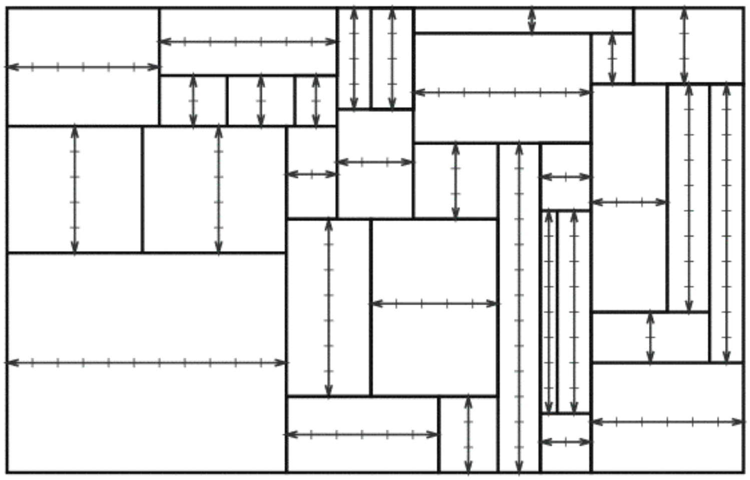
## Integers and Rectangles

A large rectangle in the plane is partitioned into smaller rectangles, each of which has either integer height or integer width (or both). Prove that the large rectangle also has this property.



## Tipping the Scale

A balance scale sits on the teacher's table, currently tipped to the right. There is a set of weights on the scales, and on each weight is the name of at least one pupil. On entering the classroom, each pupil moves all the weights carrying his or her name to the opposite side of the scale. Prove that there is *some* set of pupils that you, the teacher, can let in which will tip the scales to the left.

## Watches on the Table

Fifty accurate watches lie on a table. Prove that there exists a moment in time when the sum of the distances from the center of the table to the ends of the minute hands is more than the sum of the distances from the center of the table to the centers of the watches.

## Uses of Fuses

Simultaneously light both ends of one fuse and one end of the other; when the first fuse burns out (after half a minute), light the other end of the second. When it finishes, 45 seconds have passed.♡

This and other fuse puzzles seem to have spread like wildfire a few years ago. Recreational mathematics expert Dick Hess has put together a miniature volume called *Shoelace Clock Puzzles* devoted to them; he first heard the one above from Carl Morris of Harvard University.

Hess considers multiple fuses (shoelaces, for him) of various lengths, but lights them only at ends. If you allow midfuse ignition and arbitrary dexterity, you can do even more. For example, you can get 10 seconds from a single 60-second fuse by lighting at both ends and at two internal points, then lighting a new internal point every time a segment finishes; thus, at all times, three segments are burning at both ends and the fuse material is being consumed at six times the intended rate.

Bit of a mad scramble at the end, though. You'll need infinitely many matches to get perfect precision.

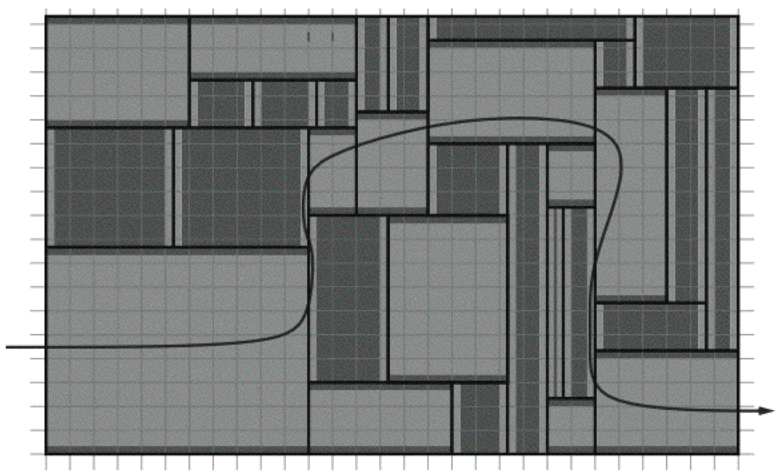
## Integers and Rectangles

This puzzle was the subject of a unique article by Stan Wagon (of Macalester College in St. Paul, MN) called "Fourteen Proofs of a Result about Tiling a Rectangle," in *The American Mathematical Monthly*, Vol. 94 (1987), pp. 601–617.

Some of Wagon's solutions make amusing use of heavy mathematical machinery; one that doesn't entails placing the lower left-hand corner of the big rectangle on the origin of a grid made up of squares of side $1/2$. Coloring the grid squares alternately black and white, as on a chessboard, we see that each small rectangle is exactly half white and half black. The same, consequently, is true for the big rectangle. However, if (say) the height of the big rectangle is not integral, the region of the big rectangle between the lines $x = 0$ and $x = 1/2$ will not be color-balanced. Hence, the width would have to be integral. ♡

Your author is responsible for the following solution, not found in Wagon's article. Letting $\varepsilon$ be less than the smallest tolerance in the partition, color each small rectangle of integral width green except for a red horizontal strip of width $\varepsilon$ across the top, and another across the bottom. Color each remaining small rectangle red, except for a green vertical strip of width $\varepsilon$ along the left side and another along the right.

Place the lower left point of the big rectangle at the origin. Either there is a green path from the left side of the big rectangle to the right side, or a red path from bottom to top; suppose the former. Every time the green path crosses a vertical border of the partition, it is at an integral coordinate; thus, the big rectangle has integral width. Similarly a red path from bottom to top forces integral height.



## Tipping the Scale

Consider all subsets of students, including the empty set and the full set. Each weight will be on the left half the time, so the total weight on the left for all these subsets is the same as the total weight on the right. Since the empty set results in a tip to the right, some other set must tip to the left.

(*Source*: Second All Soviet Union Mathematical Competition, Leningrad 1968). ♡

The "averaging" technique used here comes up often: watch for it!