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Equality in the Aleksandrov-Fenchel Inequality — Present State and New Results

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1. Introduction

It was first noted by Hermann Minkowski that the isoperimetric inequality

$$S^3 \geq 36\pi V^2 \tag{1}$$

satisfied by the surface area S and the volume V of a three-dimensional convex body K in $\mathbb{R}^3$ is a consequence of two simpler, namely quadratic, inequalities. Introducing a third functional M, the integral of mean curvature, one has the inequalities

$$M^2 \geq 4\pi S \tag{2}$$

and

$$S^2 \geq 3VM, \tag{3}$$

which together imply (1). Equality in (2) holds only if K is a ball, hence also in (1) equality takes place only in this case. However, the inequality (3) exhibits many more cases of equality. As Minkowski stated (but did not prove, see Section 3), equality holds in (3) if and only if K is a cap body of a ball. Recall that such a body is, by definition, the convex hull of a ball and (finitely or infinitely many) points $x_1, x_2, \ldots$ outside the ball such that each segment with endpoints $x_i \neq x_j$ meets the ball. The identification of the equality cases in (3) can be considered as an extremal problem with side condition: e.g., to minimize S under the condition that V and M have given values. Usually the solution of such an extremal problem for convex bodies characterizes a small class of simple sets, like balls, ellipsoids, simplices, etc. The fact that the extremal bodies have a high degree of symmetry is often the clue to a solution of the extremal problem, since symmetrization procedures may be applicable. The equality problem for inequality (3) is of an entirely different nature. Here the extremal class depends on infinitely many parameters, its elements show rather different shapes, and they cannot be characterized by symmetry properties. The common feature of the extremal bodies is a local curvature property, in a generalized sense.

The quadratic inequalities (2) and (3) are special cases of the general Aleksandrov-Fenchel inequality

(4)
$$V(K_1, K_2, K_3, \ldots, K_n)^2 \geq V(K_1, K_1, K_3, \ldots, K_n) V(K_2, K_2, K_3, \ldots, K_n)$$

for the mixed volume $V(K_1, K_2, K_3, \ldots, K_n)$ of convex bodies $K_1, \ldots, K_n$ in R^n. For the specialization, observe that $V = V(K, K, K)$, $S = 3V(K, K, B^3)$, $M = 3V(K, B^3, B^3)$, $4\pi = 3V(B^3, B^3, B^3)$ for $n = 3$. By B^n we denote the unit ball in R^n. For $n = 2$, inequality (4) reduces to Minkowski's inequality for the mixed area, and it is well known that here equality holds if and only if K_1 and K_2 are homothetic. In the following, we assume therefore that $n \geq 3$. Clearly, equality holds in (4) if K_1 and K_2 are homothetic, but in general not only in this case. The complete conditions for equality in (4) are still not known. This question is the topic of the present article. We describe the known partial results and add a new one.

As (4) represents a classical inequality of fundamental importance and with many applications, the identification of the equality cases is a problem of intrinsic geometric interest. Without its solution, the Brunn-Minkowski theory of mixed volumes remains in an uncompleted state.

2. A Conjecture

In this section, we want to repeat a conjecture on the equality cases in (4), and this requires some preliminaries. General references on convex bodies are Bonnesen-Fenchel [1934], Eggleston [1958] and, in particular, Leichtweiß [1980].

By $\mathcal{K}^n$ we denote the set of convex bodies (nonempty, compact, convex subsets) in n-dimensional Euclidean space R^n, and by $\mathcal{K}_0^n$ the subset of bodies with interior points. Mixed volumes and mixed area measures are denoted, respectively, by $V(K_1, \dots, K_n)$ and $S(K_1, \dots, K_{n-1}; \cdot)$, thus

$$(5) \qquad V(K_1, K_2, \dots, K_n) = \frac{1}{n} \int\limits_{S^{n-1}} h(K_1, u) dS(K_2, \dots, K_n; u).$$

Here $S^{n-1} = \{x \in \mathrm{R}^n : \|x\| = 1\}$ is the unit sphere of R^n, and

$$h(K, u) = \max\{\langle x, u\rangle : x \in K\} \qquad \text{for } u \in \mathrm{R}^n \backslash \{0\}$$

defines the support function of $K \in \mathcal{K}^n$.

In the following, we assume that $\mathcal{C} = (K_3, \dots, K_n)$ is a given $(n-2)$-tuple of convex bodies $K_3, \dots, K_n \in \mathcal{K}^n$, and we write $V(K, L, K_3, \dots, K_n) = V(K, L, \mathcal{C})$ and $S(K, K_3, \dots, K_n; \cdot) = S(K, \mathcal{C}; \cdot)$. Our problem then is to characterize the convex bodies $K, L \in \mathcal{K}^n$ for which equality holds in

$$(6) \qquad V(K, L, \mathcal{C})^2 \geq V(K, K, \mathcal{C}) V(L, L, \mathcal{C}).$$

Equation (5) suggests that the support of the mixed area measure $S(K, \mathcal{C}; \cdot)$, denoted by $\operatorname{supp} S(K, \mathcal{C}; \cdot)$, will play a role. One can show that

$$(7) \qquad \operatorname{supp} S(K, \mathcal{C}; \cdot) \subset \operatorname{supp} S(B^n, \mathcal{C}; \cdot)$$

for arbitrary $K \in \mathcal{K}^n$ (see Schneider [1985], Lemma 3.4). Suppose now that K and L are such that

$$(8) \qquad h(K, u) = h(\alpha L + t, u) \qquad \text{for } u \in \operatorname{supp} S(B^n, \mathcal{C}; \cdot),$$

with a constant $\alpha > 0$ and some vector $t \in \mathrm{R}^n$. Then (5) yields

$$\begin{aligned} V(K, K, \mathcal{C}) &= \frac{1}{n} \int\limits_{S^{n-1}} h(K, u) dS(K, \mathcal{C}; u) \\ &= \frac{1}{n} \int\limits_{S^{n-1}} h(\alpha L + t, u) dS(K, \mathcal{C}; u) \\ &= \alpha V(L, K, \mathcal{C}), \end{aligned}$$

and similarly one obtains $\alpha V(L,L,\mathcal{C}) = V(K,L,\mathcal{C})$. Thus equality holds in (6). One may conjecture that the converse is also true, at least if $\mathcal{C}$ consists of full-dimensional bodies.

Conjecture A. If $K_3,\ldots,K_n \in \mathcal{K}_0^n$, $\dim K > 0$, and equality holds in (6), then

$$h(K,u) = h(\alpha L + t, u) \qquad \text{for } u \in \operatorname{supp} S(B^n,\mathcal{C};\cdot),$$

with suitable $\alpha > 0$ and $t \in \mathbb{R}^n$.

Unfortunately, the support of the mixed area measure is unknown in general. To formulate a conjecture on the nature of this set, let $K \in \mathcal{K}^n$ be a convex body and let $u \in \mathbb{R}^n \setminus \{0\}$. Choose a point x in the relative interior of the support set

$$F(K,u) = \{y \in K : \langle y,u\rangle = h(K,u)\},$$

and let $N(K,x)$ be the cone of exterior normal vectors to K at x. The *touching cone* $T(K,u)$ of K at u is defined as the unique face of $N(K,x)$ that contains u in its relative interior; it does not depend upon the choice of x. For convex bodies $K_1,\ldots,K_{n-1} \in \mathcal{K}^n$ we now say that the vector u is $(K_1,\ldots,K_{n-1})$-*extreme* if there exist $(n-1)$-dimensional linear subspaces $H_1,\ldots,H_{n-1}$ of $\mathbb{R}^n$ such that

$$T(K_i,u) \subset H_i \qquad \text{for } i = 1,\ldots,n-1$$

and

$$\dim(H_1 \cap \ldots \cap H_{n-1}) = 1.$$

Conjecture B. If $K_1,\ldots,K_{n-1} \in \mathcal{K}^n$, then $\operatorname{supp} S(K_1,\ldots,K_{n-1};\cdot)$ is the closure of the set of $(K_1,\ldots,K_{n-1})$-extreme unit vectors.

Conjectures A and B lead to the following one. Here a supporting hyperplane of a convex body is called $(\mathcal{C},B^n)$-*extreme* if its exterior unit normal vector is $(\mathcal{C},B^n)$-extreme.

Conjecture C. If $\mathcal{C} = (K_3,\ldots,K_n)$ is an $(n-2)$-tuple of n-dimensional convex bodies and if $\dim K, L > 0$, then equality holds in (6) if and only if suitable homothets of K and L have the same $(\mathcal{C},B^n)$-extreme supporting hyperplanes.

With slight modifications, all three conjectures, of which A is due to A. Loritz, were formulated in Schneider [1985]. In Conjecture C, the assumption that $K_3,\ldots,K_n$ be n-dimensional cannot be omitted; this was pointed out by Ewald (1988). To give an example, let $n = 3$, let S_1, $S_2 \subset \mathbb{R}^3$ be non-parallel segments, and put $K = M + S_1$, $L = M + S_2$, $K_3 = S_1 + S_2$, where $M \in \mathcal{K}^3$ is an arbitrary convex body. Then $V(K,K,K_3) = V(L,L,K_3) = V(K,L,K_3)$ and thus $V(K,L,K_3)^2 = V(K,K,K_3)V(L,L,K_3)$, whereas the condition of Conjecture C is not satisfied.

We conclude this section with a condition equivalent to equality in (6), which has a local character and serves as a generalized Euler-Lagrange equation.

Lemma 2.1. *If $K, L, K_3, \ldots, K_n \in \mathcal{K}^n$ are convex bodies with $V(K,L,\mathcal{C}) > 0$, $\mathcal{C} = (K_3, \ldots, K_n)$, then equality holds in* (6) *if and only if*

$$S(K,\mathcal{C};\cdot) = \alpha S(L,\mathcal{C};\cdot) \tag{9}$$

with some constant $\alpha > 0$.

This was proved, independently, by Aleksandrov and by Fenchel and Jessen. References are found in Schneider [1985]. Lemma 2.5 given there contains additional equivalent conditions (however, condition (c) of Lemma 2.5 must be completed by 'for all $M \in \mathcal{K}_0^n$').

For the case where $\mathcal{C}$ consists only of polytopes, Ewald [1988] has formulated a different condition which is equivalent to equality in (6).

3. Known Results

We shall now collect the special cases in which the equality conditions for (6) are known. There are some old and a few new results. Among the old results are those which come out in Aleksandrov's two proofs of (6). Let $K, L \in \mathcal{K}^n$, $K_3, \ldots, K_n \in \mathcal{K}_0^n$, $\mathcal{C} = (K_3, \ldots, K_n)$. In each of the following cases, equality in (6) holds only if K and L are homothetic:

(a) $K_3, \ldots, K_n$ are analogous simple polytopes (see Section 4 for the definition), K and L are polytopes with the same system of facet normals as $K_3, \ldots, K_n$.

(b) $K_3, \ldots, K_n$ have twice continuously differentiable support functions and everywhere positive radii of curvature, K and L have twice continuously differentiable support functions.

(c) $K_3, \ldots, K_n$ are balls.

(d) $K_3, \ldots, K_n$ are smooth.

Here (a) and (b) are due to Aleksandrov [1937, 1938]; (b) was also treated by Favard [1938]. For (c), different proofs were given by Kubota [1925] and Favard [1933]. Case (d) was proved by Schneider [1990]. A convex body is called *smooth* if at each boundary point it has a unique supporting hyperplane. Thus (d) implies (b) and (c). Observe that in (c) and (d) the convex bodies K and L may be arbitrary. The conditions on $K_3, \ldots, K_n$ in (d) seem, at first sight, only slightly weaker than in (b); they are, however, the optimal general assumptions under which equality in (6) implies homothety of K and L. Compared to (b), case (d) requires a completely different method of proof. It relies on the following Lemma 3.1, also established in Schneider [1990]. For convex bodies $A, K \in \mathcal{K}_0^n$, we say that A is *adapted to* K if to each boundary point x of K there exists a boundary point y of A such that the normal cones at these points satisfy $N(K,x) \subset N(A,y)$.

Lemma 3.1. *Suppose that* $K_3, \ldots, K_n$, $A_3, \ldots, A_n \in \mathcal{K}_0^n$, $\mathcal{C} = (K_3, \ldots, K_n)$, $\mathcal{A} = (A_3, \ldots, A_n)$, $K, L \in \mathcal{K}^n$. *If*

$$V(K,L,\mathcal{C})^2 = V(K,K,\mathcal{C})V(L,L,\mathcal{C})$$

and if A_i *is adapted to* K_i *for* $i = 3, \ldots, n$, *then also*

$$V(K,L,\mathcal{A})^2 = V(K,K,\mathcal{A})V(L,L,\mathcal{A}).$$

In particular, A_i is adapted to K_i if A_i is a summand (or homothetic to a summand) of K_i. For this case, Lemma 3.1 was already obtained in Schneider [1985]. Lemma 3.1 permits to deduce case (d) above from case (c), since a ball is adapted to any smooth convex body. As remarked in Schneider [1990], the truth of Conjecture C would imply Lemma 3.1, and Lemma 3.1 lends strong support to that conjecture, as it shows the essential role of the relative positions of the normal cones in a discussion of equality in (6).

It is clear that Conjecture C is in agreement with cases (a) to (d). In cases (b), (c), (d), each touching cone $T(K_i, u)$ is one-dimensional, hence each vector $u \in S^{n-1}$ is $(\mathcal{C}, B^n)$-extreme. For case (a), see Section 4, where Conjecture C will be proved under the assumption that $K_3, \ldots, K_n$ are analogous simple polytopes, while K and L may be arbitrary convex bodies.

Now we describe the known results on equality in (6) which are not covered by (d) or Section 4.

In Schneider [1988] it was proved that Conjecture C is true if $K_3, \dots, K_n \in \mathcal{K}_0^n$ are zonoids and $K, L \in \mathcal{K}_0^n$ are centrally symmetric. Although this is only a very special case, it is, as remarked loc. cit., a good test for the validity of Conjecture C, since zonoids may have all kinds of singularities.

The final results to be mentioned are conveniently formulated in terms of tangential bodies. Let $K \in \mathcal{K}^n$. A supporting hyperplane of K and its exterior unit normal vector u are called *p-extreme*, for $p \in \{0, \dots, n-1\}$, if u is

$$(\underbrace{K, \dots, K}_{n-1-p}, \underbrace{B^n, \dots, B^n}_{p}) - \text{extreme},$$

equivalently, if $\dim T(K, u) \le p + 1$. Another equivalent formulation is the following. The vector u is a p-extreme normal vector of K if and only if there do not exist $p+2$ linearly independent normal vectors $u_1, \dots, u_{p+2}$ at some boundary point of K such that $u = u_1 + \dots + u_{p+2}$. Now K is called an $(n-p-1)$-*tangential body* of the convex body L if $L \subset K$ and each p-extreme supporting hyperplane of K also supports L. The 1-tangential bodies of L are precisely the cap bodies of L.

The preceding explanations make clear that the following result verifies Conjecture C in a further special case.

Proposition 3.2. *Let $K, L \in \mathcal{K}_0^n$ be n-dimensional convex bodies, let $\mathcal{C} = (K, \dots, K)$. Then equality holds in* (6) *if and only if K is homothetic to an $(n-2)$-tangential body of L.*

For $n = 3$, this result was stated by Minkowski [1903] without proof. He says (on p. 476): "Wir begnügen uns hier, das bezügliche Resultat ohne Beweis anzugeben", a formulation which does not make clear whether he had a proof or merely conjectured the result. A proof of Proposition 3.2 was given by Bol [1943]. In a different way, a special case was proved by Knothe [1949]. Concerning possible extensions of Proposition 3.2, Bol made the following remark (loc. cit., p. 56). Let $K, L \in \mathcal{K}_0^4$. Then the inequality

$$V(K, L, K, L)^2 \ge V(K, K, K, L)V(L, L, K, L) \tag{10}$$

holds with equality if K is homothetic to a cap body of L or L is homothetic to a cap body of K. But Bol conjectured that these are not the only cases.

That he was right with this conjecture (and thus Bonnesen-Fenchel [1934], p. 92, were wrong), can be seen from the following example. Let $x \in \mathbb{R}^4 \backslash B^4$ and put $K = \operatorname{conv}(B^4 \cup \{x\})$ and $L = \operatorname{conv}(B^4 \cup \{-x\})$ (where B^4 has its centre at 0). Then

$$h(K,u) = h(L,u) \qquad \text{for each } u \in \operatorname{supp} S(K,L,B^4;\cdot)$$

and hence (10) holds with equality, but neither is K a cap body of L nor is L a cap body of K.

An extension of Proposition 3.2 is known if L is a ball.

Proposition 3.3. *Let $K \in \mathcal{K}_0^n$, $i \in \{1,\dots,n-1\}$, and*

$$\mathcal{C} = (\underbrace{K,\dots,K}_{n-i-1},\underbrace{B^n,\dots,B^n}_{i-1}).$$

If K is centrally symmetric and L is a ball, then equality holds in (6) *if and only if K is an $(n-i-1)$-tangential body of a ball.*

This was proved in Schneider [1985] (Theorem 4.9). The assumption of central symmetry was made for a technical reason and is most probably superfluous.

In this section, we have always assumed that the bodies $K_3,\dots,K_n$ of the $(n-2)$-tuple $\mathcal{C}$ have interior points. We mention that Ewald and Tondorf [1990+] have studied equality in (6) for the case where $K_3,\dots,K_n$ are polytopes in parallel hyperplanes.

4. Extension of a Result of Aleksandrov

Two convex polytopes $P_1, P_2 \in \mathcal{K}^n$ are called *analogous* if for each vector $u \in S^{n-1}$ the faces $F(P_1,u)$ and $F(P_2,u)$ with exterior normal vector u have the same dimension. In a different but equivalent way this notion was introduced by Aleksandrov [1937]. In the later literature, analogous polytopes have also been called *strongly combinatorially isomorphic.* In this section we prove the following theorem.

Theorem 4.1. *Let $P_3,\dots,P_n \in \mathcal{K}_0^n$ be analogous simple polytopes. Let $\sigma \subset S^{n-1}$ be the set of all unit vectors u for which the faces $F(P_i,u)$ are* *of dimension at least $n-2$. Let $K, L \in \mathcal{K}^n$ be convex bodies of positive dimension. The inequality*

$$V(K,L,P_3,\dots,P_n)^2 \geq V(K,K,P_3,\dots,P_n)V(L,L,P_3,\dots,P_n) \tag{11}$$

holds with equality if and only if

$$h(K,u) = h(\alpha L + t, u) \quad \text{for all } u \in \sigma, \tag{12}$$

with a constant $\alpha > 0$ and a vector $t \in \mathrm{R}^n$.

In the special case where K and L are polytopes with the same facet normals as P_i, condition (12) implies $K = \alpha L + t$, that is, K and L are homothetic. Thus Theorem 4.1 includes Aleksandrov's result (see (a) in Section 3). For general convex bodies K and L, however, (12) provides only partial information on the supporting hyperplanes of K and $\alpha L + t$, and K and L need not be homothetic. Theorem 4.1 verifies Conjecture C in a special case. This can be seen as follows. If $P \in \mathcal{K}^n$ is a polytope, $u \in S^{n-1}$ and $F(P,u) = F$, then the touching cone $T(P,u)$ is equal to the normal cone $N(P,F)$ (which is defined as the normal cone $N(P,x)$ at any point x in the relative interior of F); hence, $\dim T(P,u) = n - \dim F$. Since $P_3,\dots,P_n$ are analogous, the touching cone $T(P_i,u)$ does not depend on i. It follows that σ is precisely the set of $(P_3,\dots,P_n,B^n)$-extreme unit vectors.

Proof of Theorem 4.1. The proof requires some familiarity with Aleksandrov's [1937] first proof for the Aleksandrov-Fenchel inequalities; this proof can also be found in Leichtweiß [1980] and in the author's forthcoming book on 'Convex Bodies – The Brunn-Minkowski Theory'.

First we collect some notation connected with the analogous simple polytopes $P_3,\dots,P_n$. Let P be an arbitrary polytope analogous to the P_i. Let $u_1,\dots,u_N$ be the exterior unit normal vectors of the facets of P. We write

$$F_i = F(P,u_i), \quad F_{ij} = F_i \cap F_j, \quad h_i = h(P,u_i).$$

The support numbers $h_1,\dots,h_N$ determine P. Let

$$J = \{(i,j) : i,j \in \{1,\dots,N\},\ \dim F_{ij} = n-2\}$$

and let $(i,j) \in J$. By θ_{ij} we denote the angle between u_i and u_j and by $v_{ij} \perp u_i$ the exterior unit normal vector of the $(n-1)$-polytope F_i at its $(n-2)$-face F_{ij}. The support numbers

$$h_{ij} = h(F_i, v_{ij})$$

of F_i satisfy

$$h_{ij} = h_j \operatorname{cosec} \theta_{ij} - h_i \cot \theta_{ij}. \tag{13}$$

Let σ_{ij} be the spherical image of F_{ij}, that is, the set of all exterior unit normal vectors to P at relatively interior points of F_{ij}. Further, let π_{ij} be the orthogonal projection onto the two-dimensional linear subspace L_{ij} orthogonal to F_{ij}.

We point out that $u_1, \dots, u_N, J, \theta_{ij}, v_{ij}, \sigma_{ij}, \pi_{ij}$ do not depend on the special choice of P, but only on its equivalence class of analogous polytopes. For $r = 3, \dots, n$, we define

$$F_i^{(r)} = F(P_r, u), \quad F_{ij}^{(r)} = F_i^{(r)} \cap F_j^{(r)};$$

then $\dim F_{ij}^{(r)} = n - 2$ for $(i,j) \in J$.

To prove Theorem 4.1, we may assume that $V(K, L, P_3, \dots, P_n) > 0$, since otherwise K and L are parallel segments and thus homothetic. Replacing K or L by a suitable homothet, we may also assume that

$$V(K, K, P_3, \dots, P_n) = V(L, L, P_3, \dots, P_n). \tag{14}$$

By Lemma 2.1 and in view of (14), equality in (11) holds if and only if

$$S(K, P_3, \dots, P_n; \cdot) = S(L, P_3, \dots, P_n; \cdot). \tag{15}$$

To describe these measures explicitly, we denote by v, $v^{(n-2)}$ the mixed volume in dimensions $n-1$ and $n-2$, respectively. Then

$$S(K, P_3, \dots, P_n; \{u_i\}) = v(F(K, u_i), F_i^{(3)}, \dots, F_i^{(n)}) \tag{16}$$

for $i = 1, \dots, N$, as follows immediately from the definition of the mixed area measure.

Let $(i,j) \in J$. If $\omega \subset S^{n-1}$ is a Borel set satisfying $\omega \subset \operatorname{relint} \sigma_{ij}$ (where relint denotes the relative interior), then

$$S(K, P_3, \dots, P_n; \omega) = v^{(n-2)}(F_{ij}^{(3)}, \dots, F_{ij}^{(n)}) s(\pi_{ij} K; \omega), \tag{17}$$

where s denotes the area measure in the two-dimensional subspace L_{ij}. On the other hand, if $\omega \subset S^{n-1}$ is a Borel set satisfying $\omega \cap \sigma_{ij} = \emptyset$ for all $(i,j) \in J$, then

$$S(K, P_3, \dots, P_n; \omega) = 0. \tag{18}$$

The equations (17) and (18) are obtained by computing the area measure of the Minkowski combination $\lambda K+\lambda_3 P_3+\ldots+\lambda_n P_n$ at the set ω using Fubini's theorem (where for (18) one can assume that ω is contained in the spherical image of a k-face of P_i, with $k \leq n-3$), expanding into polynomials, and comparing the coefficients. Equations (16), (17), (18) together describe the measure $S(K, P_3, \ldots, P_n; \cdot)$ completely and thus permit to take advantage of equality (15).

Let $(i,j) \in J$ and let $\omega \subset \operatorname{relint} \sigma_{ij}$ be a Borel set. From (15) and (17) we get

$$s(\pi_{ij}K;\omega) = s(\pi_{ij}L;\omega). \tag{19}$$

In a two-dimensional space, the uniqueness assertion of the Minkowski problem has a local version, which can easily be derived from the global version. This local version together with (19) yields that the convex boundary arcs of $\pi_{ij}K$ and $\pi_{ij}L$ at which there exist exterior normal vectors (in L_{ij}) belonging to relint σ_{ij}, must be translates of each other. Observing that $h(\pi_{ij}K, u) = h(K,u)$ for $u \in \sigma_{ij}$, we deduce that there exists a vector $t_{ij} \in \mathrm{R}^n$ which is orthogonal to F_{ij} and for which

$$h(K,u) - h(L,u) = \langle t_{ij}, u\rangle \qquad \text{for } u \in \sigma_{ij}. \tag{20}$$

This equality is first obtained for $u \in \operatorname{relint} \sigma_{ij}$ only, but by continuity it holds for $u \in \sigma_{ij}$.

Now we define

$$\alpha_i = h(K,u_i) - h(L,u_i) = \langle t_{ij}, u_i\rangle$$

for $i = 1, \ldots, N$ and

$$\alpha_{ij} = h(F(K,u_i), v_{ij}) - h(F(L,u_i), v_{ij})$$

for $(i,j) \in J$. Let $i \in \{1, \ldots, N\}$. From (15), (16) and (5) (applied in dimension $n-1$) we get

$$\sum_{\{j:(i,j)\in J\}} \alpha_{ij} v^{(n-2)}(F_{ij}^{(3)}, \ldots, F_{ij}^{(n)}) = 0. \tag{21}$$

In (20) we take directional derivatives and use Bonnesen-Fenchel [1934], Section 16, to get

$$\alpha_{ij} = \langle t_{ij}, v_{ij}\rangle.$$

Since t_{ij} is a linear combination of u_i and v_{ij}, we deduce that $t_{ij} = \alpha_i u_i + \alpha_{ij} v_{ij}$. Since $t_{ij} = t_{ji}$, we have

$$\alpha_i u_i + \alpha_{ij} v_{ij} = \alpha_j u_j + \alpha_{ji} v_{ji}.$$

Multiplying by u_j (which is orthogonal to v_{ji}), we get

$$\alpha_{ij} = \alpha_j \operatorname{cosec} \theta_{ij} - \alpha_i \cot \theta_{ij}, \tag{22}$$

in analogy to (13).

We are now in a position to apply an argument of Aleksandrov. Suppose that also P_1, P_2 are analogous to $P_3, \ldots, P_n$, and write

$$h_i^{(r)} = h(P_r, u_i) \qquad \text{for } i = 1, \ldots, N, \quad r = 1, \ldots, n.$$

The mixed volume of $P_1, \ldots, P_n$ has a representation

$$V(P_1, \ldots, P_n) = \sum a_{j_1 \ldots j_n} h_{j_1}^{(1)} \cdots h_{j_n}^{(n)}$$

with symmetric coefficients $a_{j_1 \ldots j_n}$ which depend only on the equivalence class (under analogy) of the P_i. One can define a symmetric bilinear form Φ on R^N by

$$\Phi(X, Y) = \sum a_{j_1 \ldots j_n} x_{j_1} y_{j_2} h_{j_3}^{(3)} \cdots h_{j_n}^{(n)}$$

for $X = (x_1, \ldots, x_N)$, $Y = (y_1, \ldots, y_N)$. Now put

$$Z = (\alpha_1, \ldots, \alpha_N).$$

Then

$$\Phi(X, Z) = \frac{1}{n} \sum_{i=1}^{N} x_i \frac{1}{n-1} \sum_{\{j:(i,j) \in J\}} \alpha_{ij} v^{(n-2)}(F_{ij}^{(3)}, \ldots, F_{ij}^{(n)}),$$

since by (22) the numbers α_{ij} are obtained from the α_i in the same way as the numbers h_{ij} from the support numbers h_i in (13). From (21) we infer that $\Phi(X, Z) = 0$. As Aleksandrov [1937], §3, Lemma III, has shown, this implies that Z is the system of support numbers of a point, that is,

$$\alpha_i = \langle z, u_i \rangle \qquad \text{for } i = 1, \ldots, N$$

with some vector $z \in \mathrm{R}^n$. Replacing K by a suitable translate, we may assume that $z = 0$. Then $\alpha_{ij} = 0$ by (22) and $t_{ij} = \alpha_i u_i + \alpha_{ij} v_{ij} = 0$, thus

$$h(K, u) = h(L, u) \qquad \text{for } u \in \sigma_{ij}$$

by (20). Since $\sigma = \bigcup_{(i,j)\in J} \sigma_{ij}$, this proves (12).

That (12) implies equality in (11), can be proved along the lines of Section 2. This completes the proof of Theorem 4.1.

In the preceding proof, we had to verify the following assertion.

Assertion. Let $P_3, \ldots, P_n \in \mathcal{K}_0^n$ be analogous polytopes, let $u_1, \ldots, u_N$ and θ_{ij} be defined as above. Suppose that numbers $\alpha_1, \ldots, \alpha_N$ are given such that the numbers α_{ij} defined by (22) satisfy (21). Then there is a vector $z \in \mathrm{R}^n$ such that $\alpha_i = \langle z, u_i \rangle$ for $i = 1, \ldots, N$.

The assumption that $P_3, \ldots, P_n$ be simple was only needed in Aleksandrov's proof of this assertion. For $n = 3$, this assumption can be avoided, as shown by Aleksandrov [1958], pp. 410 - 411. Thus we have complete information on equality in

$$V(K,L,P)^2 \geq V(K,K,P)V(L,L,P)$$

for the case where P is a three-dimensional polytope.

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