\mathrm{R}_{ac}^{2hi\,.\,agdi}(r,\,k)_n$;

for every one of the last three classes has $h\times i$ terminations of agonal axes of the n-gon loaded with monogonally reversible polygons. Wherefore

$$\Sigma\mathrm{M}\,.\,\mathrm{M}'\,.\,\Delta_{\frac{n-2h}{2h}}=\Sigma_i\{2\mathrm{R}_0^{hi\,.\,di}(r,\,k)_n+\mathrm{R}_{\frac{1}{2}c}^{2hi\,.\,di}(r,\,k)_n+\mathrm{R}_0^{2hi\,.\,agdi}(r,\,k)_n+\mathrm{R}_{ac}^{2hi\,.\,agdi}(r,\,k)_n\},$$

and $(i>0)$,

$$\mathrm{R}_0^{hdi}(r,k)_n=\tfrac{1}{2}\left\{\Sigma\mathrm{MM}'\Delta_{\frac{n-2h}{2h}}-\Sigma_i\left(2\mathrm{R}_0^{h(i+1)di}(r,k)_n+\mathrm{R}_{\frac{1}{2}c}^{2hidi}(r,k)_n+\mathrm{R}_0^{2hiagdi}(r,k)_n+\mathrm{R}_{ac}^{2hi\,.\,agdi}(r,k)_n\right)\right\}.$$

This result is useless until $\mathrm{R}_{\frac{1}{2}c}^{hdi}(r,\,k)_n$ is known.

XLII. Problem k. *To find* $\mathrm{R}_{\frac{1}{2}c}^{\mathrm{hdi}}(\mathrm{r},\,\mathrm{k})_{\mathrm{n}}$, *the number of* (1+k)*-partitioned* r*-gons built on an* n*-gonal nucleus, to have* $\frac{1}{2}$h *loaded and* $\frac{1}{2}$h *clear axes only, all diagonal axes.*

A clear diagonal axis of the r-gon is a diagonal of the nucleus, evidently. These

r-gons will have only n sides of the n-gon loaded with $(1+\varepsilon_0)$-partitioned $(3+2\alpha_0)$-gons monogonally reversible, namely the nth, the $\frac{n}{h}$th, the $\frac{2n}{h}$th, &c. sides. And half the interval between the nth and $\frac{n}{h}$th, namely $\frac{n-h}{2h}$ sides, will be loaded, the mth with a $(1+e_m)$-partitioned $(2+a_m)$-gon, which will be placed also on the $\left(\frac{n}{h}-m\right)$th side, so as to form a figure reversible about the diagonal bisecting that interval.

Our equations of condition are now

$$\left.\begin{array}{l} r=n+h(1+2\alpha_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-h}{2h}}\right) \\ k=n+h\varepsilon_0 \qquad\quad +2h\left(e_1+e_2+\ldots+e_{\frac{n-h}{2h}}\right) \end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (C'.)$$
$$a_m \geqq 0,\; e_m \ngtr a_m-1,$$

where α_0 and ε_0 are any numbers we select from our list of monogonally reversibles.

Reasoning as in the previous constructions, we deduce that the entire sum, made from every solution of (C′.), of products $M\Delta$ of

$$M=\Sigma_m R^{m.mo}(3+2\alpha_0,\ \varepsilon_0),\ \text{and}$$

$$\Delta_{\frac{n-h}{2h}}=D(2+a_1,\ e_1).D(2+a_2,\ \varepsilon_2)\ldots D\left(2+a_{\frac{n-h}{2h}},\ e_{\frac{n-h}{2h}}\right),$$

is the complete number of configurations about a loaded axis, of r-gons having $\frac{h}{2}$ equidistant loaded, and bisecting the intervals between these, $\frac{h}{2}$ clear, diagonal axes of reversion.

Now one of these configurations about a loaded axis is seen in each of $R_{\frac{1}{2}c}^{(2i+1)h.di}(r,\ k)_n$; for it has in the intervals of any equidistant $\frac{h}{2}$ loaded axis $2i+1$ clear ones, one of them bisecting each interval. But none of the class $R_{\frac{1}{2}c}^{2ih.di}(r,\ k)_n$ has that configuration, because in the equal intervals of $\frac{h}{2}$ loaded axes it has an even number of clear ones, and none bisecting an interval. And no ago-diagonally reversible can have this configuration, because the diagonal axes are in these all clear or all loaded.

Wherefore $\qquad \Sigma.M\Delta_{\frac{n-h}{2h}}=\Sigma_i R_{\frac{1}{2}c}^{(2i+1)h.di}(r,\ k)_n,$

and $\qquad R_{\frac{1}{2}c}^{hdi}(r,\ k)_n=\Sigma M.\Delta_{\frac{n-h}{2h}}-\Sigma_i R_{\frac{1}{2}c}^{(2i+3)h.di}(r,\ k)_n;\ (i\geqq 0),$

where $(2i+3)h$ is a value h' that makes $\frac{n-h'}{2h'}$ integer.

XLIII. The greatest value of h in (C′.) is $h=n$, when there is no interval between the axes. Then,

$$\frac{r-2n}{n}=2\alpha_0,\ \frac{k-n}{n}=\varepsilon_0,$$

$$\Delta_0=1=\Sigma\Delta_0 \quad M=\Sigma M,$$

and $\qquad R_{\frac{1}{2}c}^{ndi}(r,\ k)_n=\Sigma_m R^{m.mo}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right),$

which $=0$, if r and k are not multiples of n.

Next putting $\frac{n-h}{2h}=1$, we find $R_{\frac{1}{2}c}^{\frac{n}{3}di}(r, k)_n$, and so on for every even value of $h>0$ that makes

$$n-h,\ r-n-h(1+2\alpha_0),\ \text{and}\ k-n-h\varepsilon_0,$$

all divisible by $2h$.

The greatest value of h in equation (C.), XLI., is $h=\frac{1}{2}n$. Those equations become

$$\frac{r-2n}{n}=\alpha_0+\alpha_1$$

$$\frac{2k-2n}{n}=\varepsilon_0+\varepsilon_1;$$

and $\quad \Delta_0=1=\Sigma\Delta_0,$

wherefore, $\quad R_0^{n.di}(r, k)=0,$

and because (XXXIII., XXXV.)

$$R_{ac}^{n.agdi}(r, k)_n=0=R_0^{n.agdi}(r, k)_n,$$

$$R_0^{\frac{1}{2}n.di}(r, k)_n=\tfrac{1}{2}\Sigma M.M'-\tfrac{1}{2}R_{\frac{1}{2}c}^{ndi}(r, k)_n,$$

or $R_0^{\frac{1}{2}ndi}(r, k)_n=\frac{1}{2}\Sigma.\left\{\Sigma_m\left(R^{mmo}(3+2\alpha_0, \varepsilon_0).\Sigma_m R^{m.mo}(3+2\alpha_1, \varepsilon_1)\right)\right\}-\frac{1}{2}\Sigma_m\left(R^{m.mo}\left(\frac{r+n}{n}, \frac{k-n}{n}\right)\right);$

the number of products MM′ under the first Σ being that of the solutions of the above equations in $\alpha_0\,\alpha_1$, $\varepsilon_0\,\varepsilon_1$, every order of values counting for a solution. From this $R_0^{hdi}(r, k)_n$ can be found for every value of $h>1$ in equation (C.), XLI., that makes n, r, and $k-h\left(\varepsilon_0+\varepsilon_{\frac{n}{4h}}\right)$ divisible by $2h$, conditions imposed by those equations. But we cannot attempt by these formulæ to find $R_0^{di}(r, k)_n$, $(h=1)$, because (XIII.) this subclass reduces to no nucleus.

XLIV. We proceed with our partitions of the 8-gon, to find $R_{\frac{1}{2}c}^{h.di}(8, k)_n$ and $R_0^{h.di}(8, k)_n$. In equations (C′.), $n=6$ or $n=4$, for $r=8$; and $\frac{6-h}{2h}$=integer gives only $h=2$. Hence

$$r=8=6+2.(1+0)+4.0,$$

$$k=2=6+2.0+4.-1,$$

the only value of k, because $\varepsilon_0 \not> 1+2\alpha_0-1$. Wherefore

$$R_{\frac{1}{2}c}^{2di}(8, 2)_6=R^{mo}(3, 0)\Delta_1=1.$$

Next, if $n=4$, $\frac{4-h}{2h}$=integer, $=0$, gives $h=4$,

$$r=8=4+4(1+0)$$

$$k=4=4+4.0,$$

the only value of k;

$$R_{\frac{1}{2}c}^{4di}(8, 4)_4=R^{mo}(3, 0)\Delta_0=1.$$

Next, in equations (C.), $r=8$ gives $n=6$, or $n=4$, of which the first, putting $\frac{6-2h}{2h}$=integer, gives only $h=1$, the value above forbidden. But $\frac{4-2h}{2h}$= integer $=0$, gives $h=2$; whence $k=4$, and by the formula for $R_0^{\frac{1}{2}n.di}(r, k)_n$ we obtain $(\alpha_0=\varepsilon_0=0)$,

$$R_0^{2di}(8, 4)_4=\tfrac{1}{2}\{R(3, 0).R(3, 0)-R_{\frac{1}{2}c}^{4di}(8, 4)_4\}=\tfrac{1}{2}\{1-1\}=0.$$

XLV. Problem 1. *To find* $R_c^{h.di}(r, k)_n$, *the number of* $(1+k)$*-partitioned* r*-gons built on the* n*-gonal nucleus, which have* h *clear diagonal axes of reversion, and none others.*

In these all the axes are diagonal axes of the nucleus, and terminate at $2h$ equidistant angles of it. The interval of $\frac{n}{2h}$ sides between any two, as the 1st, 2nd, ... $\frac{n}{2h}$th sides of the n-gon, is loaded, the mth side with a $(1+e_m)$-partitioned $(2+a_m)$-gon, as in the previous constructions, so that all shall be reversible about every axis. We have to solve

$$\left.\begin{aligned} r&=n+2h\left(a_1+a_2+\ldots a_{\frac{n}{2h}}\right)\\ k&=n+2h\left(e_1+e_2+\ldots e_{\frac{n}{2h}}\right)\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C''.)$$

$$a_m \overline{\gtrless} 0, \quad e_m \not> a_m - 1.$$

The sum of products, one for every solution, all orders of the values counted among solutions,

$$\Sigma\Delta_{\frac{n}{2h}} = \Sigma D(2+a_1, e_1) D(2+a_2, e_2) \ldots D\left(2+a_{\frac{n}{2h}}, e_{\frac{n}{2h}}\right),$$

is the entire number of configurations about a clear diagonal axis, if r-gons have h equidistant clear diagonal axes, and built on this nucleus.

Now of these configurations

two are seen on each of $\qquad R_c^{hi.di}(r, k)_n$,

one is seen on each of $\qquad R_{\frac{1}{2}c}^{2ih.di}(r, k)_n$,

namely, that about the clear axes in these,

one is seen on each of $\qquad R_c^{2ih.agdi}(r, k)_n$,

and one on each of $\qquad R_{dc}^{2ih.agdi}(r, k)_n$,

namely, that about the diagonal axes in these. Wherefore $(i>0)$,

$$\Sigma\Delta_{\frac{n}{2h}} = \Sigma_i\{2R_c^{hi.di}(r, k)_n + R_{\frac{1}{2}c}^{2hi.di}(r, k)_n + R_c^{2ih.agdi}(r, k)_n + R_{dc}^{2ih.agdi}(r, k)_n\}, \text{ and}$$

$$2.R_c^{h.di}(r, k)_n = \Sigma\Delta_{\frac{n}{2h}} - \Sigma_i\{2R_c^{(i+1)h.di}(r, k)_n + R_{\frac{1}{2}c}^{2hi.di}(r, k)_n + R_c^{2ih.agdi}(r, k)_n + R_{dc}^{2ih.agdi}(r, k)_n\}.$$

As $r>n$ in (C''.), the greatest value of h gives $\frac{n}{2h}=1$. Those equations then become

$$\frac{r-n}{n}=a_1, \quad \frac{k-n}{n}=e_1$$

$$\Delta_1 = D\left(\frac{r+n}{n}, \frac{k-n}{n}\right) = \Sigma\Delta_1;$$

hence $\qquad R_c^{ndi}(r, k)_n = 0$,

and since $\qquad R_c^{n.agdi}(r, k) = 0$ (XVIII.),

$$R_c^{\frac{1}{2}n.di}(r, k)_n = \tfrac{1}{2}\left\{D\left(\frac{r+n}{n}, \frac{k-n}{n}\right) - R_{\frac{1}{2}c}^{n.di}(r, k)_n - R_{dc}^{n.agdi}(r, k)_n\right\}$$

a given number, by XLIII. and XXX.

And we can proceed to find $R_c^{h.di}(r, k)_n$, for every value of h down to $h=1$, which gives, in (C''.), r, n, and k, divisible by $2h$.

Thus we have completely determined

$$R^{h.di}(r, k)_n = R_0^{h.di}(r, k)_n + R_{\frac{1}{2}c}^{h.di}(r, k)_n + R_c^{h.di}(r, k)_n.$$

XLVI. When $r=8$ in equation (C''.), $n=6$, or $n=4$. If $n=6$, $\frac{n}{2h}=3$ and $h=1$ are the only values making $\frac{8}{2h}$ integer. We have then the three solutions

$$r=8=6+2(1+0+0)=6+2(0+1+0)=6+2(0+0+1),$$
$$k=2=6+2(0-1-1)=6+2(-1+0-1)=6+2(-1-1+0);$$
$$\Sigma\Delta_3=3D(3, 0)=3, \text{ and we obtain}$$
$$R_c^{di}(8, 2)_6=\tfrac{1}{2}\{3-R_{\frac{1}{2}c}^{2di}(8, 2)_6\}=1.$$

If $n=4$, $h=2$ or $h=1$; for the first of which we obtain by the formula for $R_c^{\frac{1}{2}ndi}(r, k)_n$,

$$R_c^{2di}(8, 4)_4=\tfrac{1}{2}\{D(3, 0)-R_{\frac{1}{2}c}^{4di}(8, 4)_4\}=\tfrac{1}{2}(1-1)=0,$$

$k=4$ being the only possible value.

If $h=1$, we may have $k=4$ or $k=2$; for $k=4$, we find the three solutions $\left(\frac{n}{2h}=2\right)$,

$$8=4+2(2+0)=4+2(0+2)=4+2(1+1)$$
$$4=4+2(1-1)=4+2(-1+1)=4+2(0+0);$$
$$\Sigma\Delta_2=2D(4, 1)+D(3, 0)=4+1=5,$$

whence $$R_c^{di}(8, 4)_4=\tfrac{1}{2}\{5-R_{\frac{1}{2}c}^{4di}(8, 4)_4\}=2.$$

For $k=2$, there are two solutions,

$$8=4+2(2+0)=4+2(0+2),$$
$$2=4+2(0-1)=4+2(-1+0);$$

giving $$\Sigma\Delta_2=2D(4, 0)=2,$$

and $$R_c^{di}(8, 2)_4=\tfrac{1}{2}(2-0)=1;$$

for there is nothing in our list under $R(8, 2)_4$ to subtract in the formula for $2R_c^{hdi}(r, k)_n$.

XLVII. Our next step is to determine

$$R^{h.mo}(r, k)n=R_c^{h.mo}(r, k)_n+R_0^{h.mo}(r, k)_n.$$

Problem m. *To find* $R_c^{h.mo}(r, k)_n$, *the number of* $(1+k)$*-partitioned* r*-gons, built on the* n*-gonal nucleus to have* h *clear monogonal axes of reversion, and none besides.*

The h clear axes (Theorem O) are monogonal axes of the n-gon, bisecting h equidistant sides of it; between which lie h vertices of the n-gon upon those h axes; and n is odd. The interval of $\frac{n-h}{2h}$ sides between such a vertex and bisected side is loaded the mth from the vertex with a $(1+e_m)$-partitioned $(2+a_m)$-gon, so that all is reversible about the h axes. The equations following are satisfied in the construction:

$$\left.\begin{array}{l} r=n+\quad 2h\left(a_1+a_2+\ldots+a_{\frac{n-h}{2h}}\right) \\ k=n-h+2h\left(e_1+e_2+\ldots+e_{\frac{n-h}{2h}}\right) \end{array}\right\}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(D.)}$$

$$(a_m \overline{\gtrless} 0), \quad (e_m \not> a_m-1).$$

For every solution of these equations we can make

$$D(2+a_1, e_1)D(2+a_2, e_2)\times\ldots D\left(2+a_{\frac{n-h}{2h}}, e_{\frac{n-h}{2h}}\right)=\Delta_{\frac{n-h}{2h}}$$

different arrangements of the e_1, e_2, &c. diagonals, giving as many distinct configurations about the monogonal axis through the nth side of the n-gon. And, as before, we obtain $\Sigma\Delta_{\frac{n-h}{2h}}$ different configurations in all by adding together as many such products as there are solutions of equations (D.), every order of the quantities counted as a solution; which sum is the entire number of configurations of $(1+k)$-partitioned r-gons built on this nucleus to have h clear monogonal equidistant axes of reversion.

Now one of these configurations is seen on each r-gon of the number $R_c^{2(i+1)h.mo}(r, k)_n$; for each has h equidistant monogonal axes, and on none besides, because if one monogonal axis is clear, all are clear; otherwise there would be more than two axial configurations, and the axes are odd in number (Theorem F). Wherefore

$$\Sigma\Delta_{\frac{n-h}{2h}}=\Sigma_i R_c^{(2i+1)h.mo}(r, k)_n,$$

and $(i \overline{\gtrless} 0)$ $$R_c^{h.mo}(r, k)_n=\Sigma\Delta_{\frac{n-h}{2h}}-\Sigma_i R_c^{(2i+3)h.mo}(r, k)_n.$$

XLVIII. Since $r>n$ in equations (D.), $n>h$. Let $n-h=2h$; then $h=\frac{n}{3}$ is the highest value, and

$$r=n+\frac{2n}{3}a_1, \qquad 2+a_1=\frac{3r+n}{2n},$$

$$k=\frac{2n}{3}+\frac{2n}{3}e_1, \qquad e_1=\frac{3k-2n}{2n};$$

$$\Delta_1=\Sigma\Delta_1=D\left(\frac{3r+n}{2n}, \frac{3k-2n}{2n}\right),$$

and $$R_c^{n.mo}(r, k)_n=0,$$

$$R_c^{\frac{n}{3}.mo}(r, k)_n=D\left(\frac{3r+n}{2n}, \frac{3k-2n}{2n}\right),$$

which of course $=0$ if the fractions are irreducible.

From this we can proceed to find $R_c^{h.mo}(r, k)_n$, for every odd value of h down to $h=1$ which makes $n-h$, $r-n$, and k, divisible by $2h$.

The octagon has no monogonal axis of reversion (Theorem E).

XLIX. Problem n. *To find* $R_0^{h.mo}(r, k)_n$, *the number of* $(1+k)$*-partitioned* r*-gons built on the* n*-gonal nucleus to have* h *only monogonal axes of reversion, all loaded.*

The axes will be either all monogonal or all agonal axes of the n-gon (Theorem N, O), for every axis of the r-gon carries the same two configurations (Theorem C). Let them first be all monogonal, *i. e.* let n be odd; this requires that each be loaded at one end with an agonally reversible (about one or more axes) $(1+\varepsilon_0)$-partitioned $(4+2\alpha_0)$-gon. The intervals of $\frac{n-h}{2h}$ sides will be loaded as in the last preceding construction, to satisfy

$$\left.\begin{aligned} r&=n+h(2+2\alpha_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-h}{2h}}\right),\\ k&=n+h\varepsilon_0 \qquad\quad +2h\left(e_1+e_2+\ldots+e_{\frac{n-h}{2h}}\right);\end{aligned}\right\} \quad \ldots\ldots \quad (\mathrm{D}'.)$$

$$(a_m \overline{\gtrless} 0, \quad e_m \not> a_m-1), \ (n=2m+1),$$

$(4+2\alpha_0)$ and ε_0 being numbers selected at pleasure from our register of agonally or ago-diagonally reversibles. Putting, as in XIX.,

$$\Sigma_m\{2\mathrm{R}^{m.ag}(4+2\alpha_0, \varepsilon_0)+\mathrm{R}^{2magdi}(4+2\alpha_0, \varepsilon_0)\}=\mathrm{H},$$

and forming the product $\Delta_{\frac{n-h}{2h}}$ from a solution of (D′.) which contains those numbers α_0, ε_0, we deduce, as before, that $\Sigma\mathrm{H}.\Delta_{\frac{n-h}{2h}}$, comprising as many products under Σ as those solutions of (D′.), all orders of the quantities $a_1\, e_1$, &c. being counted as solutions, is the total number of axial configurations of $(1+k)$-partitioned r-gons having h loaded monogonal axes and built on the n-gonal nucleus when n is odd.

And as one of these configurations is seen on each r-gon of the number $\mathrm{R}_0^{(2i+1)h.mo}(r, k)_n$, and only one, read from the loaded end of the axis, we obtain

$$\Sigma\left(\mathrm{H}.\Delta_{\frac{n-h}{2h}}\right)=\Sigma_i\mathrm{R}_0^{(2i+1)h.mo}(r, k)_n,$$

and

$$\mathrm{R}_0^{h.mo}(r, k)_n=\Sigma\mathrm{H}.\Delta_{\frac{n-h}{2h}}-\Sigma_i\mathrm{R}_0^{(2i+3)h.mo}(r, k)_n,$$

where n is odd, and $i \overline{\gtrless} 0$.

L. Now let the h axes be agonal axes of the n-gon; *i. e.* let n be even.

One end of every axis will carry (Theorem O) the same $(3+2\alpha_0)$-gon $(1+\varepsilon_0)$-partitioned and reversible about at least one monogonal axis. The other ends will carry the same $\left(2+2\alpha_{\frac{n}{2h}}\right)$-gon, $\left(1+\varepsilon_{\frac{n}{2h}}\right)$-partitioned and reversible about at least one agonal axis, which will be no load if $\alpha_{\frac{n}{2h}}=0$. The interval of $\frac{n-2h}{2h}$ sides between these reversibles will be loaded, on the mth side from the first, with a $(1+e_m)$-partitioned $(2+a_m)$-gon; and the equations to be satisfied are,

$$\left.\begin{aligned} r&=n+h(1+2\alpha_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{2h}}\right)+h\alpha_{\frac{n}{2h}},\\ k&=n+h\varepsilon_0 \qquad\quad +2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right)+h\varepsilon_{\frac{n}{2h}};\end{aligned}\right\} \quad \ldots\ldots \quad (\mathrm{D}''.)$$

where $a_m \overline{\gtrless} 0$, $e_m \not> a_m-1$, and α_0, ε_0, $\alpha_{\frac{n}{2h}}$, $\varepsilon_{\frac{n}{2h}}$ are chosen at pleasure, the first pair from our register of monogonally reversibles, and the second from our agonally or ago-diagonally reversibles.

Taking now any one solution of these equations, the product $\mathrm{H}''\mathrm{M}\Delta$ of

$$\mathrm{H}''=\Sigma_m\left\{2\mathrm{R}^{m.ag}\left(2+\alpha_{\frac{n}{2h}}, \varepsilon_{\frac{n}{2h}}\right)+\mathrm{R}^{2m.agdi}\left(2+\alpha_{\frac{n}{2h}}, \varepsilon_{\frac{n}{2h}}\right)\right\},$$

(which is to be considered unity when $\alpha_{\frac{n}{2h}}=0$, the load being then simply the edge of the n-gon),

$$M=\Sigma_m R^{m.mo}(3+\alpha_0, \varepsilon_0),$$

$$\Delta_{\frac{n-2h}{2h}}=D(2+a_1, e_1).D(2+a_2, e_2)\times ..D\left(2+\alpha_{\frac{n-2h}{2h}}, e_{\frac{n-2h}{2h}}\right),$$

is the number of configurations that we can produce, about the axis through the nth side of the n-gon, read from that side, of $(1+k)$-partitioned r-gons built on this nucleus from this one solution, to have h loaded monogonal axes. And the sum of these products belonging to every solution, all orders and values counting as solutions, is the exact number of all such configurations possible. Now one, and only one, of these is seen in each r-gon of the number $R_0^{(2i+1)h.mo}(r, k)_n$, for n even. That is,

$$\Sigma\left(H''.M.\Delta_{\frac{n-2h}{2h}}\right)=\Sigma_i R_0^{(2i+1)h.mo}(r, k)_n,$$

$$R_0^{h.mo}(r, k)_n=\Sigma\left(H''.M.\Delta_{\frac{n-2h}{2h}}\right)-\Sigma_i R_0^{(2i+3)h.mo}(r, k)_n,$$

where n is even, and $i\geqq 0$; and $H''=1$ whenever $\alpha_{\frac{n}{2h}}=0$.

LI. In equation (D′.), the highest value of h is $h=n$. Then those equations become

$$\frac{r-n}{n}=2+2\alpha_0, \frac{k-n}{n}=\varepsilon_0$$

$$H=\Sigma(H\Delta_0)=\Sigma_m\left\{2R^{mag}\left(\frac{r+n}{n}, \frac{k-n}{n}\right)+R^{2magdi}\left(\frac{r+n}{n}, \frac{k-n}{n}\right)\right\}=R_0^{n.mo}(r, k)_n,$$

since $\Delta_0=1$, where n is odd; but $R_0^{n.mo}(r, k)_n=0$ of course, if n is even.

From this we can proceed to find $R_0^{h.mo}(r, k)_n$, for all odd values of $h>1$, which make, in equations (D′.),

$$n-h, r-n-h(2+2\alpha_0) \text{ and } k-n-h\varepsilon_0,$$

divisible by $2h$; α_0 and ε_0 being selected from our register of agonally and ago-diagonally reversibles.

In equations (D″.), the greatest value of h gives $n-2h=0$, or $h=\frac{1}{2}n$. Those equations become

$$\frac{2r-3n}{2n}=\alpha_0+\alpha_1,$$

$$\frac{2k-2n}{n}=\varepsilon_0+\varepsilon_1,$$

$$\Delta_0=\Sigma\Delta_0=1,$$

whence, if $\frac{1}{2}n$ be an odd number (for h is always odd),

$$R_0^{\frac{n}{2}.mo}(r, k)_n=\Sigma H''.M,$$

$\left(\text{but this is }=0\text{ if }\frac{n}{2}\text{ be even}\right)$, H'' and M being defined in the preceding article, and the number of products $H''.M$ being equal to the number of solutions of the above reduced form of equations (D″.), made with numbers α_0, ε_0, $\alpha_{\frac{n}{2h}}$, $\varepsilon_{\frac{n}{2h}}$, selected from our register, as above stated (L.).

From this value of $\mathrm{R}_0^{\frac{n}{2}mo}(r, k)_n$, we can proceed to find, when n is even, $\mathrm{R}_0^{hmo}(r, k)_n$ for all values of $h>1$, which make

$$n,\ r-h\left(1+2\alpha_0+\alpha_{\frac{n}{2h}}\right),\ \text{and}\ k-h\left(\varepsilon_0+\varepsilon_{\frac{n}{2h}}\right),$$

divisible by $2h$. But we cannot find $\mathrm{R}_0^{mo}(r, k)_n$ by these formulæ, there being no such division reducible to a nucleus (VIII.).

Thus we have found (XLIX., L.) $\mathrm{R}_0^{hmo}(r, k)_n$ for any value of $n>2$.

LII. All the numbers R of article XX. have now been determined, for every value of h when the axes are not all loaded, and for every value of $h>1$ when they are all loaded, the nucleus being n-gonal, and $n \not< 3$. We have next to enumerate the doubly and singly reversibles which are built on the 2-gonal nucleus, that is, upon a line, which is of course a drawn axis of reversion of the r-gon, and also the singly reversibles which have an e-scored axis. These have no proper nucleus (Theorem K).

These r-gons now to be discussed fall into the classes following,

$$\mathrm{R}^{2agdi}(r, k)_2,\ \mathrm{R}^{2di}(r, k)_2,\ \mathrm{R}^{di}(r, k)_2,\ \mathrm{R}_0^{ag}(r, k),\ \mathrm{R}_0^{di}(r, k),\ \mathrm{R}_0^{mo}(r, k);$$

of which the first two have a drawn diagonal axis standing as a perpendicular score upon an undrawn axis (Theorem G); the third has a single drawn diagonal axis (Theorem G); and the fourth, fifth and sixth have a single undrawn and scored axis of reversion (Theorem K). The subscript 2 in the first three symbols shows that the nucleus is a drawn line, which may be considered as a 2-gon reversible about a diagonal and an agonal axis.

Problem o. *To find* $\mathrm{R}^{2agdi}(\mathrm{r}, \mathrm{k})_2$, *the* $(1+\mathrm{k})$-*partitioned* r-*gons built on a nucleus-line to have a diagonal and an agonal axis of reversion.*

If the drawn diagonal be erased, the figure will be still reversible about the same two axes undrawn, for (Theorem B, VI.) the erasure has not disturbed the symmetry about either axis. But the erasure may have restored the symmetry about some n-gonal nucleus of which that erased line is a clear diagonal axis (Theorem R, XVIII.). And every r-gon which has a clear diagonal axis perpendicular to an agonal one, that is, every one of $\mathrm{R}_c^{(4h+2)agdi}(r, k-1)_n$ and of $\mathrm{R}_{dc}^{(4h+2)agdi}(r, k-1)_n$, whatever h or n may be, if $n>2$ (*Cor.* 1, VII.), will give us one and only one (Theorem G) of $\mathrm{R}^{2agdi}(r, k)_2$, by the drawing of any diagonal axis of reversion; for all these axes bisect the same configuration. Wherefore

$$\mathrm{R}^{2agdi}(r, k)_2=\Sigma_h\Sigma_n\{\mathrm{R}_c^{(4h+2)agdi}(r, k-1)_n+\mathrm{R}_{dc}^{(4h+2)agdi}(r, k-1)_n\},$$

including all values of h, and of $n>2$ in our register; n of course being even, else it could have no diameter.

As we have nothing in our register under $\mathrm{R}^{(4h+2)agdi}(8, k)_n$, $\mathrm{R}^{2agdi}(8, k)_2=0$, whatever k may be.

LIII. Problem p. *To find* $\mathrm{R}^{2di}(\mathrm{r}, \mathrm{k})_2$, *the number of* $(1+\mathrm{k})$-*partitioned* r-*gons built on a line-nucleus, to have two diagonal axes of reversion.*

If the drawn diagonal axis be erased, the figure still remains reversible about the same axis, as in the preceding article, and we may have restored the symmetry about some n-gonal nucleus of which that erased line is a clear diagonal axis. Every r-gon of the number

$$\mathrm{R}_c^{2hdi}(r, k-1)_n$$

will give us two of $\mathrm{R}^{2di}(r, k)_2$, for we can draw a diagonal axis bisecting either of its configurations; and every r-gon of the number

$$(\mathrm{R}_{\frac{1}{2}c}^{2hdi}+\mathrm{R}_c^{4hagdi}+\mathrm{R}_{dc}^{4hagdi})(r, k-1)_n,$$

which have all a clear diagonal axis at right angles to a diagonal axis (VI. VII. *Cor.* 1), will give us one of $\mathrm{R}^{2di}(r, k)_2$, by drawing any diagonal axis of reversion, which can be drawn without meeting diagonals; *i. e.*

$$\mathrm{R}^{2di}(r, k)_2=\Sigma_h\Sigma_n\{2\mathrm{R}_c^{2h.di}(r, k-1)_n+\mathrm{R}_{\frac{1}{2}c}^{2h.di}(r, k-1)_n+\mathrm{R}_c^{4hagdi}(r, k-1)_n+\mathrm{R}_{dc}^{4hagdi}(r, k-1)_n\};$$

comprehending all the values of $h>0$ and $n>2$ in our register of reversibles.

Hence follows (XLIV.),

$$\mathrm{R}^{2di}(8, 5)_2=\mathrm{R}_{\frac{1}{2}c}^{4di}(8, 4)_4 =1,$$
$$\mathrm{R}^{2di}(8, 3)_2=\mathrm{R}_{\frac{1}{2}c}^{2di}(8, 2)_6 =1,$$
$$\mathrm{R}^{2di}(8, 1)_2=\mathrm{R}_c^{8agdi}(8, 0)_8=1.$$

LIV. Problem q. *To find* $\mathrm{R}^{\mathrm{di}}(\mathrm{r}, \mathrm{k})_2$, *the number of* $(1+\mathrm{k})$*-partitioned singly reversible* r*-gons having a drawn diagonal axis.*

If this axis be erased, the figure F remains reversible about the same diagonal axis, and the erasure may have restored the symmetry about some n-gonal nucleus of which that erased line is a clear diagonal axis, one of an odd number of axes of reversion in F; for if this number were even, there would then be an axis (VI.) perpendicular to the erased one, about which the symmetry would not be disturbed by the erasure, so that the figure before the erasure must have been at least doubly reversible, contrary to hypothesis.

Now every one of the number $\mathrm{R}_c^{(2h+1)di}(r, k-1)_n$ will give one (Theorems B, G), and only one, of $\mathrm{R}^{di}(r, k)_2$, by drawing of any of its axes of reversion, because all these carry the same configuration. Consequently

$$\mathrm{R}^{di}(r, k)_2=\Sigma_h\Sigma_n\mathrm{R}_c^{(2h+1)di}(r, k-1)_n,$$

comprehending all values of h including $h=0$, and all of n, in our register of k-partitions. Hence we obtain (XLVI.),

$$\mathrm{R}^{di}(8, 5)_2=\mathrm{R}_c^{di}(8, 4)_4=2,$$
$$\mathrm{R}^{di}(8, 3)_2=\mathrm{R}_c^{di}(8, 2)_6+\mathrm{R}_c^{di}(8, 2)_4=2.$$

LV. Problem r. *To find* $\mathrm{R}_0^{\mathrm{mo}}(\mathrm{r}, \mathrm{k})$, *the number of* $(1+\mathrm{k})$*-partitioned* r*-gons reversible only about a single scored monogonal axis.*

The number of ways in which e diagonals can be drawn at right angles across an axis of one of the r-gons of the division $R_c^{(2h+1)mo}(r, k-e)_2$ is $\left(\frac{n-3}{2}\right)^{e|-1} . \overline{|e+1}^{-1}$; for any e vertices may be chosen out of $\frac{n-3}{2}$ on one side of the monogonal axis. The whole number of operations described in Theorems P and S, by which a $(1+k)$-partitioned monogonal can arise, is the above operation performed on every $(1+k-e)$-partitioned monogonal in our register for every value of e, n, and h. The results, by Theorems E, P, and S, are all monogonally reversibles. And by Theorem Q we see that every $(1+k)$-partitioned $(2h+3)$-ly reversible about loaded monogonal axes will be among these results, whatever be its nucleus; and every singly reversible with a loaded monogonal axis will be among them; for if this be cleared, we have before us one of our subjects of operation just referred to in our register. Wherefore

$$\Sigma_h \Sigma_e \Sigma_n \left(\frac{n-3}{2}\right)^{e|-1} . \overline{|e+1}^{-1} . R_c^{(2h+1)mo}(r, k-e)_n = \Sigma_h R_0^{(2h+1)mo}(r, k),$$

and $$R_0^{mo}(r, k) = \Sigma_h \Sigma_e \Sigma_n \left(\frac{n-3}{2}\right)^{e|-1} . \overline{|e+1}^{-1} . R_c^{(2h+1)mo}(r, k-e)_n - \Sigma_h R_0^{(2h+3)mo}(r, k),$$

where the omission of the subscript n shows that all nuclei are included. This is a given number; and as no monogonally reversible can be obtained by scoring any but a monogonal axis, it is the whole number $R_0^{mo}(r, k)$.

LVI. Problem s. *To find* $R_0^{ag}(r, k)$, *the number of* $(1+k)$-*partitioned* r-*gons singly reversible about a scored agonal axis.*

$R_0^{ag}(r, k)$ and $R_0^{di}(r, k)$ are obtained by the operations of Theorems P, R, S, by scoring both $2h$-ly and $(2h+1)$-ly reversibles. We write

$$R_0^{ag}(r, k) = R_0^{ag}(r, k)' + R_0^{ag}(r, k)'',$$

the first of the right member denoting the number of those r-gons obtained by scoring one of an odd number, and the second, those by one of an even number, of axes.

The number $R_0^{ag}(r, k)'$ is obtained by Theorems P, Q, S, exactly as $R_0^{mo}(r, k)$ was obtained, except that the number of angles of the nucleus from which on one side of the clear axis e scores can be drawn, is $\frac{n-4}{2}$ in the first, instead of $\frac{n-3}{2}$ in the second. That is,

$$R_0^{ag}(r, k)' = \Sigma_e \Sigma_n \left(\frac{n-4}{2}\right)^{e|-1} . \overline{|e+1}^{-1} . R_c^{(2h+1)ag}(r, k-e)_n - \Sigma_h R_0^{(2h+3)ag}(r, k),$$

where again the omission of the subscript n shows that constructions upon all nuclei are included.

LVII. We have next to determine $R_0^{ag}(r, k)''$, obtained by scoring a clear axis out of an even number of axes (Theorem R).

Whatever e may be, the constructions of Theorem R are enumerated by subtracting the symmetrical arrangements of the e scores from the whole number of arrangements, and dividing the remainder by two; for every unsymmetrical one occurs twice in this remainder, once as read from each end of the scored axis, which readings are the two configurations of the singly reversible about the axis (Theorem C).

Let e first be even, in constructing $R_0^{ag}(r, k)''$. The entire number of e-plets eligible from the $\frac{n-4}{2}$ numerals on one side of the axis is

$$\left(\frac{n-4}{2}\right)^{e|-1}.\overline{|e+1}^{\,-1},$$

and that of the $\frac{e}{2}$-plets eligible alike on opposite sides of the centre of the axis is

$$\left(\frac{n-4}{4}\right)^{\frac{e}{2}|-1}\times\overline{\left|\frac{e}{2}+1\right.}^{\,-1},$$

when $n=4m$, and, when $n=4m+2$, it is

$$\left(\frac{n-6}{4}\right)^{\frac{e}{2}|-1}\times\overline{\left|\frac{e}{2}+1\right.};$$

wherefore, using the circulators 4_n and 4_{n-2} and 2_e,

$$\frac{2_e}{2}\left\{\left(\frac{n-4}{2}\right)^{e|-1}.\overline{|e+1}^{\,-1}-\left(\frac{n-4}{4}\right)^{\frac{e}{2}|-1}.\overline{\left|\frac{e}{2}+1\right.}.4_n-\left(\frac{n-6}{4}\right)^{\frac{e}{2}|-1}.\overline{\left|\frac{e}{2}+1\right.}.4_{n-2}\right\}. \quad . \quad (a.)$$

is the number out of $R_0^{ag}(r, k)''$, when e is even, that can be made from any $(1+k-e)$-partitioned r-gon having an n-gonal nucleus, $2h$ clear axes, and clear agonal ones, by scoring any one of these with e parallels at right angles to it.

But when e is odd, a diameter of the n-gon must be one of the scores if they be symmetrical about the centre, which requires $n=4m+2$. Hence, if $n=4m$, every odd e scores that can be drawn gives an unsymmetrical arrangement, so that there are thus

$$4_n.2_{e-1}.\tfrac{1}{2}\left(\frac{n-4}{2}\right)^{e|-1}.\overline{|e+1}^{\,-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b.)$$

out of $R_0^{ag}(r, k)''$ obtained by so scoring any clear agonal axis of a $2h$-ly reversible.

If $n=4m+2$, and e is odd, we are to subtract from the whole number of ways of drawing e scores that of the ways of doing it symmetrically, which, when we have drawn a diameter for one, is that of the ways of drawing $\frac{1}{2}(e-1)$ scores on one side of it. This difference, divided by two, is

$$4_{n-2}.2_{e-1}.\tfrac{1}{2}\left(\left(\frac{n-4}{2}\right)^{e|-1}.\overline{|e+1}^{\,-1}-\left(\frac{n-6}{4}\right)^{\frac{e-1}{2}|-1}.\overline{\left|\frac{e+1}{2}\right.}^{\,-1}\right), \quad . \quad . \quad . \quad . \quad . \quad (c.)$$

which is the number of ways of drawing an odd number e of scores across the agonal axis of a $(4m+2)$-gonal nucleus.

Uniting the expressions $(a.)$, $(b.)$, $(c.)$, we find

$$\tfrac{1}{2}\left\{\left(\frac{n-4}{2}\right)^{e|-1}.\overline{|e+1}^{\,-1}-2_e\left(4_n.\left(\frac{n-4}{4}\right)^{\frac{e}{2}|-1}.\overline{\left|\frac{e}{2}+1\right.}^{\,-1}+4_{n-2}.\left(\frac{n-6}{4}\right)^{\frac{e}{2}|-1}.\overline{\left|\frac{e}{2}+1\right.}^{\,-1}\right)\right.$$
$$\left.-2_{e-1}\left(4_{n-2}.\left(\frac{n-6}{4}\right)^{\frac{e-1}{2}|-1}:\overline{\left|\frac{e+1}{2}\right.}^{\,-1}\right)\right\}$$

for the correct number out of $R_0^{ag}(r, k)''$ that can be made by drawing e diagonals of the

n-gonal nucleus in any $2h$-ly reversible $(1+k-e)$-partitioned r-gon, across a clear agonal axis.

When the axes are all clear and agonal, we can score across either configuration, *i. e.* upon either of two adjacent axes, and thus obtain twice this number towards $R_0^{ag}(r, k)''$; but if only half are clear and agonal, we have only one configuration to score, and can obtain this number only. Hence we see that the product under $\Sigma_e\Sigma_n$ of the number above written, into

$$\Sigma_h\Sigma_e\Sigma_n\{2R_c^{2h.ag}(r, k-e)_n+R_{\frac{1}{2}c}^{2h.ag}(r, k-e)_n+R_c^{2h.agdi}(r, k-e)_n+R_{ac}^{2hagdi}(r, k-e)_n\},$$

is the entire sum $R_0^{ag}(r, k)''$, of r-gons in $R_0^{ag}(r, k)$ obtainable by scoring one of an even number of axes, by the construction of Theorem R.

LVIII. Adding, then, to this entire sum the portion of $R_0^{ag}(r, k)$ obtained by scoring one of an odd number of axes (LVI.), we express the complete result thus:—

$$R_0^{ag}(r, k)=\Sigma_h\Sigma_e\Sigma_n\Bigg[R_c^{(2h+1)ag}(r, k-e)_n.\left(\frac{n-4}{2}\right)^{e|-1}.\overline{|e+1}^{\,-1}-R_0^{(2h+3)ag}(r, k)$$
$$+\tfrac{1}{2}\{2R^{2hag}(r, k-e)_n+R_{\frac{1}{2}c}^{2hag}(r, k-e)_n+R_c^{2h.agdi}(r, k-e)_n+R_{ac}^{2h.agdi}(r, k-e)_n\}$$
$$\times\left\{\left(\frac{n-4}{2}\right)^{e|-1}.\overline{|e+1}^{\,-1}-2_e\left(4_n.\left(\frac{n-4}{4}\right)^{\frac{e}{2}|-1}.\overline{|\tfrac{e}{2}+1}^{\,-1}+4_{n-2}.\left(\frac{n-6}{4}\right)^{\frac{e}{2}|-1}\overline{|\tfrac{e}{2}+1}^{\,-1}\right)\right.$$
$$\left.-2_{e-1}\left(4_{n-2}.\left(\frac{n-6}{4}\right)^{\frac{e-1}{2}|-1}.\overline{\left|\frac{e+1}{2}\right.}^{\,-1}\right)\right\}\Bigg],$$

comprising all the values of e, h and n on our register of $(1+k-e)$-partitioned reversibles specified in the right member: $(n>2,\ e>0,\ h\gtreqless 0)$.

When $k=e$, the r-gon scored is the simple r-gon, or $n=r$.

The only numbers of those in the right member of this expression that our list contains for $r=8$, are

$$R_c^{8agdi}(8, 0)_8=1,\ R_c^{ag}(8, 2)_6=1,\ R_c^{ag}(8, 4)_4=1,\ R_{\frac{1}{2}c}^{2ag}(8, 2)_4=1$$

(XXVIII. XXXVIII.), of which the two last disappear by the multiplier $n-4$, since e always >0. Therefore

$$R_0^{ag}(8, 1)=\tfrac{1}{2}\left(R_c^{8agdi}(8, 0)_8\times\frac{8-4}{2.1}\right)=1$$

$$R_0^{ag}(8, 2)=\tfrac{1}{2}\left\{R_c^{8.agdi}(8, 0)_8\times\left((\tfrac{4}{2})^{2|-1}.\overline{|3}^{\,-1}-\frac{4}{4.1}\right)\right\}=0$$

$$R_0^{ag}(8, 3)=R_c^{ag}(8, 2)_6\times\left(\frac{6-4}{2}\cdot 1\right)=1$$

$$R_0^{ag}(8, 4)=R_c^{ag}(8, 2)_6\times(\tfrac{2}{2})^{2|-1}.\overline{|3}^{\,-1}=0.$$

LIX. Problem t. *To find* $R_0^{di}(r, k)$, *the number of* $(1+k)$-*partitioned* r-*gons singly reversible about a scored diagonal axis.*

We write again in the sense of art. LVI.,

$$R_0^{di}(r, k)=R_0^{di}(r, k)'+R_0^{di}(r, k)''.$$

The number of angles of the nucleus from which, on one side of a clear diagonal axis, e scores can be drawn, is $\frac{1}{2}(n-2)$. Wherefore, by the reasoning of LV.,

$$R_0^{di}(r,\ k)'=\Sigma_e\Sigma_n\left(\frac{n-2}{2}\right)^{e|-1}.\lfloor\overline{e+1}^{\,-1}.R_c^{(2h+1)di}(r,\ k-e)_n-\Sigma_h R_0^{(2h+3)di}(r,\ k).$$

LX. When the scored axis is a diagonal one out of an even number of axes, the constructions $R_0^{di}(r,\ k)''$ of Theorem R are enumerated by a subtraction and division like those of LVII. Let e first be even.

The entire number of e-plets eligible from the $\frac{n-2}{2}$ numerals on one side of this diagonal axis, is $\left(\frac{n-2}{2}\right)^{e|-1}\times\lfloor\overline{e+1}^{\,-1}$; and that of $\frac{e}{2}$-plets eligible alike on opposite sides of the centre of the axis is $\left(\frac{n-2}{4}\right)^{\frac{e}{2}|-1}\times\lfloor\overline{\frac{e}{2}+1}^{\,-1}$, when $n=4m+2$, and $\left(\frac{n-4}{4}\right)^{\frac{e}{2}|-1}.\lfloor\overline{\frac{e}{2}+1}^{\,-1}$, when $n=4m$: wherefore

$$\tfrac{1}{2}\left\{\left(\frac{n-2}{2}\right)^{e|-1}.\lfloor\overline{e+1}^{\,-1}-\left(\frac{n-2}{4}\right)^{\frac{e}{2}|-1}.\lfloor\overline{\frac{e}{2}+1}^{\,-1}.4_{n-2}-\left(\frac{n-4}{4}\right)^{\frac{e}{2}|-1}.\lfloor\overline{\frac{e}{2}+1}^{\,-1}.4_n\right\}.2_e \quad . \quad (a.)$$

is the number out of $R_0^{di}(r,\ k)''$, when e is even, that can be made by e-scoring a clear diagonal axis of a $2h$-ly reversible $(1+k-e)$-partitioned r-gon.

But when e is odd, a diameter of the r-gon must be one of the e scores, if all is symmetrical about the centre, which requires $r=4m$. Then when $n=4m+2$, every odd e scores that can be drawn gives an unsymmetrical configuration, so that there are thus

$$4_{n-2}.2_{e-1}.\left\{\frac{1}{2}.\left(\frac{n-2}{2}\right)^{e|-1}.\lfloor\overline{e+1}^{\,-1}\right\},\ \text{out of } R_0^{di}(r,\ k)'', \quad . \quad . \quad . \quad . \quad . \quad (b.)$$

obtained by so scoring a clear diagonal axis of a $2h$-ly reversible. And when $n=4m$, we have merely to subtract from this entire number of ways as before, the number of symmetrical ways, which, after drawing a diameter, is that of drawing $\frac{1}{2}(e-1)$ scores on one side of the centre, and then to divide by two, as before, thus:—

$$\tfrac{1}{2}.2_{e-1}.4_n\left(\left(\frac{n-2}{2}\right)^{e|-1}\lfloor\overline{e+1}^{\,-1}-\left(\frac{n-4}{4}\right)^{\frac{e-1}{2}|-1}.\lfloor\overline{\frac{e}{2}+1}^{\,-1}\right). \quad . \quad . \quad . \quad . \quad . \quad (c.)$$

Then uniting the expressions $(a.)$, $(b.)$, $(c.)$, we find

$$\tfrac{1}{2}\left\{\left(\frac{n-2}{2}\right)^{e|-1}.\lfloor\overline{e+1}^{\,-1}-2_e\left(4_{n-2}.\left(\frac{n-2}{4}\right)^{\frac{e}{2}|-1}.\lfloor\overline{\frac{e}{2}+1}^{\,-1}+4_n.\left(\frac{n-4}{4}\right)^{\frac{e}{2}|-1}.\lfloor\overline{\frac{e}{2}+1}^{\,-1}\right)\right.$$
$$\left.-2_{e-1}\left(4_n.\left(\frac{n-4}{4}\right)^{\frac{e-1}{2}|-1}.\lfloor\overline{\frac{e+1}{2}}^{\,-1}\right)\right\},$$

for the exact number out of $R_0^{di}(r,\ k)''$ that can be made from any one $2h$-ly reversible $(1+k-e)$-partitioned r-gon, by drawing e scores upon a clear diagonal axis.

When the axes of the r-gon are all clear and diagonal, we obtain twice this number, for we can score an axis carrying either of two configurations; but we obtain it once only, when only half the axes are clear and diagonal. That is, if we multiply this number, under $\Sigma_e\Sigma_n$, by

$$\Sigma_h\Sigma_e\Sigma_n\{2\mathrm{R}_c^{2hdi}(r,\,k-e)_n+\mathrm{R}_{\frac{1}{2}c}^{2h\cdot di}(r,\,k-e)_n+\mathrm{R}_c^{2h\cdot agdi}(r,\,k-e)_n+\mathrm{R}_{dc}^{2h\cdot agdi}(r,\,k-e)_n\},$$

we have the full number $\mathrm{R}_0^{di}(r,\,k)''$, out of $\mathrm{R}_0^{di}(r,\,k)$, that are made by scoring one of an even number of axes.

LXI. Adding the expression in LIX. to this product, we obtain

$$\mathrm{R}_0^{di}(r,\,k)=\Sigma_h\Sigma_e\Sigma_n\Big[\mathrm{R}_c^{(2h+1)di}(r,\,k-e)_n\cdot\left(\frac{n-2}{2}\right)^{e|-1}\cdot\overline{|e+1}^{\,-1}-\mathrm{R}_0^{(2h+3)di}(r,\,k)$$

$$+\tfrac{1}{2}\{2\mathrm{R}_c^{2hdi}(r,\,k-e)_n+\mathrm{R}_{\frac{1}{2}c}^{2hdi}(r,\,k-e)_n+\mathrm{R}_c^{2h\cdot agdi}(r,\,k-e)_n+\mathrm{R}_{dc}^{2h\cdot agdi}(r,\,k-e)_n\}$$

$$\times\left\{\left(\frac{n-2}{2}\right)^{e|-1}\cdot\overline{|e+1}^{\,-1}-2_e\left(4_{n-2}\cdot\left(\frac{n-2}{4}\right)^{\frac{e}{2}|-1}\cdot\overline{\left|\frac{e}{2}+1\right.}^{\,-1}+4_n\left(\frac{n-4}{4}\right)^{\frac{e}{2}|-1}\cdot\overline{\left|\frac{e}{2}+1\right.}^{\,-1}\right)\right.$$

$$\left.-2_{e-1}\left(4_n\cdot\left(\frac{n-4}{4}\right)^{\frac{e-1}{2}|-1}\cdot\overline{\left|\frac{e+1}{2}\right.}^{\,-1}\right)\right\}\Big],$$

comprising all the values of e, h, and n, in our register of $(1+k-e)$-partitioned r-gons here specified ($h\geqq 0$, $e>0$, $n>2$).

Thus we have determined the six classes of doubly and singly reversibles of art. LII., which are not reducible to a polygonal nucleus, and can register them in order as well as the twelve classes of reversibles of art. XX., which are so reducible.

The only numbers in the right member of the above equation contained in our list are

$$\mathrm{R}_c^{8agdi}(8,\,0)_8=1,\ \mathrm{R}_{\frac{1}{2}c}^{2di}(8,\,2)_6=1,\ \mathrm{R}_c^{di}(8,\,2)_6=1,\ \mathrm{R}_c^{di}(8,\,4)_4=2,\ \mathrm{R}_{\frac{1}{2}c}^{4di}(8,\,4)_4=1,$$

(XLIV. and XLVI.). Wherefore

$$\mathrm{R}_0^{di}(8,\,1)=\tfrac{1}{2}\mathrm{R}_c^{8agdi}(8,\,0)_8\left(\frac{8-2}{2}\cdot 1-\left(\frac{8-4}{4}\right)^0\right)\qquad =1,$$

$$\mathrm{R}_0^{di}(8,\,2)=\tfrac{1}{2}\mathrm{R}_c^{8agdi}(8,\,0)_8\left\{\frac{3.2}{1.2}-\frac{8-4}{4}\right\}\qquad =1,$$

$$\mathrm{R}_0^{di}(8,\,3)=\mathrm{R}_c^{di}(8,\,2)_6.\frac{6-2}{2}+\mathrm{R}_c^{di}(8,\,2)_4.\frac{4-2}{2}+\tfrac{1}{2}\left\{\mathrm{R}_{\frac{1}{2}c}^{2di}(8,\,2)_6.\frac{6-2}{2}\right\}\qquad =4,$$

$$\mathrm{R}_0^{di}(8,\,4)=\mathrm{R}_c^{di}(8,\,2)_6.\left(\frac{6-2}{2}\right)^{2|-1}\cdot\frac{1}{1.2}+\tfrac{1}{2}\left\{\mathrm{R}_{\frac{1}{2}c}^{2di}(8,\,2)_6\left(\left(\frac{6-2}{2}\right)^{2|-1}\cdot\frac{1}{1.2}-\frac{6-2}{4}\right)\right\}=1.1+0=1,$$

$$\mathrm{R}_0^{di}(8,\,5)=\mathrm{R}_c^{di}(8,\,4)_4.1+\tfrac{1}{2}.\mathrm{R}_{\frac{1}{2}c}^{4di}(8,\,4)_4(1-0^0)\qquad =2.$$

This completes our register of reversible partitions of the octagon. Collecting them from XXVIII., XXXVIII., XLIV., XLVI., LIII., LIV., LVIII., we add to those above written,

$$\begin{aligned}\mathrm{R}_c^{8agdi}(8,\,0)_8=1,\ &\mathrm{R}_c^{ag}(8,\,2)_6=1,\ \mathrm{R}_c^{ag}(8,\,4)_4=1,\ \mathrm{R}_{\frac{1}{2}c}^{2ag}(8,\,2)_4=1,\\ &\mathrm{R}_{\frac{1}{2}c}^{2di}(8,\,2)_6=1,\ \mathrm{R}_{\frac{1}{2}c}^{4di}(8,\,4)_4=1,\\ &\mathrm{R}_c^{di}(8,\,2)_6=1,\ \mathrm{R}_c^{di}(8,\,4)_4=2,\ \mathrm{R}_c^{di}(8,\,2)_4=1,\\ &\mathrm{R}^{2di}(8,\,5)_2=1,\ \mathrm{R}^{2di}(8,\,3)_2=1,\ \mathrm{R}^{2di}(8,\,1)_2=1,\\ &\mathrm{R}^{di}(8,\,5)_2=2,\ \mathrm{R}^{di}(8,\,3)_2=2,\\ &\mathrm{R}_0^{ag}(8,\,1)=1,\ \mathrm{R}_0^{ag}(8,\,3)=1.\end{aligned}$$

That is,

$R(8, 1)=2$, $R(8, 2)=4$, $R(8, 3)=7$, $R(8, 4)=4$, $R(8, 5)=4$, $R^2(8, 1)=1$, $R^2(8, 2)=2$ $R^2(8, 3)=1$, $R^2(8, 5)=1$, $R^4(8, 4)=1$, $R^8(8, 0)=1$.

The singly reversibles $R(r, k)$ form seven classes,

$$R_c^{ag}(r, k)_n,\ R_c^{di}(r, k)_n,\ R_c^{mo}(r, k)_n,\ R^{di}(r, k)_2,\ R_0^{ag}(r, k),\ R_0^{di}(r, k),\ R_0^{mo}(r, k).$$

LXII. We have now to investigate the number of irreversible $(1+k)$-partitions of the r-gon. And when we have determined $I^{m+2}(r, k)_n$, $m \overline{\gtrless} 0$, the equations of art. XXIII. will enable us to complete the tedious solution of our problem. It is necessary to demonstrate the theorems following.

THEOREM V. *Every* h*-ly irreversible partition of an* r*-gon, if* $h>2$, *and every doubly irreversible partition in which a diameter of the* r*-gon is not drawn, is regularly built on a polygonal nucleus.*

For in each of the h sequences of configuration (art. III.) in the circuit of the r-gon, we see f marginal faces. These being all erased, we see an r'-gon $(r'<r)$ with fh fewer diagonals, and still h sequences, for those of the r-gon have been treated alike. If this r'-gon is reversibly partitioned, it is not singly reversible, because it has repeated sequences (V.); therefore the theorem is proved by XII.; for there is no drawn diameter by hypothesis. If the r'-gon be still irreversible, it has h' sequences $(h' \nless h)$, from which marginal faces can be removed as before; and thus we shall finally reduce the figure, either to a reversibly-partitioned r''-gon, or to one having no marginal faces, *i. e.* to an n-gonal nucleus. And by reversing the process of reduction, the r-gon can be regularly constructed on that nucleus. Q. E. D.

LXIII. THEOREM W. *If the nucleus line of a doubly irreversible* r*-gon built on it be erased, the figure becomes* 2h*-ly reversible or else* 2h*-ly irreversible, and has no drawn diameter.*

For it has still two irreversible sequences, occupying each half the circuit of the r-gon, since the two sequences terminated by the nucleus-line have been treated alike by the erasure; wherefore it can neither be oddly reversible nor oddly irreversible (*Obs.* 1, 3, IV.; and 5, VII.). And evidently the r-gon cannot have a drawn diameter meeting its nucleus line.

Cor. The r-gon, after the erasure, is one of those reducible to a polygonal nucleus.

THEOREM X. *If in any* 2m*-ly reversible, or* 2m*-ly irreversible,* k*-partition of an* r*-gon, a diameter be drawn which is not an axis of reversion, and which meets no diagonal, the figure becomes a doubly irreversible* (1+k)*-partition of the* r*-gon, built on that drawn line.*

For such a drawn line disturbs the symmetry about every axis of a reversible, because it is (VI.) perpendicular to none of them; therefore the result is not reversible. And every $2m$-ly reversible or $2m$-ly irreversible has an irreversible sequence occupying half its circuit (III, V.), beginning at any angle not on an axis of reversion; therefore the result is not singly irreversible. And as there is no other sequence terminated by or exhibiting a drawn diameter, since two cannot be drawn, the result has no sequence more than twice repeated, and is doubly irreversible. And evidently the figure is built on the drawn line, by applying to it the same polygon on both sides, so as not to give a reversible result.

LXIV. Problem u. *To find* $\mathrm{I}^{\mathrm{h}}(\mathrm{r}, \mathrm{k})_{\mathrm{n}}$, ($\mathrm{h}>1$, $\mathrm{n}>2$), *the number of* h-*ly irreversible* $(1+\mathrm{k})$-*partitions of the* r-*gon, which are built on the* n-*gonal nucleus.*

The construction differs from that of reversibles, in that the $(1+e_m)$-partitioned $(2+a_m)$-gon which is laid on the mth side of the n-gon in the first interval of $\frac{n}{h}$ sides, is laid also on the mth side of every interval, so that the series of loads imposed is h times repeated, and not reverted. The result will be an r-gon in whose circuit a sequence occupying $\frac{n}{h}$ sides of the nucleus is h times repeated.

The equations to be satisfied are

$$\left.\begin{aligned} r&=n+h\left(a_1+a_2+\ldots+a_{\frac{n}{h}}\right) \\ k&=n+h\left(e_1+e_2+\ldots+e_{\frac{n}{h}}\right) \end{aligned}\right\}, \qquad \text{(E.)}$$
$$a_m \overline{\gtrless} 0, \quad e_m \not> a_m - 1;$$

exactly as in our previous constructions.

We take any one solution, and lay on the polygons, and then make every possible change in the arrangement of the diagonals of the imposed polygons; this gives us

$$\mathrm{D}(2+a_1, e_1)\mathrm{D}(2+a_2, e_2)\ldots\mathrm{D}\left(2+a_{\frac{n}{h}}, e_{\frac{n}{h}}\right)=\Delta_{\frac{n}{h}}$$

different sequences read in the r-gon upon the first $\frac{n}{h}$ sides of its nucleus, which are all h times repeated in the same order in the circuit of the r-gon. We then take every other solution of (E.) that can arise from changing either the values or order of the numbers $a_1\, a_2\ldots$, $e_1\, e_2\ldots$; and forming a similar product from each solution, products which will some of them be equal to each other, we shall have enumerated in the sum $\Sigma\Delta_{\frac{n}{h}}$, every possible sequence that can be read on $\frac{n}{h}$ successive sides of the n-gonal nucleus of any $(1+k)$-partitioned r-gon, which has that sequence h times repeated in its circuit. And no two of these sequences can be alike, because they are either made from different solutions of (E.), or have a different arrangement of diagonals in the $\frac{n}{h}$ imposed polygons.

Now every hi-ly irreversible $(1+k)$-partition of the r-gon has $\frac{2n}{hi}$, and no more, of these sequences; for it has a different i-ple sequence of $\frac{n}{h}$ vertices commencing at every one of $\frac{n}{hi}$ successive vertices, and no more, reading always in one direction, because the sequence beginning at the $\left(\frac{n}{hi}+1\right)$th vertex is identical with that commencing at the first; and this i-ple sequence of $\frac{n}{h}$ vertices is h times repeated in the circuit, because a simple sequence is hi times repeated. And as the hi-ly irreversible has the same sequences reversed on its other face, giving $\frac{n}{hi}$ different aspects which we have all constructed as read from the first side of the n-gon in one direction, we have had this same hi-ly irreversible $2n{:}hi$ times before us among our results.

Again, every hi-ly reversible $(1+k)$-partitioned r-gon has been $\frac{n}{hi}$ times brought before us in our $\Sigma\Delta_{\frac{n}{h}}$ constructions. For this has a different i-ple sequence reversible or irreversible of $\frac{n}{h}$ vertices of the n-gonal nucleus commencing at any one of $\frac{n}{hi}$ successive vertices, and we never see a sequence repeated in it till we have read over an interval equal to that between two alternate axes, that is an interval of $\frac{n}{hi}$ sides of the n-gon (art. V.). This i-ple sequence is h times repeated in the circuit of the r-gon; so that we have counted this hi-ly reversible $\frac{n}{hi}$ times among those constructed from equation (E.). Wherefore

$$\Sigma\Delta_{\frac{n}{h}}=\Sigma_i\left\{\frac{2n}{hi}\mathrm{I}^{hi}(r,\ k)_n+\frac{n}{hi}\mathrm{R}^{hi}(r,\ k)_n\right\},\ \text{and}$$

$$\mathrm{I}^h(r,\ k)_n=\frac{h}{2n}\left(\Sigma\Delta_{\frac{n}{h}}-\Sigma_i\left(\frac{2n}{h(i+1)}\mathrm{I}^{(i+1)h}(r,\ k)_n+\frac{n}{hi}\mathrm{R}^{hi}(r,\ k)_n\right)\right),\ (i>0).$$

LXV. The highest value of h in (E.) comes from putting the subindex $\frac{n}{h}=1$, or $h=n$. This gives

$$\mathrm{I}^n(r,\ k)_n=\tfrac{1}{2}\left\{\mathrm{D}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right)-\mathrm{R}^n(r,\ k)_n\right\},$$

a given number, since $\mathrm{R}^n(r,\ k)_n$ is found in our register.

Next putting $\frac{n}{h}=2$, or whatever next factor n may have, we can obtain $\mathrm{I}^h(r,\ k)_n$ for every value of $h>1$ that is a divisor of the three numbers r, k and n.

Since $h>1$, $n>2$, r, k and n are none of them prime; the only values of $\mathrm{I}^h(8,\ k)_n$ that give them all divisible by h, and $k \not> 8-3$, are

$$\mathrm{I}^2(8,\ 2)_6,\ \mathrm{I}^2(8,\ 2)_4,\ \mathrm{I}^2(8,\ 4)_4\ \text{and}\ \mathrm{I}^4(8,\ 4)_4.$$

By the last written formula,

$$\mathrm{I}^4(8,\ 4)_4=\tfrac{1}{2}\{\mathrm{D}(3,\ 0)-\mathrm{R}^4(8,\ 4)_4\}=\tfrac{1}{2}(1-1)=0,$$

for $\mathrm{I}^2(8,\ 2)_6$ we have three solutions, $\left(\frac{n}{h}=3\right)$

$$8=6+2(1+0+0)=\ \&\text{c}.$$

$$2=6+2(0-1-1)=\ \&\text{c}.$$

giving $\Sigma\Delta_3=3\mathrm{D}(3,\ 0)=3$, and as $\mathrm{R}^2(8,\ 2)_6=1$,

$$\mathrm{I}^2(8,\ 2)_6=\tfrac{2}{8}\{3-\tfrac{6}{2}\mathrm{R}^2(8,\ 2)_6\}=\tfrac{1}{4}(3-3)=0.$$

For $I^2(8, 2)_4$ we have $\left(\frac{n}{h}=2\right)$,

$$8=4+2.(1+1)=4+2(2+0)=4+2(0+2)$$
$$2= \qquad\qquad 4+2(0-1)=4+2(-1+0)$$
$$\Sigma\Delta_2=2D(4, 0)=2, \text{ and } R^2(8, 2)_4=1,$$
$$\therefore \quad I^2(8, 2)_4=\tfrac{2}{8}\left\{2-\frac{4}{2.1}\cdot 1\right\}=0.$$

For $I^2(8, 4)_4$ we have $\left(\frac{n}{h}=2\right)$,

$$8=4+2(1+1)=4+2(0+2)=4+2(2+0)$$
$$4=4+2(0+0)=4+2(-1+1)=4+2(1-1)$$
$$\Sigma\Delta_2=D(3, 0)D(3, 0)+2D(4, 1)=1+4=5,$$

and $R^4(8, 4)_4=1$, whence

$$I^2(8, 4)_4=\tfrac{2}{8}\left\{5-\frac{4}{2.2}\cdot 1\right\}=1,$$

which is the only form of $I^h(8, k)_n$, for $h>1$, $n>2$.

LXVI. Problem w. *To find* $I^2(r, k)_2$, *the doubly irreversible* $(1+k)$-*partitions of the* r-*gon, which have a nucleus line.*

It is proved in Theorems W and X, that these are all obtained by drawing a diameter δ which is not an axis of reversion in one of the r-gons $I^{2m}(r, k-1)_n$ and $R^{2m}(r, k-1)_n$; and that any such diameter drawn in any one of these gives us one of $I^2(r, k)_2$. And it is evident that δ is a diameter of the nucleus about which all is symmetrical (*Cor.* LXIII.).

When no diameter of the nucleus is a diagonal axis of reversion of the r-gon, every diameter that can be drawn is such a line δ. Now in $I^{2h}(r, k-1)_n$ you may draw δ from any one θ of $\frac{n}{2h}$ vertices of the nucleus, and from no more, because the configuration about θ is repeated $2h$ times, about a vertex in each of the $2h$ repeated sequences. In a reversible partition no diameter of the nucleus is a diagonal axis of reversion, unless it be a clear axis. Therefore, in

$$R_0^{2h\cdot agdi}(r, k-1)_n+R_{ac}^{2hagdi}(r, k-1)_n+R^{2hag}(r, k-1)_n,$$

whether the axes be clear or not in the last one, you can draw δ from any one of $\frac{n}{4h}$ vertices of the r-gon; for none of the partitions in these classes have a clear diagonal axis, and each vertex ε in one of the $4h$ intervals between two adjacent axes begins a different sequence, which does not occur reverted till you come to ε', equidistant with ε from the axis between them. Wherefore

$$I^2(r, k)_2=\Sigma_h\Sigma_n\left\{\frac{n}{2h}I^{2h}(r, k-1)_n+\frac{n}{4h}\left(R_0^{2hagdi}+R_{ac}^{2hagdi}+R^{2hag}\right)(r, k-1)_n+\right.$$

After the $+$ must come, first,

$$\frac{n-2h}{4h}(\mathrm{R}^{2hagdi}_{dc}+\mathrm{R}^{2hagdi}_{c}+\mathrm{R}^{2hdi}_{\frac{1}{2}c})(r,\ k-1)_n,$$

because δ cannot be drawn from any of the $2h$ terminations of clear diagonal axes, but may be drawn to begin a certain sequence $4h$ times from other vertices, once in the interval between the terminations of every adjacent pair of axes; and secondly,

$$+\frac{n-4h}{4h}\mathrm{R}^{2hdi}_{c}(r,\ k-1)_n$$

must be added, because, omitting the $4h$ terminations of clear diagonal axes, you can draw δ to begin a certain sequence $4h$ times from the remaining vertices of the nucleus. Therefore, finally,

$$\begin{aligned}\mathrm{I}^2(r,\ k)_2=\Sigma_h\Sigma_n\Big\{&\frac{n}{2h}\mathrm{I}^{2h}(r,\ k-1)_n+\frac{n}{4h}\Big(\mathrm{R}^{2hagdi}_0(r,\ k-1)_n+\mathrm{R}^{2hagdi}_{ac}(r,\ k-1)_n+\mathrm{R}^{2hag}(r,\ k-1)_n\Big)\\ &+\frac{n-2h}{4h}\Big(\mathrm{R}^{2hagdi}_{dc}(r,\ k-1)_n+\mathrm{R}^{2hagdi}_{c}(r,\ k-1)_n+\mathrm{R}^{2hdi}_{\frac{1}{2}c}(r,\ k-1)_n\Big)+\frac{n-4h}{4h}\mathrm{R}^{2hdi}_{c}(r,\ k-1)_n\Big\};\end{aligned}$$

where every value of $h>0$ and $n>2$ is included, that is found in our register, filled up by the formulæ already given.

LXVII. Hence, since we have found

$$\begin{aligned}&\mathrm{R}^{8agdi}(8,\ 0)_8=\mathrm{R}^{2ag}_{\frac{1}{2}c}(8,\ 2)_4=\mathrm{R}^{2di}_{\frac{1}{2}c}(8,\ 2)_6=\mathrm{R}^{4di}_{\frac{1}{2}c}(8,\ 4)_4=\mathrm{I}^2(8,\ 4)_4=1,\\ &\mathrm{I}^2(8,\ 1)_2\quad=0\,.\,\mathrm{R}^{8agdi}(8,\ 0)_8=0,\\ &\mathrm{I}^2(8,\ 3)_2\quad=\tfrac{4}{4}\mathrm{R}^{2ag}(8,\ 2)_4+\frac{6-2}{4}\mathrm{R}^{2di}_{\frac{1}{2}c}(8,\ 2)_6=2,\\ &\mathrm{I}^2(8,\ 5)_2\quad=\frac{4-4}{8}\mathrm{R}^{4di}_{\frac{1}{2}c}(8,\ 4)_4+\tfrac{4}{2}\,.\,\mathrm{I}^2(8,\ 4)_4=2.\end{aligned}$$

LXVIII. Problem x. *To find* I(r, k), *the number of singly irreversible* (1+k)-*partitions of the* r-*gon.*

By the formula of XXIII. we have

$$\mathrm{I}(r,\ k)=\frac{1}{2r}\Big\{\mathrm{D}(r,\ k)-\Sigma_m\Big(\frac{r}{m}\,.\,\mathrm{R}^m(r,\ k)+\frac{2r}{m+1}\,.\,\mathrm{I}^{m+1}(r,\ k)\Big)\Big\},$$

which completes the solution of our problem of the (1+k)-partitions of the polygon, or of the polyace; for these are analytically and numerically the same. Hence follows,

$$\begin{aligned}&\mathrm{I}(8,\ 0)=\tfrac{1}{16}\{\mathrm{D}(8,\ 0)-1\,.\,\mathrm{R}^8(8,\ 0)\} && && =\ 0,\\ &\mathrm{I}(8,\ 1)=\tfrac{1}{16}\{\mathrm{D}(8,\ 1)-\tfrac{8}{2}\,.\,\mathrm{R}^2(8,\ 1)-8\mathrm{R}(8,\ 1)\} && =\tfrac{1}{16}\{20-4-16\} && =\ 0,\\ &\mathrm{I}(8,\ 2)=\tfrac{1}{16}\{\mathrm{D}(8,\ 2)-\tfrac{8}{2}\mathrm{R}^2(8,\ 2)-8\,.\,\mathrm{R}(8,\ 2)\} && =\tfrac{1}{16}\{120-4\cdot2-8\cdot4\} && =\ 5,\\ &\mathrm{I}(8,\ 3)=\tfrac{1}{16}\{\mathrm{D}(8,\ 3)-\tfrac{8}{2}\mathrm{R}^2(8,\ 3)-8\mathrm{R}(8,\ 3)-\tfrac{16}{2}\mathrm{I}^2(8,\ 3)\} && =\tfrac{1}{16}\{300-4\cdot1-8\cdot7-8\cdot2\} && =14,\\ &\mathrm{I}(8,\ 4)=\tfrac{1}{16}\{\mathrm{D}(8,\ 4)-\tfrac{8}{4}\mathrm{R}^4(8,\ 4)-8\,.\,\mathrm{R}(8,\ 4)-\tfrac{16}{2}\mathrm{I}^2(8,\ 4)\} && =\tfrac{1}{16}\{330-2\cdot1-8\cdot4-8\cdot1\} && =18,\\ &\mathrm{I}(8,\ 5)=\tfrac{1}{16}\{\mathrm{D}(8,\ 5)-\tfrac{8}{2}\mathrm{R}^2(8,\ 5)-8\,.\,\mathrm{R}(8,\ 5)-\tfrac{16}{2}\mathrm{I}^2(8,\ 5)\} && =\tfrac{1}{16}\{132-4\cdot1-8\cdot4-8\cdot2\} && =\ 5,\end{aligned}$$

which, along with the results in LXI., LXV., LXVII., complete the enumeration of the partitions of the octagon.

LXIX. It may be useful to collect into one view all our formulæ for $R^m(r, k)$ and $I^m(r, k)$.

Problem b. To find $R_c^{2hagdi}(r, k)_n$ $(n>2)$, we have in XXIV., XXVIII.,

$$\left.\begin{aligned} r&=n \qquad\quad +4h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{4h}}\right)\\ k&=n-2h+4h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{4h}}\right)\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (A.)$$

$$(a_m \overline{\overline{>}} 0, \quad e_m \not> a_m-1,$$

which is always to be understood in what follows), for the equations of condition. Here $r-n$ and k are both multiples of $4h$.

$$D(2+a_1, e_1)\times\ldots\times D\left(2+a_{\frac{n-2h}{4h}}, e_{\frac{n-2h}{4h}}\right)=\Delta_{\frac{n-2h}{4h}},$$

$$R_c^{2h.agdi}(r, k)_n=\Sigma\Delta_{\frac{n-2h}{4h}}-\Sigma_i R_c^{2(i+1)h.agdi}(r, k)_n, \ (i>0);$$

$$R_c^{n.agdi}(r, k)=0=R_c^{\frac{n}{2}agdi}(r, k)_n;$$

$$R_c^{\frac{n}{3}agdi}(r, k)_n=D\left(\frac{3r+n}{2n}, \frac{3k-2n}{2n}\right).$$

Problem c. To find $R_{dc}^{2hagdi}(r, k)_n$ (XXIX., XXX.),

$$\left.\begin{aligned} r&=n+2h(2+2\alpha_0)+4h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{4h}}\right)\\ k&=n+2h\varepsilon_0 \qquad\qquad +4h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{4h}}\right)\end{aligned}\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (A'.)$$

$$H=\Sigma_m\{2R^{mag}(4+2\alpha_0, \varepsilon_0)+R^{2magdi}(4+2\alpha_0, \varepsilon_0)\},$$

$$R_{dc}^{2hagdi}(r, k)_n=\Sigma\left(H.\Delta_{\frac{n-2h}{4h}}\right)-\Sigma_i R_{dc}^{2h(i+1)agdi}(r, k)_n, \ (i>0);$$

$$R_{dc}^{n.agdi}(r, k)_n=\Sigma_m\left\{2R^{m.ag}\left(\frac{r+n}{n}, \frac{k-n}{n}\right)+R^{2m.agdi}\left(\frac{r+n}{n}, \frac{k-n}{n}\right)\right\},$$

$$R_{dc}^{\frac{n}{2}agdi}(r, k)_n=0,$$

$$R_{dc}^{\frac{n}{3}agdi}(r, k)_n=\Sigma(H.\Delta_1)-R_{dc}^{n.agdi}(r, k)_n.$$

Problem d. To find $R_{ac}^{2h.agdi}(r, k)_n$ (XXXII., XXXIII.),

$$\left.\begin{aligned} r&=n \qquad\quad +4h\left(a_1+a_2+\ldots+a_{\frac{n-4h}{4h}}\right)+2h\left(2\alpha_{\frac{n}{4h}}+1\right)\\ k&=n-2h+4h\left(e_1+e_2+\ldots+e_{\frac{n-4h}{4h}}\right)+2h.\varepsilon_{\frac{n}{4h}}\end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad (A''.)$$

$$M=\Sigma_m R^{m.mo}\left(3+2\alpha_{\frac{n}{4h}}, \varepsilon_{\frac{n}{4h}}\right),$$

$$R_{ac}^{2h.agdi}(r, k)_n=\Sigma\left(M\Delta_{\frac{n-4h}{4h}}\right)-\Sigma_i R_{ac}^{2h(i+1)agdi}(r, k)_n, \ (i>0),$$

$$R_{ac}^{n.agdi}(r, k)_n=0,$$

$$R_{ac}^{\frac{n}{2}agdi}(r, k)_n=\Sigma_m R^{m.mo}\left(\frac{2r}{n}, \frac{2k-n}{n}\right),$$

$$R_{ac}^{\frac{n}{4}agdi}(r, k)_n=\Sigma M\Delta_1-\Sigma_m R^{m.mo}\left(\frac{2r}{n}, \frac{2k-n}{n}\right).$$

Problem e. To find $R_0^{2hagdi}(r, k)_n$ (XXXIV., XXXV.),

$$\left.\begin{array}{l} r=n+2h(2+2\alpha_0)+4h\left(a_1+a_2+\ldots+a_{\frac{n-4h}{4h}}\right)+2h\left(2\alpha_{\frac{n}{4h}}+1\right) \\ k=n+2h\varepsilon_0 \qquad +4h\left(e_1+e_2+\ldots+e_{\frac{n-4h}{4h}}\right)+2h\varepsilon_{\frac{n}{4h}} \end{array}\right\}, \quad \ldots \quad (\mathrm{A}''')$$

$$H=\Sigma_m\left(2R^{m.ag}(4+2\alpha_0, \varepsilon_0)+R^{2magdi}(4+2\alpha_0, \varepsilon_0)\right),$$

$$M=\Sigma_m R^{m.mo}\left(3+2\alpha_{\frac{n}{4h}}, \varepsilon_{\frac{n}{4h}}\right),$$

$$R_0^{2hagdi}(r, k)_n=\Sigma\left(H.M.\Delta_{\frac{n-4h}{4h}}\right)-\Sigma_i R_0^{2(i+1)h.agdi}(r, k)_n, \ (i>0),$$

$$R_0^{nagdi}(r, k)_n=0,$$

$$R_0^{\frac{1}{2}n.agdi}(r, k)_n=\Sigma(H.M),$$

$$R_0^{\frac{1}{4}n.agdi}(r, k)_n=\Sigma(HM\Delta_1)-R_0^{\frac{1}{2}n.agdi}(r, k)_n.,$$

Problem f. To find $R_c^{h.ag}(r, k)_n$ (XXXVI., XXXVIII.),

$$\left.\begin{array}{l} r=n+2h \qquad \left(a_1+a_2+\ldots+a_{\frac{n-2h}{2h}}\right) \\ k=n-2h+2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right) \end{array}\right\} \quad \ldots \ldots \quad (\mathrm{B.})$$

$$R_c^{hag}(r, k)_n=\tfrac{1}{2}\left\{\Sigma\Delta_{\frac{n-2h}{2h}}(i>0)-\Sigma_i\left(2R_c^{h(i+1)ag}(r, k)_n+R_{\frac{1}{2}c}^{2hi.ag}(r, k)_n+R_{ac}^{2hiagdi}(r, k)_n+R_c^{2hiagdi}(r, k)_n\right)\right\},$$

$$R_c^{nag}(r, k)_n=R_c^{\frac{1}{2}nag}(r, k)_n=0,$$

$$R_c^{\frac{1}{4}nag}(r, k)_n=\tfrac{1}{2}\left\{D\left(\frac{2r}{n}, \frac{2k-n}{n}\right)-\Sigma_m\left\{(2R^{m.ag}+R^{2magdi})\left(\frac{2r}{n}, \frac{2k-n}{n}\right)\right\}\right\}.$$

Problem g. To find $R_{\frac{1}{2}c}^{hag}(r, k)_n$ (XXXVII., XXXVIII.),

$$\left.\begin{array}{l} r=n+h(2+2\alpha_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{2h}}\right) \\ k=n-h-h\varepsilon_0 \qquad +2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right) \end{array}\right\} \quad \ldots \ldots \quad (\mathrm{B}')$$

$$H=\Sigma_m\{2R^{mag}(4+2\alpha_0, \varepsilon_0)+R^{2magdi}(4+2\alpha_0, \varepsilon_0)\}$$

$$R_{\frac{1}{2}c}^{hag}(r, k)_n=\Sigma\left(H.\Delta_{\frac{n-2h}{2h}}\right)-\Sigma_i R_{\frac{1}{2}c}^{(2i+1)h.ag}(r, k)_n, \ (i>0),$$

$$R_{\frac{1}{2}c}^{nag}(r, k)_n=0,$$

$$R_{\frac{1}{2}c}^{\frac{1}{2}n.ag}(r, k)_n=\Sigma_m\left\{(2R^{mag}+R^{2magdi})\left(\frac{2r}{n}, \frac{2k-n}{n}\right)\right\}.$$

Problem h. To find $R_0^{hag}(r, k)_n$ (XXXIX. XL.),

$$\left.\begin{array}{l} r=n+h(2+2\alpha_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{2h}}\right)+h.\left(2+2\alpha_{\frac{n}{2h}}\right), \\ k=n+h\varepsilon_0 \qquad +2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right)+h\varepsilon_{\frac{n}{2h}}; \end{array}\right\} \quad \ldots \quad (\mathrm{B}'')$$

$$H=\Sigma_m\{2R^{mag}(4+2\alpha_0, \varepsilon_0)+R^{2magdi}(4+2\alpha_0, \varepsilon_0)\},$$

$$H'=\Sigma_m\left\{2R^{mag}\left(4+2\alpha_{\frac{n}{2h}}, \varepsilon_{\frac{n}{2h}}\right)+R^{2magdi}\left(4+2\alpha_{\frac{n}{2h}}, \varepsilon_{\frac{n}{2h}}\right)\right\};$$

$$R_0^{hag}(r, k)_n=\tfrac{1}{2}\left\{\Sigma\left(H.H'.\Delta_{\frac{n-2h}{2h}}\right)-\Sigma_i\left(2R_0^{h(i+1)ag}(r, k)_n+R_{\frac{1}{2}c}^{2hiag}(r, k)_n+R_0^{2hiagdi}(r, k)_n+R_{dc}^{2hiagdi}(r, k)_n\right)\right\},$$

where $i>0$:

$$\mathrm{R}_0^{nag}(r,k)_n=0,$$

$$\mathrm{R}_0^{\frac{1}{2}nag}(r,k)_n=\tfrac{1}{2}\{\Sigma(\mathrm{H}.\mathrm{H}')-\mathrm{R}_{dc}^{nagdi}(r,k)_n\}.$$

Problem i. To find $\mathrm{R}_0^{hdi}(r,k)_n$ (XLI., XLIII.),

$$\left.\begin{aligned} r&=n+h(1+2\alpha_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{2h}}\right)+h\left(1+2\alpha_{\frac{n}{2h}}\right),\\ k&=n+h\varepsilon_0 \qquad +2h\left(e_1+e_2+\ldots+e_{\frac{n-2n}{2n}}\right)+h\varepsilon_{\frac{n}{2n}};\end{aligned}\right\}\quad\ldots\ldots\quad(\mathrm{C.})$$

here r and n are both even.

$$\mathrm{M}=\Sigma_m\mathrm{R}^{m.mo}(3+2\alpha_0,\ \varepsilon_0),$$

$$\mathrm{M}'=\Sigma_m\mathrm{R}^{m.mo}\left(3+2\alpha_{\frac{n}{2h}},\ \varepsilon_{\frac{n}{2h}}\right),$$

$$\mathrm{R}_0^{hdi}(r,k)_n=\tfrac{1}{2}\left\{\Sigma\left(\mathrm{MM}'\Delta_{\frac{n-2h}{2h}}\right)-\Sigma_i(2\mathrm{R}_0^{h(i+1)di}+\mathrm{R}_{\frac{1}{2}c}^{2hi}+\mathrm{R}_0^{2hiagdi}+\mathrm{R}_{ac}^{2hiagdi})(r,k)_n\right\},\ (i>0).$$

$$\mathrm{R}_0^{ndi}(r,k)_n=0$$

$$\mathrm{R}_0^{\frac{1}{2}ndi}(r,k)_n=\tfrac{1}{2}\Sigma\left\{\Sigma_m\left(\mathrm{R}^{m.mo}(3+2\alpha_0,\ \varepsilon_0)\right).\Sigma_m\left(\mathrm{R}^{m.mo}(3+2\alpha_1,\ \varepsilon_1)\right)\right\}-\tfrac{1}{2}\Sigma_m\mathrm{R}^{m.mo}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right),$$

where $\dfrac{r-2n}{n}=\alpha_0+\alpha_1$, and $\dfrac{2k-2n}{n}=\varepsilon_0+\varepsilon_1$.

Problem k. To find $\mathrm{R}_{\frac{1}{2}c}^{h.di}(r,k)_n$ (XLII., XLIII.);

here h is even, consequently n is even.

$$\begin{aligned} r&=n+h(1+2\alpha_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-h}{2h}}\right),\\ k&=n+h\varepsilon_0 \qquad 2h\left(e_1+e_2+\ldots+e_{\frac{n-h}{2h}}\right);\end{aligned}$$

and r is divisible by $2h$.

$$\mathrm{M}=\Sigma_m.\mathrm{R}^{mmo}(3+2\alpha_0,\ \varepsilon_0)$$

$$\mathrm{R}_{\frac{1}{2}c}^{hdi}(r,k)_n=\Sigma\left(\mathrm{M}\Delta_{\frac{n-h}{2h}}\right)-\Sigma_i\mathrm{R}_{\frac{1}{2}c}^{(2i+3)h.di}(r,k)_n,\ i\gtreqless 0.$$

$$\mathrm{R}_{\frac{1}{2}c}^{ndi}(r,k)_n=\Sigma_m\mathrm{R}^{m.mo}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right).$$

Problem l. To find $\mathrm{R}_c^{hdi}(r,k)_n$ (XLV., XLVI.),

$$\left.\begin{aligned} r&=n+2h\left(a_1+a_2+\ldots+a_{\frac{n}{2h}}\right),\\ k&=n+2h\left(e_1+e_2+\ldots+e_{\frac{n}{2h}}\right);\end{aligned}\right\}\quad\ldots\ldots\quad(\mathrm{C}'.)$$

here r, k and n are all even and divisible by $2h$.

$$\mathrm{R}_c^{hdi}(r,k)_n=\tfrac{1}{2}\Sigma\Delta_{\frac{n}{2h}}-\tfrac{1}{2}\Sigma_i\{(2\mathrm{R}_c^{(i+1)hdi}+\mathrm{R}_{\frac{1}{2}c}^{2hidi}+\mathrm{R}_c^{2ih.agdi}+\mathrm{R}_{dc}^{2ihagdi})(r,k)_n\};\ (\mathrm{i}>0),$$

$$\mathrm{R}_c^{ndi}(r,k)_n=0$$

$$\mathrm{R}_c^{\frac{1}{2}ndi}(r,k)_n=\tfrac{1}{2}\left\{\mathrm{D}\left(\frac{r+n}{n},\ \frac{k-n}{n}\right)-\mathrm{R}_{\frac{1}{2}c}^{ndi}(r,k)_n-\mathrm{R}_{dc}^{nagdi}(r,k)_n\right\}.$$

Problem m. To find $R_c^{hmo}(r, k)_n$ (XLVII., XLVIII.); here n, h and r are odd numbers.

$$\left.\begin{aligned} r&=n \qquad +2h\left(a_1+a_2+\ldots+a_{\frac{n-h}{2h}}\right),\\ k&=n-h+2h\left(e_1+e_2+\ldots+e_{\frac{n-h}{2h}}\right),\end{aligned}\right\} \quad \text{(D.)}$$

$$R_c^{h.mo}(r, k)_n=\Sigma\Delta_{\frac{n-h}{2h}}-\Sigma_i R_c^{(2i+3)mo}(r, k)_n, \ (i \gtreqless 0).$$

$$R_c^{nmo}(r, k)_n=0$$

$$R_c^{\frac{1}{3}n.mo}(r, k)_n=D\left(\frac{3r+n}{2n}, \frac{3k-2n}{2n}\right).$$

Problem n. To find $R_0^{h.mo}(r, k)_n$ (XLIX., L.), for n odd and n even,

$$\left.\begin{aligned} r&=n+h(2+2\alpha_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-h}{2h}}\right),\\ k&=n+h\varepsilon_0 \qquad +2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right),\end{aligned}\right\} \quad \text{(D}'.\text{)}$$

$$H=\Sigma_m\{2R^{mag}(4+2\alpha_0, \varepsilon_0)+R^{2magdi}(4+2\alpha_0, \varepsilon_0)\}$$

$$R_0^{hmo}(r, k)_n=\Sigma H.\Delta_{\frac{n-h}{2h}}-\Sigma_i.R_0^{h(2i+3)mo}(r, k)_n, \ (i \gtreqless 0),$$

when n is odd. And, when n is even,

$$r=n+h(1+2\alpha_0)+2h\left(a_1+a_2+\ldots+a_{\frac{n-2h}{2h}}\right)+h\alpha_{\frac{n}{2h}}$$

$$k=n+h\varepsilon_0 \qquad +2h\left(e_1+e_2+\ldots+e_{\frac{n-2h}{2h}}\right)+h\varepsilon_{\frac{n}{2h}};$$

$$H''=\Sigma_m\left(2R^{mag}+R^{2magdi}\left(2+\alpha_{\frac{n}{2h}}, e_{\frac{n}{2h}}\right)\right)$$

$$M=\Sigma_m R^{mmo}(3+\alpha_0, \varepsilon_0);$$

$$R_0^{h.mo}(r, k)_n=\Sigma\left(H''M.\Delta_{\frac{n-2h}{2h}}\right)-\Sigma_i R_0^{(2i+3)h.mo}(r, k)_n, \ (i \gtreqless 0),$$

$$R_0^{n.mo}(r, k)_n=\Sigma_m\left\{(2R^{mag}+R^{2magdi})\left(\frac{r+n}{n}, \frac{k-n}{n}\right)\right\} \ (n \text{ odd});$$

$$R_0^{nmo}(r, k)_n=0, \text{ for } n \text{ even};$$

$$R_0^{\frac{1}{2}nmo}(r, k)_n=\Sigma H''M, \text{ if } \frac{n}{2} \text{ be odd, and } =0, \text{ if } \frac{n}{2} \text{ be even.}$$

Problem o. To find $R^{2agdi}(r, k)_2$ (LII.),

$$R^{2agdi}(r, k)_2=\Sigma_h\Sigma_n\{R_c^{(4h+2)agdi}(r, k-1)_n+R_{dc}^{(4h+2)agdi}(r, k-1)_n\}.$$

Here r and $k-1$ are both even, by Problems b, c, as well as n.

Problem p. To find $R^{2di}(r, k)_2$ (LIII.),

$$R^{2di}(r, k)_2=\Sigma_h\Sigma_n\{(2R_c^{2hdi}+R_{\frac{1}{2}c}^{2hdi}+R_c^{4hagdi}+R_{dc}^{4hagdi})(r, k-1)_n\}.$$

Here r and $k-1$ are both even (Problems b, c, k, l).

Problem q. To find $R^{di}(r, k)_2$ (LIV.),

$$R^{di}(r, k)_2=\Sigma_h\Sigma_n R_c^{(2h+1)di}(r, k-1)_n.$$

Here r and $k-1$ are both even, by Problem l.

Problem r. To find $R_0^{mo}(r, k)$ (LV.),

$$R_0^{mo}(r, k)=\Sigma_h\Sigma_e\Sigma_n\left(\frac{n-3}{2}\right)^{e|-1}.\overline{e+1}^{\,-1}.R_c^{(2h+1)mo}(r, k-e)_n-\Sigma_h R_0^{(2h+3)mo}(r, k).$$

Problem s. To find $R_0^{ag}(r, k)$ (LVI., LVII.),

$$\begin{aligned}R_0^{ag}(r, k)=\Sigma_h\Sigma_e\Sigma_n\Bigg[&R_c^{(2h+1)ag}(r, k-e)_n.\left(\frac{n-4}{2}\right)^{e|-1}.\overline{e+1}^{\,-1}-R_0^{(2h+3)ag}(r, k)\\&+\tfrac{1}{2}\{2R^{2hag}(r, k-e)_n+R_{\frac{1}{2}c}^{2hag}(r, k-e)_n+R_c^{2h.agdi}(r, k-e)_n+R_{ac}^{2h.agdi}(r, k-e)_n\}\\&\times\left\{\left(\frac{n-4}{2}\right)^{e|-1}.\overline{e+1}^{\,-1}-2_e\left(4_n\left(\frac{n-4}{4}\right)^{\frac{e}{2}|-1}.\overline{\frac{e}{2}+1}^{\,-1}+4_{n-2}.\left(\frac{n-6}{4}\right)^{\frac{e}{2}|-1}.\overline{\frac{e}{2}+1}^{\,-1}\right)\right.\\&\left.-2_{e-1}\left(4_{n-2}\left(\frac{n-6}{4}\right)^{\frac{e-1}{2}|-1}.\overline{\frac{e+1}{2}}^{\,-1}\right)\right\}\Bigg].\end{aligned}$$

Problem t. To find $R_0^{di}(r, k)$ (LXI.),

$$\begin{aligned}R_0^{di}(r, k)=\Sigma_h\Sigma_e\Sigma_n\Bigg[&R_c^{(2h+1)di}(r, k-e)_n.\left(\frac{n-2}{2}\right)^{e|-1}.\overline{e+1}^{\,-1}-R_0^{(2h+3)di}(r, k)\\&+\tfrac{1}{2}\{2R_c^{2hdi}(r, k-e)_n+R_{\frac{1}{2}c}^{2hdi}(r, k-e)_n+R_c^{2hagdi}(r, k-e)_n+R_{dc}^{2hagdi}(r, k-e)_n\}\\&\times\left\{\left(\frac{n-2}{2}\right)^{e|-1}.\overline{e+1}^{\,-1}-2_e\left(4_{n-2}\left(\frac{n-2}{4}\right)^{\frac{e}{2}|-1}.\overline{\frac{e}{2}+1}^{\,-1}+4_n\left(\frac{n-4}{4}\right)^{\frac{e}{2}|-1}.\overline{\frac{e}{2}+1}^{\,-1}\right)\right.\\&\left.-2_{e-1}\left(4_n\left(\frac{n-4}{4}\right)^{\frac{e-1}{2}|-1}.\overline{\frac{e+1}{2}}^{\,-1}\right)\right\}\Bigg].\end{aligned}$$

Problem u. To find $I^h(r, k)_n$, $h>1$, $n>2$ (LXIV.),

$$r=n+h\left(a_1+a_2+\ldots+a_{\frac{n}{h}}\right),$$
$$k=n+h\left(e_1+e_2+\ldots+e_{\frac{n}{h}}\right).$$

Here h divides r, n, and k.

$$I^h(r, k)_n=\frac{h}{2n}\left\{\Sigma\Delta_{\frac{n}{h}}-\Sigma_i\left(\frac{.2n}{h(i+1)}I^{(i+1)h}(r, k)_n+\frac{n}{hi}R^{hi}(r, k)_n\right)\right\}\ (i>0),$$

$$I^n(r, k)_n=\tfrac{1}{2}\left\{D\left(\frac{r+n}{n}, \frac{k-n}{n}\right)-R^n(r, k)_n\right\}.$$

Problem w. To find $I^2(r, k)_2$ (LXVI.),

$$\begin{aligned}I^2(r, k)_2=\Sigma_h\Sigma_n\Bigg\{&\frac{n}{2h}I^{2h}(r, k-1)_n+\frac{n}{4h}.\left(R_0^{2hagdi}(r, k-1)_n+R_{ac}^{2h.agdi}(r, k-1)_n+R^{2h.ag}(r, k-1)_n\right)\\&+\frac{n-2h}{4h}\Big(R_{dc}^{2h.agdi}(r, k-1)_n+R_c^{2hagdi}(r, k-1)_n+R_{\frac{1}{2}c}^{2hdi}(r, k-1)_n\\&+\frac{n-4h}{4h}R_c^{2hdi}(r, k-1)_n\Big)\Bigg\}.\end{aligned}$$

Here r and $k-1$ are both even.

Problem x. To find I(r, k) (LXVIII.),

$$\mathrm{I}(r, k)=\frac{1}{2r}\left\{\mathrm{D}(r, k)-\Sigma_m\left(\frac{r}{m}\mathrm{R}^m(r, k)+\frac{2r}{m+1}\mathrm{I}^{m+1}(r, k)\right)\right\}.$$

LXX. As a further illustration of these formulæ and their use, and for a comparison of methods, it will be worth our while to deduce the seven partitions of the 9-gon, and the eight partitions of the 10-gon, which are obtained by way of example in the memoir referred to, in the first article.

First, to find $\mathrm{R}^h(9, 6)$ and $\mathrm{I}^h(9, 6)$.

As the 9-gon is divided into triangles, there can be only two nuclei, a triangle or a line. We ask, then, what are $\mathrm{R}^h(9, 6)_3$ and $\mathrm{R}^h(9, 6)_2$?

By Theorem E, the axes are monogonal only, and $\mathrm{R}^{hmo}(9, 6)_3$ is what we are seeking. By Problem m,

$$\mathrm{R}_c^{3mo}(9, 6)_3=0,$$

$$\mathrm{R}_c^{1.mo}(9, 6)_3=\mathrm{D}(5, 2)=5;$$

and by Problem n,

$$\mathrm{R}_0^{3mo}(9, 6)_3=(2\mathrm{R}^{mag}+\mathrm{R}^{2magdi})(4, 1)=0,$$

$\mathrm{R}^{2di}(4, 1)_2=1$ being the only entry in our register under (4, 1).

For $\mathrm{R}_0^{mo}(9, 6)_3$, we have, by equations (D′.), $\frac{n-h}{2h}=1=h$.

$$9=3+2+2\alpha_0+2a_1,$$

$$6=3+\quad\varepsilon_0+2e_1,$$

$$4+2\alpha_0=8-2a_1,\qquad\qquad \Delta_1=\mathrm{D}(2+a_1, e_1),$$

$$\varepsilon_0=3-2e_1,$$

$$\mathrm{R}_0^{mo}(9, 6)_3=\Sigma.\{\Sigma(2\mathrm{R}^{mag}+\mathrm{R}^{2magdi})(8-2a_1, 3-2e_1)\}.\mathrm{D}(2+a_1, e_1).$$

The only values of e_1 are 0 or 1; if $e_1=1$, $a_1\nless 2$, if $e_1=0$, $a_1\nless 1$. And as we have nothing in our register under $(\mathrm{R}^{mag}$ or $\mathrm{R}^{magdi})\big((4, 1)$ or $(6, 3)\big)$,

$$\mathrm{R}_0^{mo}(9, 6)_3=0,$$

and (Problems o, p, q) $\qquad \mathrm{R}(9, 6)_2\ =0;$

wherefore $\qquad \mathrm{R}(9, 6)\ \ =\mathrm{R}_c^{mo}(9, 6)_3=5.$

Next to find $\mathrm{I}^h(9, 6)_3$; by Problem u,

$$\mathrm{I}^3(9, 6)_3=\tfrac{1}{2}\{\mathrm{D}(4, 1)-\mathrm{R}^3(9, 6)_3\}=\tfrac{1}{2}(2-0)=1,$$

the subtracted term having just been proved to be zero. And as 9 is odd (Problem w),

$$\mathrm{I}^2(9, 6)_2=0, \text{ wherefore (Problem x),}$$

$$\mathrm{I}(9, 6)=\tfrac{1}{18}\{\mathrm{D}(9, 6)-9\mathrm{R}(9, 6)-\tfrac{18}{3}\mathrm{I}^3(9, 6)=\tfrac{1}{18}\{429-9\cdot 5-6\cdot 1\}=21.$$

This agrees with the results at p. 409 of my former memoir, saving an error of transcription in the seventh line from the bottom, in which $+1.ii(8, 2, 0)$ is omitted from the value of I(10, 2) as it stands above (line 14). If the indulgent reader will kindly correct the sixth and seventh lines thus,

$$\mathrm{I}(10, 2)=2.\tfrac{1}{2}\{ii(8, 2, 0).ii(4, 1, 0)\}+1.ii(8, 2, 0)$$
$$=2.\tfrac{1}{2}\{4-2\}+4=6,$$

he will find the sum

$$\mathrm{I}(10, 2)+\mathrm{I}(10, 3)+\mathrm{I}(10, 4)=6+13+2=21,$$
$$\mathrm{I}^3(10, 3)=1,$$

and $$\mathrm{R}(10, 2)+\mathrm{R}(10, 4)=5,$$

the 7-partitions of the nonagon, in that notation, as we have just found them: vide (I) of this memoir.

LXXI. We shall for a final example determine the 8-partitions of the decagon. There is no nucleus but 3 and 2; we want then

$$\mathrm{R}^h(10, 7)_3\, \mathrm{R}^2(10, 7)_2,\ \mathrm{R}^{di}(10, 7)_2,\ \mathrm{R}_0(10, 7),\ \mathrm{I}^h(10, 7)_3\, \mathrm{I}^2(10, 7)_2.$$

By Theorems M, E (XIV., IX), no axis can be agonal, or monogonal; we seek then $\mathrm{R}^{hdi}(10, 7)_3$, $\mathrm{R}^{2di}(10, 7)_2$, $\mathrm{R}^{di}(10, 7)_2$, and $\mathrm{R}_0^{di}(10, 7)$.

$\mathrm{R}_0^{hdi}(10, 7)_3=0$ by Problem i, because n is odd; and $\mathrm{R}_{\frac{1}{2}c}^{hdi}(10, 7)_3=0$ for the same reason, as is also $\mathrm{R}_c^{hdi}(10, 7)_3$. Therefore $\mathrm{R}^{hdi}(10, 7)_3=0$.

By Problem p,

$$\mathrm{R}^{2di}(10, 7)_2=\Sigma_h\Sigma_n\{2\mathrm{R}_c^{2hdi}+\mathrm{R}_{\frac{1}{2}c}^{2hdi}+\mathrm{R}_c^{4hagdi}+\mathrm{R}_{dc}^{4hagdi})(10, 6)_n\}=0;\ \text{for}$$

R_c^{2hdi} $(10, 6)_n=0$, because 10 is not a multiple of $4h$ (Problem l);
$\mathrm{R}_{\frac{1}{2}c}^{2hdi}$ $(10, 6)_n=0$, for the same reason (Problem k);
$\mathrm{R}_c^{4hagdi}(10, 6)_n=0$, because 10 is no multiple of $8h$ (Problem b);
$\mathrm{R}_{dc}^{4hagdi}(10, 6)_n=0$, because $10-n$, $(n>2)$ is no multiple of $8h$ (Problem c).

R^{di} $(10, 7)_2=\Sigma_h\Sigma_n\mathrm{R}_c^{(2h+1)di}(10, 6)_n=\mathrm{R}_c^{di}(10, 6)_n$, because 10 is a multiple of $4h+2$ (Problems l, q);

$$\mathrm{R}_0^{di}\ (10, 7)=\Sigma_h\Sigma_e\Sigma_n\left[\mathrm{R}_c^{(2h+1)di}(10, 7-e)_n\left\{\left(\frac{n-2}{2}\right)^{e|-1}\cdot\overline{e+1}^{\,-1}\right\}^2+\&c.\right](\text{Problem t}),$$

all the remaining part of the expression vanishing; for we shall presently see that $n=4$ only in both these equations; and $\mathrm{R}_0^{(2h+3)di}(10, 7)$ has been proved above to be $=0$, as have $\mathrm{R}_c^{2hdi}(10, 7-e)_n$ and $\mathrm{R}_{\frac{1}{2}c}^{2hdi}(r, 7-e)_n$ also; and $\mathrm{R}_c^{2hagdi}(10, 7-e)_n=0$, because $7-e$ and $10-n$ are not multiples of $4h$, unless $h=1, n=6$; $(n>2)$, which makes equations (A.) $\left(\frac{n-2h}{4h}=1\right)$,

$$10=6+4.1,\ \text{and}\ 7-e=6-2+4.-1,$$

$\therefore e=7$; but $\mathrm{R}^{2agdi}(10, 0)_6$ has no existence, for if there are no diagonals the nucleus is the 10-gon itself. $\mathrm{R}_{dc}^{2hagdi}(10, 7-e)_n=0$; for this can only be $\mathrm{R}_{dc}^{2agdi}(10, 7-e)_6$, for a reason just given; but (Problem c) $7-e-6$ is even, therefore $e=1$, and as $h=1$, $=\frac{n-2a}{4h}$, equations (A′.) become

$$10=6+.2(2+0)+4.0,\ \text{and}\ k=6=6+2.2+4.-1,$$

i. e. in H, $\alpha_0=0$, $\varepsilon_0=2$, or a 4-gon has two diagonals not crossing, which is absurd.

Therefore $R_c^{(2h+1)di}(10, 7-e)_n$ is what we have to determine for every value of h, e, and n.

Now in $R_c^{hdi}(10, 7-e)_n$ (Problem 1), when h is odd, 10, $7-e$ and n are not all divisible by $2h$ if $n>6$; for $n<10$. And in $R_c^{hdi}(10, 7-e)_6$, h is either 1 or 3. For $h=3$ the formula for $R^{\frac{1}{2}ndi}$ gives $R^{3di}(10, 7-e)_6=0$, because $\frac{10+6}{6}$ is not integer; and for $h=1$, $\frac{n}{2h}=3$,

$$10=6+2.(a_1+a_2+a_3)$$
$$7-e=6+2.(e_1+e_2+e_3);$$

the solutions are

$$\frac{10-6}{2}=2=1+1+0, \text{ or } =2+0+0$$
$$1-e=2(0+0-1), \quad \text{or } =2(1-1-1), \text{ or } =2(0-1-1),$$

the second line of which is absurd, if $e<3$. But for $e>2$, the multiplier $\left(\frac{6-2}{2}\right)^{3|-1}=0$. Therefore n is not 6.

Let then $n=4$, and seek $R^{hdi}(10, 7-e)_4$, for h odd. If $\frac{4}{2h}=1$, the formula for $R^{\frac{1}{2}ndi}$ gives $R_c^{2di}(10, 7-e)_4=0$, because $\frac{10+4}{4}$ is not integer. Then $\frac{4}{2h}=2$, $h=1$, and we have for $e=1$,

$$10=4+2.(2+1)=4+2(3+0)$$
$$6=4+2\ (1+0)=4+2(2-1),$$

giving four solutions; and for $e=3$, the multiplier $\left(\frac{4-2}{2}\right)^{e|-1}$ vanishes. These four are then the only solutions, and

$$\Sigma(\Delta_2)=2D(4, 1)D(3, 0)+2D(5, 2)=2(2+5)=14;$$

whence $\quad R_c^{di}(10, 6)_4=\frac{1}{2}\{14-\Sigma_i(2R_c^{(i+1)di}+R_{\frac{1}{2}c}^{2di}+R_c^{2i.agdi}+R_{dc}^{2iagdi})(10, 6)_4\}$

where $(i>0)$.

Now

$R_c^{(i+1)di}(10, 6)_4=0$ (Problem 1), because 10 and 6 and 4 have no common measure $2(i+1)$.

$R_{\frac{1}{2}c}^{2di}\ (10, 6)_4=0$ (Problem k), because 10 is not divisible by 4.

$R_c^{2i.agdi}(10, 6)_4=0$ (Problem b), because 10 is no multiple of $4i$.

$R_{dc}^{2iagdi}\ (10, 6)_4=0$ (Problem c), because $10-4$ is no multiple of $4i$.

Therefore $\quad R_c^{di}(10, 6)_4=\frac{1}{2}14=7=R_0^{di}(10, 7)$,

and $\quad R(10, 7)=R^{di}(10, 7)_2+R_0^{di}(10, 7)=2R_c^{di}(10, 6)_4=14.$

LXXII. We have next to determine $I^h(10, 7)$, $h>1$.

The formulæ for I^n (Problem u) give

$$I^3(10, 7)_3=0, \text{ because } \frac{10+3}{3} \text{ is no integer.}$$

And (Problem w)

$$I^2(10, 7)_2=\Sigma_h\Sigma_n\left\{\frac{h}{2h}I^{2h}(10, 6)_n+\frac{n}{4h}R_{ac}^{2hagdi}(10, 6)_n\right\},$$

the quantities here omitted from the formula being

$R_0^{2hagdi}(10, 6)_n=0$, because 10 is not divisible by $4h$ (Problem e).

$R^{2hag}(10, 6)_n=0$, for the same reason (Problems f, g, h).

$R_{dc}^{2hagdi}(10, 6)_n=0$, as it has been proved in the preceding article.

$R_c^{2hagdi}(10, 6)_n=0$, because 10 is not divisible by $4h$ (Problem b).

$R_{\frac{1}{2}c}^{2hdi}=R_c^{2hdi}=0$, for the same reason (Problems k, l).

Now the equations of Problem u for $I^{2h}(10, 6)_n$ are those of Problem l for $R_c^h(10, 6)_n$, and we have proved in the preceding article that $h=1$ and $n=4$ are the only values affording a solution; which gives

$$\Sigma\left(\Delta_{\frac{n}{2h}}\right)=\Sigma(\Delta 2)=14.$$

Wherefore (Problem w) ($i>0$),

$$I^2(10, 6)_4=\tfrac{2}{8}\left\{14-\Sigma_i\left(\frac{8}{2(i+1)}I^{(i+1)2}(10, 6)_4+\frac{4}{2i}R^{2i}(10, 6)_4\right)\right\}.$$

The formula for $I^n(r, k)_n$ shows that $I^4(10, 6)_4=0$, for $\frac{10+4}{4}$ is not integer, and we have just proved that $R^{2h}(10, 6)_n=0$ under every form except $R_{ac}^{2hagdi}(10, 6)_n$, which remains to be determined.

In Problem d we see that $r-n-2h$ is divisible by $4h$; wherefore $n=4$ and $h=1$. Therefore, by the formula for R_{ac}^{in},

$$R_{ac}^{2agdi}(10, 6)_4=\Sigma_m R^{m\,.\,mo}(5, 2)=R^{mo}(5, 2)=1,$$

as is easily verified. Consequently

$$I^2(10, 6)_4=\tfrac{2}{8}\{14-\tfrac{4}{2}R^{2agdi}(10, 6)_4\}=\frac{14-2}{4}=3,$$

and $\quad I^2(10, 7)=I^2(10, 7)_2=\tfrac{4}{2}I^2(10, 6)_4+\tfrac{4}{4}.R^{2agdi}(10, 6)_4=6+1=7,$

whence, finally (Problem x),

$$I(10, 7)=\tfrac{1}{20}\{D(10, 7)-\tfrac{10}{1}R(10, 7)-\tfrac{20}{2}I^2(10, 7)\}=\tfrac{1}{20}(1430-10.14-10.7)=61.$$

And this enumeration of the 8-partitions of the 10-gon agrees with that at p. 410 of my memoir mentioned in the first article.

It is evident that this last example would have caused far less trouble, if we had had our register of partitions filled up for inspection up to the 7-partitions of the 10-gon.

There is little difficulty, as I have verified to some extent, in framing, by the aid of the results here given, algebraical expressions, containing circulating functions for $R^h(r, k)$ and $I^h(r, k)$, in terms of r and k only, and thence by addition, complete expressions of the $(1+k)$-partitions of the r-gon. But the subject has been pursued far enough for one communication.

LXXIII. The notation above employed for the partitions of the r-gon is applicable, with hardly any change, to express those of the r-ace.

An h-ly reversible $(1+k)$-partition of the r-ace has h axial planes of reversion, which are *achorial*, *diachorial*, or *monochorial*, according as they cut none, two, or one only of the faces ($\chi\omega\rho\acute{\iota}\alpha$) about the r-ace. The k partitioning lines may conveniently be called *diapipeds*, or shorter, *diapeds*, being each in two planes about the r-ace, as a diagonal is in two summits of the r-gon.

Putting Ac, Di, and Mo, for achorial, diachorial and monochorial, we have the following account to give, N standing for the nucleus n-ace, of the $(1+k)$-partitions of the r-ace:

$$\begin{aligned}
&\mathrm{R}^{2h\,.\,\mathrm{A}c\,.\,\mathrm{D}i}(r,\,k)_{\mathrm{N}}=\mathrm{R}^{2h\,.\,ag\,.\,di}(r,\,k)_n,\\
&\mathrm{R}^{h\,.\,\mathrm{A}c}\quad(r,\,k)_{\mathrm{N}}=\mathrm{R}^{h\,.\,ag}\quad(r,\,k)_n,\\
&\mathrm{R}^{h\,.\,\mathrm{D}i}\quad(r,\,k)_{\mathrm{N}}=\mathrm{R}^{h\,.\,di}\quad(r,\,k)_n,\\
&\mathrm{R}^{h\,.\,\mathrm{M}o}\quad(r,\,k)_{\mathrm{N}}=\mathrm{R}^{h\,.\,mo}\quad(r,\,k)_n,\\
&\mathrm{I}^{h}\qquad(r,\,k)_{\mathrm{N}}=\mathrm{I}^{h}\qquad(r,\,k)_n.
\end{aligned}$$