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THE INNER PLETHYSM OF SYMMETRIC FUNCTIONS AND SOME OF ITS APPLICATIONS

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Abstract

We review the fundamental properties of inner plethysm, and give applications to the computation of Poincaré series, and to the root-number functions in symmetric groups.

1. Introduction

The aim of this paper is to give a coherent presentation of some of the very few known facts about the so-called *inner plethysms*, and to point out some applications.

The name *inner plethysm* has been given by D.E. Littlewood **[Li1]** to the operation on symmetric functions corresponding to the composition of a representation $\mathfrak{S}_n \longrightarrow GL(m, \mathbb{C})$ of the symmetric group $\mathfrak{S}_n$ by a representation $GL(m, \mathbb{C}) \longrightarrow GL(q, \mathbb{C})$ of a linear group, the correspondence between symmetric group representations and symmetric functions being given, as usual, by the Frobenius characteristic map (see section 2). In other words, the study of inner plethysm is the same as the study of the standard λ-ring structure of the complex representation ring $R(\mathfrak{S}_n)$, as described for example in **[Kn]**. In what follows, we shall emphasize the λ-ring formalism and use it to give short proofs of some classical results and to derive a number of interesting facts from Aitken's elementary computation of the exterior powers of the representation by permutation matrices. For example, we give short derivations of the Poincaré series of the Weil algebras $S(V) \otimes \Lambda(V)$ and of the space $\mathcal{H}_n$ of $\mathfrak{S}_n$-harmonic polynomials, recently computed by Kirillov and Pak **[KP]** and Kirillov **[KM]** (section 4). In section 5, we make a detailed study of the Adams operations, and use them to give a simple theory of the root number functions in symmetric groups.

2. Notations and background

We denote by $R(\mathfrak{S}_n)$ the *complex Grothendieck ring* of the symmetric group $\mathfrak{S}_n$, that is, the ring generated by the isomorphism classes of complex finite dimensional representations of $\mathfrak{S}_n$, addition and multiplication being induced by direct sum and tensor product. It is well known (see *e.g.* **[Kn]**) that $R(\mathfrak{S}_n)$ is isomorphic with the ring $CF(\mathfrak{S}_n)$ of central functions on $\mathfrak{S}_n$ (generated by the irreducible characters), the isomorphism being given by the map χ which associates to the class $[\rho]$ of a representation ρ its character χ_ρ. The canonical λ-ring structure of $R(\mathfrak{S}_n)$, defined by $\Lambda^k([\rho]) = [\Lambda^k(\rho)]$, where Λ^k is the k-th exterior power, induces via this isomorphism a λ-ring structure on $CF(\mathfrak{S}_n)$, whose *Adams operations*, denoted ψ^k, are given, for $\xi \in CF(\mathfrak{S}_n)$ and $\sigma \in \mathfrak{S}_n$, by the formula (*cf.* **[Kn]**)

$$(\psi^k(\xi))(\sigma) = \xi(\sigma^k).$$

The ring $CF(\mathfrak{S}_n)$ is endowed with its usual scalar product

$$(\xi, \eta) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \bar{\xi}(\sigma)\eta(\sigma)$$

and the adjoint of the linear operator ψ^k for this scalar product is denoted by $\psi_k^\dagger$.

On the $\mathbb{Z}$-module $R = \oplus_{n\geq 0} R(\mathfrak{S}_n)$, we define the *induction product*, also called *outer tensor product* and denoted by a dot, as follows: for $[\rho] \in R(\mathfrak{S}_m)$ and $[\eta] \in R(\mathfrak{S}_n)$,

$$[\rho]\cdot[\eta] = \left[\operatorname{ind}_{\mathfrak{S}_m\times\mathfrak{S}_n}^{\mathfrak{S}_{m+n}}(\rho\times\eta)\right]$$

(*cf.* **[Mcd]** p.60 or **[Kn]** p.127). With this product, R becomes a commutative ring, which is isomorphic to a ring of symmetric polynomials via the Frobenius characteristic, as we will describe now.

Let us denote by $\mathfrak{Sym}$ or $\mathfrak{Sym}(A)$ the ring of symmetric polynomials with coefficients in $\mathbb{Z}$ in an infinite set of indeterminates $A = \{a_n | n \in \mathbb{N}^*\}$, and let $\mathfrak{Sym}^n$ be its homogeneous component of degree n. We denote by Λ_n (or $\Lambda_n(A)$) the *elementary symmetric functions* of A, defined by the generating series

$$\lambda_z(A) = \prod_{a\in A}(1+za) = \sum_{n\geq 0}\Lambda_n(A)z^n$$

(where z is an extra indeterminate), and by S_n or $S_n(A)$ the *complete symmetric functions*, defined by

$$\sigma_z(A) = \prod_{a\in A}(1-za)^{-1} = \sum_{n\geq 0} S_n(A)z^n.$$

The *power-sums* ψ_k $(k\geq 1)$ are defined by $\psi_k(A) = \sum_{a\in A} a^k$.

A *partition* is for us a finite non-decreasing sequence of positive integers, $I = (i_1 \leq i_2 \leq \ldots \leq i_r)$. We shall also write $I = (i_1^{\alpha_1} i_2^{\alpha_2}\ldots)$, α_m being the number of parts i_k which are equal to m. The *weight* of I is $|I| = \sum_k i_k$, and its *length* is its number of (nonzero) parts $\ell(I) = r$.

For a partition I, we set $\psi^I = \psi_1^{\alpha_1}\psi_2^{\alpha_2}\cdots$, $\Lambda^I = \Lambda_1^{\alpha_1}\Lambda_2^{\alpha_2}\cdots$, and $S^I = S_1^{\alpha_1}S_2^{\alpha_2}\cdots$. For $I \in \mathbb{Z}^r$, not necessarily a partition, we define the *Schur function*

$$S_I = \det\left(S_{i_k+k-h}\right)_{1\leq h,k\leq r}$$

where $S_j = 0$ for $j < 0$. The Schur functions indexed by partitions form a $\mathbb{Z}$-basis of $\mathfrak{Sym}$, and we endow $\mathfrak{Sym}$ with a scalar product $(\cdot,\cdot)$ for which this basis is orthonormal. The ψ^I form an orthogonal $\mathbb{Q}$-basis of $\mathfrak{Sym}$, with $(\psi^I,\psi^I) = 1^{\alpha_1}\alpha_1!2^{\alpha_2}\alpha_2!\cdots$, so that for a partition I of weight n, $n!/(\psi^I,\psi^I)$

is the cardinality of the conjugacy class of $\mathfrak{S}_n$ whose elements have α_k cycles of length k ($k = 1, \dots, n$). A permutation σ of this class will be said *of type* I, and we shall write $T(\sigma) = I$.
The *Frobenius characteristic map* $\mathcal{F} : CF(\mathfrak{S}_n) \longrightarrow \mathfrak{Sym}^n$ associates to a central function ξ the symmetric function

$$\mathcal{F}(\xi) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \xi(\sigma) \psi^{T(\sigma)} = \sum_{|I|=n} \xi(I) \frac{\psi^I}{(\psi^I, \psi^I)}$$

where $\xi(I)$ is the common value of the $\xi(\sigma)$ for $T(\sigma) = I$. We can also consider $\mathcal{F}$ as a map from $R(\mathfrak{S}_n)$ to $\mathfrak{Sym}^n$ by setting $\mathcal{F}([\rho]) = \mathcal{F}(\chi_\rho)$ (we use the same letter $\mathcal{F}$ for the two maps since this will not lead to ambiguities). Glueing these maps together, we get a linear map $\mathcal{F} : R \longrightarrow \mathfrak{Sym}$, which turns out to be an isomorphism of graded rings, isometric for the two scalar products previously defined.
We denote by $[I]$ the class of the irreducible representation of $\mathfrak{S}_n$ associated with the partition I, and by χ_I its character. We have then $\mathcal{F}(\chi_I) = S_I$ (see *e.g.* **[Mcd]** p.62).
The ring $\mathfrak{Sym}$ has a natural λ-ring structure, coming from the interpretation of symmetric functions as characters of general linear groups. In fact, this structure is the restriction to $\mathfrak{Sym}(A) \subset \mathbb{C}[A]$ of the standard λ-ring structure of the polynomial ring $\mathbb{C}[A] = \mathbb{C}[a_1, a_2, \dots]$, which is defined as follows (see **[LS]** for more details and other applications):
for $P = \sum_{\alpha \in \mathbb{N}^{(\mathbb{N})}} c_\alpha a^\alpha$, where $c_\alpha \in \mathbb{C}$ and $a^\alpha = a_1^{\alpha_1} a_2^{\alpha_2} \cdots$, we set

$$\lambda_z(P) = \prod_\alpha (1 + z a^\alpha)^{c_\alpha} = \sum_{n \geq 0} \Lambda_n(P) z^n$$

and for $F = f(\Lambda_1, \Lambda_2, \dots) \in \mathfrak{Sym}$, $F(P) = f(\Lambda_1(P), \Lambda_2(P), \dots)$. For example, $S_n(P)$ has the generating series $\prod_\alpha (1 - z x^\alpha)^{-c_\alpha}$, and $\psi_n(P) = \sum_\alpha c_\alpha a^{n\alpha}$. The polynomial $F(P)$ is called the *plethysm* of P by F ($P \otimes F$ in Littlewood's notation).The restriction to $\mathfrak{Sym}$ of this kind of plethysm is called *outer plethysm*, and corresponds to the composition of representations of $GL(n, \mathbb{C})$. More precisely, if $P(a_1, \dots, a_n)$ is the character $\chi(A) =$

$\mathrm{tr}(\rho(A))$ of a representation ρ of $GL(n,\mathbb{C})$ expressed as a symmetric function of the eigenvalues $a_1,\ldots,a_n$ of A, $(\Lambda_k(P))(a_1,\ldots,a_n)$ is the character of the k-th exterior power $\Lambda^k(\rho)$ of ρ.

We can also define a λ-ring structure on each homogeneous component $\mathfrak{Sym}^n$ of $\mathfrak{Sym}$ by transporting through the $\mathbb{Z}$-modules isomorphism $\mathcal{F}$ the canonical λ-ring structure of $R(\sigma_n)$. The product $*$, defined by

$$\mathcal{F}([\rho]\otimes[\eta]) = \mathcal{F}(\chi_\rho\chi_\eta) = \mathcal{F}([\rho]) * \mathcal{F}([\eta])$$

is called the *internal product*. The λ-ring operators induced by $\mathcal{F}$ on $\mathfrak{Sym}^n$ are written with hats (*e.g.* $\hat{\Lambda}_k$, $\hat{\psi}_k$) to be distinguished from those described above. Thus, if $P = \mathcal{F}([\rho])$, $\hat{\Lambda}_k(P) = \mathcal{F}([\Lambda^k(\rho)])$, and if $F = f(\Lambda_1,\Lambda_2,\ldots) = \sum_I c_I\Lambda^I \in \mathfrak{Sym}$, then

$$\hat{F}(P) = \sum_I c_I\,\hat{\Lambda}_{i_1}(P) * \hat{\Lambda}_{i_2}(P) * \cdots * \hat{\Lambda}_{i_r}(P).$$

The symmetric polynomial $\hat{F}(P)$ is called the *inner plethysm* of P by F ($P \odot F$ in Littlewood's notation).

The individual λ-ring structures of the $\mathfrak{Sym}^n$ can be collected to give a λ-ring structure on the completed ring $\mathfrak{Sym}^\wedge$ of symmetric formal series, if we make the convention that $P * Q = 0$ for $P \in \mathfrak{Sym}^p$ and $Q \in \mathfrak{Sym}^q$ with $p \neq q$. The unit element is then the infinite series $\sigma_1 = \sum_{n\geq 0} S_n$ (recall that $S_n = \mathcal{F}(\chi_n)$, where χ_n is the trivial character).

The elements of rank 1 of the λ-ring $\mathfrak{Sym}^n$ are S_n and Λ_n, corresponding to the two one-dimensional representations $[n]$ and $[1^n]$. We clearly have $\hat{S}_m(S_n) = S_n$ for all m, and to determine $\hat{S}_m(\Lambda_n)$, we can for example make use of the identity $S_I = \Lambda_{I^\sim}$, where $I^\sim$ is the conjugate partition of I, and $\Lambda_J = \det(\Lambda_{j_k+k-h})$. Then,

$$\hat{S}_m(\Lambda_n) = \hat{\Lambda}_{1^m}(\Lambda_n) = \det{}^{(*)}(\mathrm{diag}(\hat{\Lambda}_1(\Lambda_n),\ldots,\hat{\Lambda}_1(\Lambda_n)))$$

(where $\det^{(*)}$ means that the determinant must be expanded with internal products), since $\hat{\Lambda}_k(\Lambda_n) = 0$ for $k > 1$. Thus, $\hat{S}_m(\Lambda_n)$ is equal to S_n for m even and to Λ_n for m odd.

Also, making use of the general formula

$$S_I(A.B) = \sum_K S_K(A).(S_I * S_K)(B)$$

valid for A, B in any λ-ring with product $\cdot$, and using the fact that $\Lambda_n * F = F^{\sim}$, where $F \mapsto F^{\sim}$ is the linear map defined by $(S_I)^{\sim} = S_{I^{\sim}}$, we have, for $|I| = m$ and $|J| = n$,

$$\hat{S}_I(S_{J^{\sim}}) = \hat{S}_I(\Lambda_n * S_J) = \sum_{|K|=m} \hat{S}_K(\Lambda_n) * (\widehat{S_I * S_K}(S_J))$$

and Λ_n being of rank 1, $\hat{S}_K(\Lambda_n)$ is nonzero only for $K = (m)$, so that

$$\hat{S}_I(S_{J^{\sim}}) = \begin{cases} \hat{S}_I(S_J) & m \text{ even} \\ (\hat{S}_I(S_J))^{\sim} & m \text{ odd} \end{cases}$$

This result is given by King [**Ki**].

3. Exterior and symmetric powers of the fundamental representations

Let $\rho : \mathfrak{S}_n \longrightarrow GL(n, \mathbb{C})$ be the standard representation of $\mathfrak{S}_n$ by permutation matrices (*i.e.* if $(e_i)_{1 \le i \le n}$ is the canonical basis of $\mathbb{C}^n$, one has $\rho(\sigma)(e_i) = e_{\sigma(i)}$), and let χ_ρ be its character. As is well-known, $\mathcal{F}(\chi_\rho) = S_1 S_{n-1}$. One way to see this is as follows: $\chi_\rho(\sigma) = \operatorname{tr} \rho(\sigma)$ is just the number of fixed points of σ, so that if $T(\sigma) = I = (1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n})$, then $\chi_\rho(\sigma) = \alpha_1$. Hence,

$$\mathcal{F}(\chi_\rho) = \sum_{\alpha_1 + 2\alpha_2 + \cdots + \alpha_n = n} \alpha_1 \frac{\psi_1^{\alpha_1} \psi_2^{\alpha_2} \cdots \psi_n^{\alpha_n}}{1^{\alpha_1} \alpha_1! 2^{\alpha_2} \alpha_2! \cdots n^{\alpha_n} \alpha_n!}$$

$$= \psi_1 \sum \frac{\psi_1^{\alpha_1 - 1} \psi_2^{\alpha_2} \cdots}{1^{\alpha_1 - 1} (\alpha_1 - 1)! 2^{\alpha_2} \alpha_2! \cdots} = \psi_1 \sum_{|J| = n-1} \frac{\psi^J}{(\psi^J, \psi^J)}$$

$$= \psi_1 S_{n-1} = S_1 S_{n-1}.$$

Next, we can easily compute the characters of the exterior and symmetric powers of ρ, by an elegant method due to Aitken [**A1-2**]. Indeed, if $T(\sigma) = I = (1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n})$, the characteristic polynomial of the matrix $\rho(\sigma)$ is

$$P_\sigma(x) = (1-x)^{\alpha_1}(1-x^2)^{\alpha_2}\cdots(1-x^n)^{\alpha_n}$$

since the characteristic polynomial of a p-cycle is $1-x^p$. This can also be written

$$P_\sigma(x) = \psi^I(1-x)$$

and since

$$P_\sigma(x) = \det(1 - x\rho(\sigma)) = \sum_{k=0}^{n}(-1)^k \operatorname{tr}\Lambda^k(\rho)(\sigma)x^k$$

$$= \sum_{k=0}^{n}(-1)^k x^k(\hat{\Lambda}^k(S_1S_{n-1}), \psi^I)$$

we see that

$$(\hat{\lambda}_{-x}(S_1S_{n-1}), \psi^I) = \psi^I(1-x)$$

whence

$$\hat{\lambda}_{-x}(S_1S_{n-1}) = \sum_{|I|=n}\frac{\psi^I(1-x)\psi^I(A)}{(\psi^I,\psi^I)} = \sum_I \frac{\psi^I((1-x)A)}{(\psi^I,\psi^I)}$$

$$= S_n((1-x)A). \tag{3.1}$$

Similarly,

$$\sum_{k\geq 0}\operatorname{tr} S^k(\rho)(\sigma)x^k = \det(1-x\rho(\sigma))^{-1} = \frac{1}{P_\sigma(x)}$$

$$= \frac{1}{\psi^I(1-x)} = \psi^I\left(\frac{1}{1-x}\right)$$

which yields

$$\hat{\sigma}_x(S_1S_{n-1}) = S_n\left(\frac{A}{1-x}\right). \tag{3.2}$$

From this, we can easily derive Littlewood's results **[Li2]** for the symmetric and exterior powers of ρ, and for the exterior powers of $[1, n-1]$. Indeed,

$$\hat{\lambda}_{-x}(S_1 S_{n-1}) = S_n((1-x)A) = S_n(A - xA) = \sum_{k=0}^{n} S_{n-k}(A) S_k(-xA)$$

$$= \sum_{k=0}^{n} (-1)^k x^k S_{n-k}(A) \Lambda_k(A),$$

so that

$$\hat{\Lambda}^k(S_1 S_{n-1}) = \Lambda_k S_{n-k} = S_{1^k, n-k} + S_{1^{k-1}, n-k+1}. \tag{3.3}$$

Also,

$$\hat{\Lambda}^k(S_1 S_{n-1}) = \hat{\Lambda}^k(S_{1,n-1} + S_n) = \sum_{k=0}^{n} \hat{\Lambda}^k(S_{1,n-1}) \hat{\Lambda}^{n-k}(S_n)$$

$$= \hat{\Lambda}^k(S_{1,n-1}) + \hat{\Lambda}^{k-1}(S_{1,n-1})$$

(since $[n]$ is 1-dimensional) and from this we get immediately

$$\hat{\Lambda}^k(S_{1,n-1}) = S_{1^k, n-k}. \tag{3.4}$$

One can also write, as in **[A2]** or **[CGR]**,

$$\hat{\lambda}_{-x}(S_1 S_{n-1}) = S_n\big((1-x)A\big) = \sum_{|I|=n} S_I(1-x) S_I(A)$$

(by Cauchy's formula)

$$= \sum_{k=0}^{n-1} (-x)^k (1-x) S_{1^k, n-k}(A)$$

since $S_I(1-x)$ is zero if I is not a hook, and $S_{1^k,n-k}(1-x) = (-x)^k(1-x)$, as is easily seen from the determinantal expression of Schur functions. On the other hand,

$$\hat{\lambda}_{-x}(S_1 S_{n-1}) = \hat{\lambda}_{-x}(S_{1,n-1} + S_n) = \hat{\lambda}_{-x}(S_{1,n-1}) \hat{\lambda}_{-x}(S_n)$$

$$= (1-x)\hat{\lambda}_{-x}(S_{1,n-1})$$

so that

$$\hat{\lambda}_{-x}(S_{1,n-1}) = \sum_{k=0}^{n-1}(-x)^k S_{1^k,n-k}.$$

The result for symmetric powers is a bit more complicated: denoting by $[x^r]F$ the coefficient of x^r in the polynomial F, we have

$$\hat{S}^r(S_1S_{n-1}) = [x^r]S_n(A(1+x+x^2+\cdots))$$

$$= [x^r]S_n(A+xA+x^2A+\cdots+x^rA)$$

$$= [x^r] \sum_{k_0+k_1+\cdots+k_r=n} S_{k_0}(A)S_{k_1}(xA)\cdots S_{k_r}(x^rA)$$

$$= \sum_{\substack{k_0+k_1+\cdots+k_r=n \\ k_1+2k_2+\cdots+rk_r=r}} S^{k_0k_1\cdots k_r}(A). \tag{3.5}$$

4. Applications to Poincaré series

Let V be some $\mathfrak{S}_n$-module, and for a partition I of n, denote by $m(I,V)$ the multiplicity of the irreducible representation of type I of $\mathfrak{S}_n$ in V. If V is a graded module $V = \oplus_{k\geq 0}V^k$, the *Poincaré series* $P_I(t;V)$ of the isotypical component of type I of V is defined as

$$P_I(t;V) = \sum_{k\geq 0} m(I,V^k)t^k.$$

We shall also need the graded version $F(t;V)$ of the Frobenius characteristic of V, defined by

$$F(t;V) = \sum_{|I|=n} P_I(t;V)S_I(A).$$

These series can be easily obtained for several interesting modules, by merely combining the observations of section 3 with the Littlewood-Stanley expressions of some specialisations of Schur functions, that is:

LEMMA 4.1. *Let q and z be elements of rank one in some λ-ring R. Then,*

$$S_I\left(\frac{1-z}{1-q}\right) = q^{\|I\|} \prod_{(i,j)\in I} \frac{1-zq^{c_{ij}}}{1-q^{h_{ij}}}$$

where $\|I\| = (n-1)i_1 + (n-2)i_2 + (n-3)i_3 + \cdots + 0i_n$ (the partition I of n being written as a non-decreasing vector of $\mathbb{N}^n$), c_{ij} is the content of the box $(i,j) \in I$ (here we denote the Ferrers diagram of the partion I by the same letter I) and h_{ij} its hook-length (see [**Mcd**] p. 28, ex. 3 *for details).*

First of all, consider the natural action of $\mathfrak{S}_n$ on the algebra $\mathcal{P}_n = K[x_1, \ldots, x_n]$ of polynomials over a field K of zero characteristic. Of course, $\mathcal{P}_n$ is isomorphic to the symmetric algebra $S(V)$ of the vector space V spanned by the variables $x_1, \ldots, x_n$, which affords the standard permutation representation of $\mathfrak{S}_n$, and whose characteristic is, as we know, the symmetric function $S_1 S_{n-1}$. Thus, setting for short

$$p_I(t) = P_I(t;\mathcal{P}_n) = \sum_{k\geq 0} m(I, \mathcal{P}_n^k)t^k,$$

we see that

$$F(t;\mathcal{P}_n) = \sum_{|I|=n} p_I(t) S_I(A) = \hat{\sigma}_t(S_1 S_{n-1})(A)$$

$$= S_n\left(\frac{A}{1-t}\right) = \sum_{|I|=n} S_I\left(\frac{1}{1-t}\right) S_I(A),$$

and using lemma 4.1, we get

THEOREM 4.2.

$$p_I(t) = t^{\|I\|} \prod_{(i,j)\in I} (1-t^{h_{ij}})^{-1}. \quad \square$$

This result is given by Kirillov in [**Kr**] (Theorem 1).

Next, we can consider, as in [**KP**] the so-called *Weil algebra*

$$E(V) = S(V) \otimes \Lambda(V)$$

of our n-dimensional vector space V on which $\mathfrak{S}_n$ acts by permutation of coordinates. The space $E(V)$ is bigraded by $E^{p,q}(V) = S^p(V) \otimes \Lambda^q(V)$, and this time the Poincaré series take the form

$$p_I(t,s) = \sum_{p,q \geq 0} m(I, E^{p,q}(V)) t^p s^q .$$

Using as above the results of section 3, we have

$$\sum_{|I|=n} p_I(t,-s) S_I(A) = \hat{\sigma}_t(S_1 S_{n-1}) * \hat{\lambda}_{-s}(S_1 S_{n-1})$$

$$= S_n\left(\frac{A}{1-t}\right) * S_n((1-s)A)$$

$$= \sum_{|I|=n} \psi^I\left(\frac{1}{1-t}\right) \psi^I(1-s) \frac{\psi^I(A)}{(\psi^I, \psi^I)}$$

$$= S_n\left(\frac{1-s}{1-t}A\right) = \sum_{|I|=n} S_I\left(\frac{1-s}{1-t}\right) S_I(A)$$

(by Cauchy's formula), whence

$$p_I(t,-s) = S_I\left(\frac{1-s}{1-t}\right) = t^{\|I\|} \prod_{(i,j)\in I} \frac{1-st^{c_{ij}}}{1-t^{h_{ij}}},$$

so that

THEOREM 4.3.

$$p_I(t,s) = t^{\|I\|} \prod_{(i,j)\in I} \frac{1+st^{c_{ij}}}{1-t^{h_{ij}}} = \prod_{(i,j)\in I} \frac{t^i + st^j}{1-t^{h_{ij}}} . \quad \square$$

This is Theorem 1 of [**KP**].

Finally, we can derive in the same way Kirillov's formula for the Poincaré series of the space of harmonic polynomials for $\mathfrak{S}_n$ [**KM**]. This formula is of particular interest in that it provides a q-analog of the celebrated Hook Formula (see *e.g.* [**JK**]) for the dimensions of the irreducible representations of $\mathfrak{S}_n$.

So, let V be an n-dimensional complex vector space, and denote its dual by V^*. The symmetric algebra $S(V^*)$ can be seen as the algebra of polynomial functions on V, and the elements of $S(V)$ as differential operators with constant coefficients acting on $S(V^*)$. Let $S(V)^G$ denote the algebra of G–invariant differential operators for some group G of linear transformations of V, and $S(V)^G_+$ the ideal of operators without constant term. Then, the polynomials of $S(V^*)$ which are annihilated by $S(V)^G_+$ are called the harmonic polynomials for G [**Ko**] (see also [**He**]). The space of G-harmonic polynomials is denoted by $\mathcal{H}(G)$. It has been shown by Chevalley that when G is a finite Coxeter group, $S(V^*)$ is isomorphic to $\mathcal{H}(G)\otimes S(V^*)^G$ as a G-module. This is in particular true for $G=\mathfrak{S}_n$, the case which we will now consider. We can then set $V^* = Vect\langle x_1,\ldots,x_n\rangle$, $S(V^*)=\mathbb{C}[x_1,\ldots,x_n]=\mathcal{P}_n$, $S(V^*)^{\mathfrak{S}_n}=\mathfrak{Sym}_n$ and $\mathcal{H}(\mathfrak{S}_n)=\mathcal{H}_n$. From the isomorphism of $\mathfrak{S}_n$-modules

$$\mathcal{P}_n \simeq \mathcal{H}_n \otimes \mathfrak{Sym}_n$$

we see that their graded characteristics satisfy the identity

$$F(t;\mathcal{P}_n) = F(t;\mathcal{H}_n) * F(t;\mathfrak{Sym}_n)$$

where

$$F(t;\mathfrak{Sym}_n) = \sum_{k\geq 0} t^k \sum_{|I|=n} m(I,\mathfrak{Sym}_n^k) S_I(A)$$

$$= \sum_{k\geq 0} t^k p(k;n) S_n(A)$$

(where $p(k;n)$ is the number of partitions of k in at most n parts)

$$= \frac{S_n(A)}{(1-t)(1-t^2)\cdots(1-t^n)}$$

and

$$F(t;\mathcal{P}_n) = S_n\left(\frac{A}{1-t}\right) = \sum_{|I|=n} S_I\left(\frac{1}{1-t}\right) S_I(A)$$

as we already know. Hence,

$$\sum_{|I|=n} S_I\left(\frac{1}{1-t}\right) S_I(A) = \left[\sum_{|I|=n} P_I(t;\mathcal{H}_n) S_I(A)\right] * \frac{S_n(A)}{(1-t)\cdots(1-t^n)}$$

$$= \prod_{i=1}^{n}(1-t^i)^{-1} \sum_{|I|=n} P_I(t;\mathcal{H}_n) S_I(A)$$

so that

$$P_I(t;\mathcal{H}_n) = \prod_{i=1}^{n}(1-t^i)\ S_I\left(\frac{1}{1-t}\right).$$

Using Lemma 4.1, we get Kirillov's formula:

THEOREM 4.4.

$$P_I(t;\mathcal{H}_n) = t^{\|I\|} \prod_{k=1}^{n}(1-t^k) \prod_{(i,j)\in I} (1-t^{h_{ij}})^{-1}. \quad \square$$

As one can see for example by looking at [Mcd] p. 130, ex. 2, $P_I(t;\mathcal{H}_n)$ is the *Kostka-Foulkes polynomial* $K_{I^\sim,(1^n)}(t)$. One can also prove this directly by writing $A = (t-1)B$, so that $S_n(A/(1-t)) = (-1)^n \Lambda_n(B)$, and using the fact that $(1-t)(1-t^2)\cdots(1-t^n)\Lambda_n(B) = Q_{(1^n)}(B)$ (Hall-Littlewood function, see [Mcd] p.105-106). Since the Kostka-Foulkes polynomials can be defined by

$$Q_I(B) = \sum_H K_{HI}(t) S_H((1-t)B),$$

the conclusion follows. Further connections between Kostka-Foulkes polynomials and harmonic polynomials are described in [La].

Various informations can be easily extracted from the equation

$$F(t;\mathcal{H}_n) = (t;t)_n S_n\left(\frac{A}{1-t}\right).$$

To take another example from [KM], let $K = (k_1,\ldots,k_m)$ be a partition of n, and suppose that we want the Poincaré series of the graded module

$\mathcal{H}_n^{\mathfrak{S}^K}$ of fixed points of the Young subgroup $\mathfrak{S}^K = \mathfrak{S}_{k_1} \times \mathfrak{S}_{k_2} \times \cdots \times \mathfrak{S}_{k_m}$ of $\mathfrak{S}_n$, that is, the series

$$G_K(t) = \sum_{k \geq 0} t^k \dim(\mathcal{H}_n^k)^{\mathfrak{S}^K}.$$

We have, denoting by $\downarrow$ and $\uparrow$ restriction and induction,

$$\dim(\mathcal{H}_n^k)^{\mathfrak{S}^K} = \dim \hom_{\mathfrak{S}^K}(\mathcal{H}_n^k \downarrow \mathfrak{S}^K, id_{\mathfrak{S}^K})$$

where id is the identity representation, and by Frobenius reciprocity, this is equal to

$$\dim \hom_{\mathfrak{S}_n}(\mathcal{H}_n^k, id_{\mathfrak{S}^K} \uparrow \mathfrak{S}_n).$$

But it is well-known that $\mathcal{F}(id_{\mathfrak{S}^K} \uparrow \mathfrak{S}_n) = S^K = S_{k_1} \cdots S_{k_m}$, so that

$$G_K(t) = (F(\mathcal{H}_n; t), S^K)$$

and using the classical identity

$$S_n(AB) = \sum_I S^I(A) \psi_I(B)$$

where the ψ_I are the monomial functions (see **[Mcd]** p. 33), together with the facts that $(S^I, \psi_J) = \delta_{IJ}$ and $S_n(1/(1-t)) = 1/(t;t)_n$, we get

$$G_K(t) = (t;t)_n S^K \left(\frac{1}{1-t} \right) = \frac{\prod_{k=1}^n (1-t^k)}{\prod_{i=1}^m \prod_{j=1}^{k_i} (1-t^j)} .$$

5. Adams operations and roots of permutations

We denote by $\hat{\psi}_k^\dagger$ the adjoint of the Adams operator $\hat{\psi}_k$ of the λ-ring $\mathfrak{Sym}^n$. The action of $\hat{\psi}_k^\dagger$ on a product of power-sums ψ^I is easily described **[Ho]**. Indeed, recall that for a central function $\xi \in CF(\mathfrak{S}_n)$ and a permutation $\sigma \in \mathfrak{S}_n$, one has $\xi(\sigma) = (\mathcal{F}(\xi), \psi^{T(\sigma)})$. Now, if σ is of type I, with $I =$

$(1^{\alpha_1}2^{\alpha_2}\dots n^{\alpha_n})$, σ^k will have for each i, $\alpha_i\delta_i$ cycles of length i/δ_i, where $\delta_i = i \wedge k = \text{g.c.d.}(i,k)$. Thus

$$\hat{\psi}_k^\dagger(\psi^I) = \prod_i \left(\psi_{i/\delta_i}\right)^{\alpha_i\delta_i}. \tag{5.1}$$

From this equation, it is immediate that $\hat{\psi}_k^\dagger(\psi^I\psi^J) = \hat{\psi}_k^\dagger(\psi^I)\hat{\psi}_k^\dagger(\psi^J)$, *i.e.* the $\hat{\psi}_k^\dagger$ are algebra morphisms for the ordinary product. The images of the S^I, and even of the S_I, can then be computed from those of the S_m, using the Littlewood-Richardson rule. The following result **[Th]**, which answers a question raised by P. Hoffman in **[Ho]** p.71, provides an expression of the $\hat{\psi}_k^\dagger(S_m)$ in terms of outer plethysms.

THEOREM 5.1. — *Consider, for $r \geq 1$, the symmetric function*

$$\ell_r = \frac{1}{r}\sum_{d|r}\mu(d)\psi_d^{r/d}$$

where μ is the Möbius function. Then,

$$\sum_{n\geq 0}\hat{\psi}_k^\dagger(S_n) = \prod_{d|k}\sum_{m\geq 0} S_m(\ell_d)$$

which can also be written more concisely, using the series $\sigma_1 = \sum_{n\geq 0} S_n$

$$\hat{\psi}_k^\dagger(\sigma_1) = \sigma_1\Big(\sum_{d|k}\ell_d\Big).$$

Proof. We start from the expression $\sigma_1 = \exp\sum_{j\geq 1}\psi_j/j$. Since the $\hat{\psi}_k^\dagger$ are algebra morphisms and are continuous for the usual topology on formal series, we get

$$\hat{\psi}_k^\dagger(\sigma_1) = \exp\sum_{j\geq 1}\frac{1}{j}\hat{\psi}_k^\dagger(\psi_j).$$

Now, let p be a prime number. Then, $\hat{\psi}_p^\dagger(\psi_j)$ is equal to $\psi_{j/p}^p$ when p divides j and to ψ_j otherwise, whence

$$\hat{\psi}_p^\dagger(\sigma_1) = \exp\sum_{i\geq 1}\frac{\psi_i^p - \psi_{ip}}{ip}\exp\sum_{j\geq 1}\frac{\psi_j}{j} = \sigma_1(\ell_p)\sigma_1(\ell_1) = \sigma_1(\ell_1+\ell_p)$$

(using the fact that $\psi_p(\psi_q) = \psi_{pq}$). Next, let m be any positive integer not divisible by p. Then, we claim that

$$\hat{\psi}_p^\dagger(\sigma_1(\ell_m)) = \sigma_1(\ell_m + \ell_{pm}). \tag{5.2}$$

Indeed,

$$\hat{\psi}_p^\dagger(\sigma_1(\ell_m)) = \exp\sum_{i\geq 1}\hat{\psi}_p^\dagger \circ \frac{\psi_i}{i} \circ \Big(\frac{1}{m}\sum_{d|m}\mu(d)\psi_d^{m/d}\Big)$$

$$= \exp\sum_{i\geq 1}\frac{1}{im}\hat{\psi}_p^\dagger\Big(\sum_{d|m}\mu(d)\psi_{id}^{m/d}\Big)$$

$$= \exp\sum_{i\geq 1}\frac{1}{im}\sum_{d|m}\mu(d)\hat{\psi}_p^\dagger(\psi_{id})^{m/d}$$

$$= \sigma_1(\ell_m)\exp\sum_{j\geq 1}\frac{1}{jmp}\sum_{d|m}\mu(d)\Big[\psi_{jd}^{mp/d} - \psi_{jdp}^{m/d}\Big]$$

$$= \sigma_1(\ell_m)\exp\sum_{j\geq 1}\frac{\psi_j}{j}\circ\Big[\frac{1}{mp}\sum_{d|m}\big(\mu(d)\psi_d^{mp/d} + \mu(dp)\psi_{dp}^{mp/dp}\big)\Big]$$

(since p does not divide d)

$$= \sigma_1(\ell_m)\sigma_1\Big(\frac{1}{mp}\sum_{d|mp}\mu(d)\psi_d^{mp/d}\Big) = \sigma_1(\ell_m + \ell_{mp}).$$

It remains to compute $\hat{\psi}_p^\dagger(\sigma_1(\ell_m))$ in the case $m = pr$. In this case, we have

$$\hat{\psi}_p^\dagger(\sigma_1(\ell_m)) = \exp\sum_{i\geq 1}\hat{\psi}_p^\dagger\Big(\frac{1}{im}\sum_{d|m}\mu(d)\psi_{id}^{m/d}\Big)$$

$$= \exp\sum_{i\geq 1}\hat{\psi}_p^\dagger\Big(\frac{1}{im}\Big[\sum_{\substack{d'|m\\ p\nmid d'}}\mu(d')\psi_{id'}^{m/d'} + \sum_{\substack{d''|m\\ p|d''}}\mu(d'')\psi_{id''}^{m/d''}\Big]\Big)$$

$$= \exp\Big(\sum_{i\geq 1}\frac{1}{im}\sum_{\substack{d'|m\\ p\nmid d'}}\mu(d')\hat{\psi}_p^\dagger(\psi_{id'}^{m/d'})\Big)\exp\Big(\sum_{i\geq 1}\frac{1}{im}\sum_{\substack{d''|m\\ p|d''}}\mu(d'')\psi_{id''/p}^{mp/d''}\Big)$$

In the second factor we can set $d'' = dp$, with $d|r$ so that the last expression is

$$\sigma_1\Big(\frac{1}{m}\sum_{\substack{d|m\\ p\nmid d}} \mu(d)\psi_d^{m/d}\Big) \times$$

$$\times \exp\sum_{j\geq 1}\frac{1}{jmp}\sum_{\substack{d'|m\\ p\nmid d'}}\Big(\mu(d')\psi_{jd'}^{mp/d'} - \mu(d')\psi_{jpd'}^{mp/d'p}\Big)\sigma_1\Big(\frac{1}{m}\sum_{d|r}\mu(dp)\psi_d^{m/d}\Big)$$

and since in the last factor $\mu(dp)$ is zero when $p|d$, this factor can be grouped with the first to yield

$$\hat{\psi}_p^\dagger(\sigma_1(\ell_m)) = \sigma_1\Big(\frac{1}{m}\sum_{\substack{d|r\\ p\nmid d}}(\mu(d)+\mu(dp))\psi_d^{m/d}\Big)\sigma_1\Big(\frac{1}{mp}\sum_{d|mp}\mu(d)\psi_d^{mp/d}\Big).$$

The first factor being equal to 1, we get finally

$$\hat{\psi}_p^\dagger(\sigma_1(\ell_m)) = \sigma_1(\ell_{mp}) \tag{5.3}$$

in the case where p divides m.

Formulas (5.2) and (5.3) allows us to conclude the proof of theorem 5.1 by induction on the number of prime factors of k. Indeed, using the fact that $\hat{\psi}_{pm}^\dagger = \hat{\psi}_p^\dagger \circ \hat{\psi}_m^\dagger$, we have by our induction hypothesis

$$\hat{\psi}_{pm}^\dagger(\sigma_1) = \hat{\psi}_p^\dagger\Big(\sigma_1\Big(\sum_{d|m}\ell_d\Big)\Big).$$

In the case $p \nmid m$ this is, by (5.2)

$$\sigma_1\Big(\sum_{d|m}\ell_d + \ell_{dp}\Big) = \sigma_1\Big(\sum_{d|mp}\ell_d\Big)$$

and if $m = pr$ we get

$$\hat{\psi}_{pm}^\dagger(\sigma_1) = \hat{\psi}_p^\dagger\Big(\sigma_1\Big(\sum_{d|r}\ell_{dp}\Big)\sigma_1\Big(\sum_{d|m, p\nmid d}\ell_d\Big)\Big)$$

$$= \sigma_1\Big(\sum_{d|r} \ell_{dp^2}\Big)\sigma_1\Big(\sum_{d|m, p\nmid d} \ell_d + \ell_{dp}\Big)$$

$$= \sigma_1\Big(\sum_{\substack{d|pm \\ p^2|d}} \ell_d + \sum_{\substack{d|pm \\ p|d, p^2\nmid d}} \ell_d + \sum_{\substack{d|pm \\ p\nmid d}} \ell_d\Big) = \sigma_1\Big(\sum_{d|pm} \ell_d\Big). \qquad \square$$

Since $\hat{\psi}_k^\dagger(S_n) \in \mathfrak{Sym}^n$, we can compute it by extracting the terms of weight n in the series for $\hat{\psi}_k^\dagger(\sigma_1)$. This can be done efficiently by first expressing the symmetric function $\sum_{d|k} \ell_d = \sum_j S_{H_j}$ as a sum of Schur functions (the H_j are not necessarily distinct). This is possible since ℓ_d is the Frobenius image of a representation of $\mathfrak{S}_d$ (see [**Fo**]). Then, if we set $|H_i| = h_i$,

$$\hat{\psi}_k^\dagger(S_n) = \sum_{m_1h_1+m_2h_2+\cdots=n} \prod_j S_{m_j}(S_{H_j}).$$

For example,

$$\hat{\psi}_3^\dagger(\sigma_1) = \sigma_1\Big(\psi_1 + \frac{\psi_1^3 - \psi_3}{3}\Big) = \sigma_1(S_1 + S_{12})$$

so that

$$\hat{\psi}_3^\dagger(S_7) = S_7 + S_{12}S_4 + S_2(S_{12})S_1$$

$$= S_7 + S_{16} + 2S_{25} + S_{115} + S_{1114} + S_{34} + 3S_{124}$$

$$+ S_{133} + 2S_{223} + 2S_{1123} + S_{11113} + S_{1222}.$$

Since there exist now good algorithms for the expansion of plethysms of the form $S_m(S_H)$ (see [**Ca1**],[**Ca2**]), the above procedure gives an efficient method for the computation of $\hat{\psi}_k^\dagger(S_n)$. Also, since $\hat{\psi}_k^\dagger(S_n)$ is a sum of products of plethysms of functions which are Frobenius images of characters, $\hat{\psi}_k^\dagger(S_n)$ is itself the Frobenius image of a character of $\mathfrak{S}_n$. As we shall see, this character has a simple interpretation.

For a partition I, let $R_k(I)$ be the set of partitions H such that

$$T(\sigma) = H \Longrightarrow T(\sigma^k) = I.$$

Then,

$$\hat\psi_k(\psi^I) = \sum_H (\psi^I, \hat\psi_k^\dagger(\psi^H)) \frac{\psi^H}{(\psi^H, \psi^H)} = (\psi^I, \psi^I) \sum_{H\in R_k(I)} \frac{\psi^H}{(\psi^H, \psi^H)}$$

so that, using the fact that $S_n = \sum_K \psi^K/(\psi^K, \psi^K)$,

$$(\hat\psi_k^\dagger(S_n), \psi^I) = (S_n, \hat\psi_k(\psi^I)) = (\psi^I, \psi^I) \sum_{H\in R_k(I)} \frac{1}{(\psi^H, \psi^H)}$$

$$= \frac{(\psi^I, \psi^I)}{n!} \sum_{H\in R_k(I)} \frac{n!}{(\psi^h, \psi^H)} = \frac{1}{|C_I|} \sum_{H\in R_k(I)} |C_H|$$

where $|C_H|$ is the cardinality of the conjugacy class C_H associated to the partition H. Thus, $(\hat\psi_k^\dagger(S_n), \psi^I) = (\hat\psi_k^\dagger(\sigma_1), \psi^I)$ is equal to the nuber $r_k^{(n)}(I)$ of k-th roots of a permutation of type I, *i.e.* we have $\hat\psi_k^\dagger(S_n) = \mathcal{F}(r_k^{(n)})$, and from the above considerations, we get the following result (conjectured by A.Kerber and first proved by T. Scharf, using a different method, see **[Sc1]**,**[Sc2]**,**[Ke]**):

COROLLARY 5.2. — *The central function $r_k^{(n)}$ is a proper character of $\mathfrak{S}_n$.*

Many other properties of the the character $r_k^{(n)}$ can be easily deduced from theorem 5.1. For example, considering the case $k = 2$, we find

$$\hat\psi_2^\dagger(\sigma_1) = \sigma_1\Big(\psi_1 + \frac{\psi_1^2 - \psi_2}{2}\Big) = \prod_{i\geq 1} \frac{1}{1-a_i} \prod_{i<j} \frac{1}{1-a_ia_j}$$

which is equal, by a well-known identity of Littlewood (**[Li1]** p.238, see **[LP]** for a simple proof), to $\sum_I S_I$ (sum over all partitions). We thus recover the classical result:

COROLLARY 5.3. — *The character $r_2^{(n)}$ is equal to the sum of all irreducible characters of $\mathfrak{S}_n$.*

Finally, one can, as in **[De]**, rearrange the summation in the formula

$$\hat\psi_k^\dagger(\sigma_1) = \exp \sum_{j\geq 1} \frac{1}{j} \sum_{d|k} \frac{1}{d} \sum_{a|d} \mu(a) \psi_{ja}^{d/a}$$

in order to collect the terms involving ψ_p. This yields, denoting by A_p the sum of these terms,

$$A_p = \sum_{\substack{aj=p \\ d|k,a|d}} \frac{\mu(a)}{dj} \psi_p^{d/a} = \sum_{\substack{d|k \\ ab=d,aj=p}} \frac{\mu(a)}{bp} \psi_p^b = \frac{1}{p} \sum_{abc=k,a|p} \frac{\psi_p^b}{b} \sum_{a|c,a|p} \mu(a)$$

$$= \frac{1}{p} \sum_{\substack{b|k \\ p\wedge(k/b)=1}} \frac{\psi_p^b}{b},$$

whence the following formula:

THEOREM 5.4. — *The series $\hat{\psi}_k^\dagger(\sigma_1)$ can be factored as a product of series, each factor involving only one variable ψ_p:*

$$\hat{\psi}_k^\dagger(\sigma_1) = \prod_{p\geq 1} \exp\left(\frac{1}{p} \sum_{\substack{b|k \\ p\wedge(k/b)=1}} \frac{\psi_p^b}{b}\right).$$

From this last result, we can easily recover the generating series of **[BC]** and the explicit expressions of **[Sc1]** for the values of the character $r_k^{(n)}$ on a given class. First, we see that this character is 'multiplicative', in the sense that

$$r_k^{(n)}(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}) = r_k^{(\alpha_1)}(1^{\alpha_1}) r_k^{(2\alpha_2)}(2^{\alpha_2}) \cdots r_k^{(n\alpha_n)}(n^{\alpha_n}).$$

Indeed,

$$r_k^{(n)}(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}) = (\hat{\psi}_k^\dagger(\sigma_1), \prod_{p\geq 1} \psi_p^{\alpha_p})$$

$$= (\prod_{p\geq 1} f_p(\psi_p), \prod_{p\geq 1} \psi_p^{\alpha_p}) = \prod_{p\geq 1} (f_p(\psi_p), \psi_p^{\alpha_p}) = \prod_{p\geq 1} (\hat{\psi}_k^\dagger(\sigma_1), \psi_p^{\alpha_p})$$

$$= r_k^{(\alpha_1)}(1^{\alpha_1}) r_k^{(2\alpha_2)}(2^{\alpha_2}) \cdots r_k^{(n\alpha_n)}(n^{\alpha_n}).$$

Thus, it is sufficient to compute the values of $r_k^{(n)}$ on rectangular partitions, that is, of the form (p^q). This is easily done:

$$r_k^{(pq)}(p^q) = \left(\exp\left\{\frac{1}{p} \sum_{\substack{b|k \\ p\wedge(k/b)=1}} \frac{\psi_p^b}{b}\right\},\ \psi_p^q\right)$$

$$= \sum_{H=(1^{\beta_1}2^{\beta_2}\cdots q^{\beta_q})\models_k q} \left(\frac{1}{p}\right)^{\beta_1+\beta_2+\cdots+\beta_q} \frac{1}{1^{\beta_1}\beta_1!2^{\beta_2}\beta_2!\cdots q^{\beta_q}\beta_q!}(\psi_p^q, \psi_p^q)$$

$$= \sum_{H\models_k q} \frac{q!}{(\psi^H, \psi^H)}\, p^{q-\ell(H)}$$

where $H \models_k q$ means that H is a partition of q, with all the parts h_i of H dividing k and $h_i \wedge (k/h_i) = 1$.

Also, Theorem 5.4 allows the computation of $r_k(p^q)$ by means of linear recurrence relations, and in some cases leads to a closed form. For example, with $k = 4$,

$$\hat{\psi}_4^{\dagger}(\sigma_1) = \prod_{k\geq 1} \exp\left\{\frac{1}{2k-1}\left(\psi_{2k-1} + \frac{\psi_{2k-1}^2}{2} + \frac{\psi_{2k-1}^4}{4}\right)\right\} \exp\left(\frac{\psi_{2k}^4}{8k}\right).$$

Let

$$f_t(x) = \exp\left(t\left(x + \frac{x^2}{2} + \frac{x^4}{4}\right)\right) = \sum_{n\geq 0} a_n(t)\frac{x^n}{n!}.$$

Then, $f_t'(x) = (1 + x + x^3)f_t(x)$, so that

$$a_n = t\left[a_{n-1} + (n-1)a_{n-2} + (n-1)(n-2)(n-3)a_{n-4}\right],$$

with

$$f_t(x) = 1 + t\frac{x}{1!} + (t + t^2)\frac{x^2}{2!} + (3t^2 + t^3)\frac{x^3}{3!} + O(x^4)$$

whence $a_0 = 1$, $a_1 = t$, $a_2 = t + t^2$ and $a_3 = 3t^2 + t^3$. For p odd we can compute recursively

$$r_4(p^q) = (\hat{\psi}_4^{\dagger}(\sigma_1), \psi_p^q) = p^q a_q(1/p)$$

and for p even, the second factor leads immediately to the explicit expressions $r_4(p^q) = 0$ for $q \not\equiv 0$ [4] and $r_4((2m)^{4r}) = 2^r m^{4r}(4r)!/r!$.

6. Final comments

The actual computation of a general inner plethysm of Schur functions is rather difficult. At present time, the best method seems to make use of the character table, together with the formula

$$\hat{\psi}_k(S_I) = \sum_J (S_I, \hat{\psi}_k^{\dagger}(\psi^J)) \frac{\psi^J}{(\psi^J, \psi^J)}$$

which, thanks to (5.1), gives immediately the expansion in the basis (ψ^I). One can then use the Newton formulas to compute recursively the exterior and symmetric powers in this same basis, and the character table to expand the result in the Schur basis. For a general plethysm $\hat{S}_H(S_I)$, one has to expand first S_H is the basis (ψ^J) by means of the character table. This is essentially the method used by Esper **[E1-3]** who has published tables (partially reproduced in **[JK]**) and studied the stability properties of the expansions.

These stability properties come from the well known fact (see **[Ke]**) that the values $\chi_{(i_1,\ldots,i_r,m)}(1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n})$ of a character are given by a polynomial $P_I(\alpha_1,\ldots,\alpha_s)$ independent of the last part m. These polynomials P_I are called *character polynomials.* The Adams operations of the characters rings $CF(\mathfrak{S}_n)$ induce Adams operations on the ring $\mathbb{Q}[\alpha_1,\alpha_2,\ldots]$ generated over $\mathbb{Q}$ by the character polynomials, and it is easy to check that $\psi^k(\alpha_1) = \sum_{d|k} d\alpha_d$, so that with this structure, $\mathbb{Q}[\alpha_1,\alpha_2,\ldots]$ is generated as a λ-ring by α_1, which is the character polynomial associated to the representation by permutation matrix. In fact, the ring generated over $\mathbb{Z}$ by the character polynomials is also generated as a λ-ring by α_1, or, which amounts to the same, by $\alpha_1 - 1$, the character of the fundamental representation $[1, n-1]$. This fact is established in **[Bo]**. One can get a much simpler proof by observing that in view of the results of section 3, this would be the case if any Schur function could be expressed by an integral linear combination of internal products of hook Schur functions, and this is indeed proved in an elementary way in **[Bu]**.

Writing

$$\sigma_z(\alpha_1) = \exp \sum_{k\geq 1} \frac{z^k}{k} \sum_{d|k} d\alpha_d,$$

taking logarithms and rearranging the summation, one gets the elegant formula

$$\sigma_z(\alpha_1) = \prod_{p\geq 1} (1 - z^p)^{-\alpha_p}.$$

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