

Theorem. For $n \geq 2$,

$$(i) \quad w_n = L_{2n} - 2,$$

$$(ii) \quad f_n = F_{2n},$$

$$(iii) \quad b_n = F_{2n} + L_{2n} - 2,$$

$$(iv) \quad e_n = F_{2n-1},$$

$$(v) \quad s_n = F_{2n}.$$

Proof: It is easy to check that each of (i)-(v) is true for $n = 2$ or 3 . Let $m > 3$ and suppose the results (i)-(v) are true when $n < m$. Then using known relationships for (f_n) and (L_n) , (ii), (iv), (v) follow immediately for $n = m$. These (i) and then (ii) also follow for $n = m$. The theorem now follows by induction.

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MATROIDS: UNIMODAL CONJECTURES AND MOTZKIN'S THEOREM

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1. SOME CONJECTURES

This paper arises from an interest in the unimodal conjecture for matroids, and variations of it. The conjecture was first posed by Rota in 1964, and appears in [8]. Since then it has been altered and adjusted in an attempt to find a proof.

When stating the main conjecture, we need some notation. Substitutions of the matroid terms can be found in most articles on matroids and in particular, in the original paper on the subject by Whitney [11].

Definition A matroid on a finite set S is denoted by M , and will have rank r . The family of independent sets of cardinality (and rank) k is written I_k and has I_k members. We use W_k to denote the number of closed subsets (flats) of rank k , and these are known as the Whitney numbers of the second kind. Notice that $W_0 = W_r = 1$. The closed subsets of rank $r-1$ are also called hyperplanes. We will use the term k set to mean a set with cardinality k .

Examples of matroids to which we will refer are:

- (i) Finite subsets of vector spaces with independence meaning linear independence.
- (ii) The free matroid on S in which every set is independent.

(iii) The cycle matroid of the complete graph K_n , whose independent sets are the sets of edges forming forests.

(iv) Hartmannis or paving matroids on S with rank $d+1$, in which all subsets of size d or smaller are independent, and the hyperplanes form a family of subsets of S such that each d -set is contained in a unique hyperplane.

All of the conjectures include, as special cases, conjectures about the edges of finite graphs, where a k -independent set is a forest with k edges.

By $\text{ave } [p \in A] w(p)$ we mean the average value of $w(p)$ over elements p in A , and $\max [p \in A] w(p)$ means the maximum value of $w(p)$ for elements p in A .

We can now state several conjectures concerning the flats of a matroid M . Conjecture 1A is the unimodal conjecture (or more accurately, the unimodal conjecture for Whitney numbers of the second kind).

Conjecture 1A

$$W_k \geq \min \{W_{k-1}, W_{k+1}\} \quad 2 \leq k \leq r-1$$

The difficulty in trying to tackle this conjecture seems to be in getting some useful algebraic expression of unimodality. Examination of special well known examples (vector spaces over finite fields, free matroids, the cycle matroid of K_n [2] and paving matroids) reveals that there is a possibility that the following conjecture holds:

Conjecture 1B

$$W_k^2 \geq W_{k-1} W_{k+1}.$$

This conjecture can be stated succinctly by saying that the Whitney numbers are log-convex. There is further evidence for this conjecture. We define the Whitney Polynomial of M to be

$$W_M(x) = \sum W_k x^k \quad \{0 \leq k \leq r\}.$$

Now, if M and N are matroids with ranks r_M and r_N respectively on disjoint sets S and T , then the Sum $M + N$ of M and N is a matroid with rank $r_M + r_N$ on $S \cup T$ whose independent sets are of the form $A_M \cup A_N$ with A_M and A_N independent in M and N respectively. It is easy to check that

$$W_{M+N}(x) = W_M(x) W_N(x).$$

Thus, assuming Conjecture 1B, the set of Whitney polynomials must be closed under multiplication. The following lemma adds support to this:

Lemma 1. The product of two log-convex polynomials is itself log-convex.

The proof of lemma 1 requires making use of a number of consequences of log-convex sequences, and then carefully arranging the coefficients. The details are not particularly instructive, and are omitted.

All of the mentioned examples satisfy Conjecture 1B with strict inequality. In an attempt to strengthen the Conjecture further, and based on the intuitive feeling that the free matroid on n points minimises the ratio $\frac{W_{k-1} W_{k+1}}{W_k^2}$ amongst

all matroids with n flats of rank 1, we have the two further conjectures:

Conjecture 1C

$$W_k^2 \geq \frac{k+1}{k} W_{k-1} W_{k+1}.$$

Conjecture 1D

$$W_k^2 \geq \frac{k+1}{k} \cdot \frac{n-k+1}{n-k} W_{k-1} W_{k+1}, \text{ where } n=W_1.$$

It is clear that $1D \Rightarrow 1C \Rightarrow 1B \Rightarrow 1A$. The evidence for these conjectures is inversely proportional to their strength, and in particular evidence for 1D is based largely on a number of examples with $W_1 \leq 10$, chosen to be intuitively most likely to produce a counter example, as well as the standard examples mentioned earlier.

Matroids form a mathematical structure which generalise a number of mathematical situations, particularly geometric ones. An interesting form of Conjecture 1C has been circulating recently, in an attempt to make some progress on the general conjectures.

Conjecture 1E

Given P points in euclidean 3-space, let L lines and π planes be determined by these points. (2 distinct points determine a line, 3 non collinear points determine a plane.) Then

$$p^2 \geq \frac{3}{2} L\pi.$$

Before indicating what is known about these numbers, there are some related conjectures which should be much easier to cope with. These involve the independent sets of a matroid. The first is Welsh's Problem 13 [10].

Conjecture 2A

$$I_k \geq \text{Min} \{I_{k-1}, I_{k+1}\} \quad (2 \leq k \leq r-1).$$

Conjecture 2B

$$I_k^2 \geq I_{k-1} I_{k+1}$$

Conjecture 2C

$$I_k^2 \geq \frac{k+1}{k} I_{k-1} I_{k+1}$$

Conjecture 2D

$$I_k^2 \geq \frac{k+1}{k} \cdot \frac{n-k+1}{n-k} I_{k-1} I_{k+1} \text{ where } n = I_1.$$

The evidence for these is slimmer than for the other, because much less effort has gone into examining them. On the other hand, the exchange property which holds for independent sets seems to demand that a result such as these must hold.

2. KNOWN RESULTS

In 1951, Motzkin [7] proved the following:

Theorem 2.1 Any n distinct points in Euclidean d -space, not all on one hyperplane, determine at least n hyperplanes. (By hyperplane, Motzkin means a hyper-flat, or coset of a $d-1$ subspace.)

Motzkin's result is a generalisation of the famous result that n distinct points in the plane, not all collinear, determine at least n lines. Many proofs of this have appeared, starting in 1938. The proof given by Erdos and De Bruijn [3], in 1948 is the most relevant, since it almost certainly

contained in B . In any Matroid, I_k is a clutter, each of whose elements has the same cardinality, and this we call an equicardinal clutter.

Theorem 2.5 Let C be an equicardinal clutter on the finite set S , whose elements have cardinality r , and let C_k be the number of subsets of S of cardinality k , contained in a member of C . Then $C_t \geq \binom{r}{k} C_k$ for $k \leq t \leq r - k$.

The proof is a trivial consequence of the remark following Theorem 3.1 in the next section.

It is easy to construct examples of equicardinal clutters for which the sequence C_k is not unimodal, as in example 1.

Example 1 Let K_N be the complete graph on the set of N vertices V . For each d -subset D of V , let S_D be a set of r new vertices, and adjoin all the edges needed to make $S_D \cup D$ a complete graph on $r + d$ vertices. Thus there are $N + \binom{N}{d} r$ vertices in the final graph G . Let C be the family sets formed by the vertices of the complete $r + d$ subgraphs of G . Then

$$C_k = \binom{N}{k} + \binom{N}{d} \left[\binom{d+r}{k} - \binom{d}{k} \right].$$

Putting $k = d$, $r = d - 1$ and choosing $N \gg k$, we find

$$C_k < C_{k-1}, C_{k+1}.$$

Theorem 2.5 when applied to the independent sets of a matroid tells us that the sequence I_k can only fail to be unimodal for $2k > r$ where r is the rank of M . Despite the fact that this theorem makes no use of matroid structure,

influenced Motzkin and it is also the starting point for the general theory of (λ, μ) designs. (Ryser [9], Woodall [12]). Motzkin was able to isolate a combinatorial lemma (Corollary 3.2) which is the bulk of the argument in his proof of our theorem 2.1, and which actually provides the proof of the more general "Hyperplane Theorem", theorem 2.3.

Theorem 2.3 (The Hyperplane Theorem) In any matroid, $W_1 \leq W_k$.

This theorem has been proved independently by Basterfield and Kelly [1], Greene [4], and Heron [5]. The Basterfield and Kelly proof is notable because of its slick algebraic proof. That proof can be used to extend Motzkin's theorem to the more general situation in theorem 3.1 which we will discuss in the next section. Greene proves somewhat more than the others by establishing matchings in the geometric lattice and in its complement, and in [6], I extended Greene's matchings to prove the following theorem. (A proper chain in a geometric lattice does not contain the rank 0 flat or the rank r flat.)

Theorem 2.4 In any finite geometric lattice there exist W_1 pairwise disjoint maximal proper chains.

As far as the unimodal conjectures are concerned, the hyperplane theorem does not provide much assistance, except to verify Conjecture 1A when the matroid M has rank 3. In section 3, we discuss some results which arose in an attempt to strengthen Motzkin's result, and make a general combinatorial tool out of it.

Turning to the independent set conjectures, the following general theorem is easy to prove. By a clutter, we mean a family of subsets of a given finite set such that for any pair of distinct members of the family, A and B , A is not

it is fairly sharp in the sense that it guarantees

$$C_k \leq C_{r-k} \quad (k \leq r/2),$$

and in general, it is false that $C_k \leq C_{r-k+1}$. By using the structure of matroids, we can get the following result and corollary. The proof is a straightforward application of Theorem 3.1.

Theorem 2.6 Let M be a matroid on S with rank r , and for each base B of M , and element $x \in S \setminus B$, let $I(B, x)$ be the number of independent sets contained in $S \setminus B$ but containing x . Let λ be the minimum value of $I(B, x)$ over all bases B and elements x . Then $rI_r \geq \lambda I_1$.

Corollary 2.7 If M is a matroid on S with rank $(S-B) = \text{rank } S$ for all bases B , then $I_r \geq \frac{2r-1}{r} I_1$.

Further evidence for the possibility of the unimodal conjectures comes from looking more closely at the bipartite graphs which represent the incidence relations between flats or independent sets. Thus, the geometric lattice of flats of a matroid M , has the following property, first shown to me by D.J.A. Welsh. The proof is a straightforward application of standard Matroid ideas.

Theorem 2.8 Let F be a k -flat, G a $k+1$ flat in the matroid M , with $F \subset G$. Then

- (i) The number of $k-1$ flats in F is less than the number of k flats in G
- (ii) The number of $k+1$ flats containing F is greater than the number of $k+2$ flats containing G .

The same result holds for equicardinal clutters C with

the word flat replaced by "set contained in a member of C ", and so in particular, holds for the Independent sets of a matroid M . This theorem in itself, offers little hope towards the conjectures, but Conjecture 2C can be cast in a form which is very similar to Theorem 2.8.

The Conjecture claims that

$$k \frac{C_k}{C_{k-1}} \geq \frac{(k+1)C_{k+1}}{C_k}$$

This is a statement about the bipartite graph Γ_1 on $C_{k-1} \times C_k$ and Γ_2 on $C_k \times C_{k+1}$ with $(A, B) \in \Gamma_1$ if and only if $A \subset B$. Since each $A \in C_k$ has degree k in Γ_1 , and each $B \in C_{k+1}$ has degree $k+1$ in Γ_2 , the conjecture states that the average degree in Γ_1 of an element of C_{k-1} is not less than the average degree in Γ_2 of an element in C_k . This seems plausible for matroids because of their exchange property, although it is false for arbitrary equicardinal clutters. (See example 1)

3. MOTZKIN'S THEOREM AND RELATED RESULTS

Motzkin's "lemma" seems to have been generally overlooked, the only reference to it being in Greene [4]. The only application of the lemma that I know of, is the proof of the hyperplane theorem. This is surprising, because it appears to have the potential of a useful combinatorial technique. One obvious area to look for places where it could be used to simplify results, is in (λ, μ) -designs, which were motivated by the Erdos - De Bruijn paper. It turns out however that the basic results of Ryser and Woodall require only a trivial result in the same vein as Motzkin's, which is stated as a remark following Theorem 3.1.

In this section we will state and prove one of a number of

This contradiction completes the main result. The case of equality requires equality in every step of the argument, with $t = 0$, so $a_h = \alpha \bar{h}$ for each $h \in H$ and $(p, h) \notin \Gamma$ implies $\bar{p} = a_h = \alpha \bar{h}$.

The trivial Motzkin-like result, referred to earlier as being useful in the (λ, μ) design results, is as follows. If $\bar{p} \geq c$ for all $p \in P$, and $a = \text{ave } \{h \in H\} (\bar{h})$ then $am \geq cn$.

Corollary 3.2 (Motzkin) Let $\Gamma \subseteq P \times H$ be a bipartite graph satisfying

- (i) $(p, h) \notin \Gamma$ implies $\bar{p} \geq \bar{h}$
- (ii) $h \in H$ implies $\bar{h} < n$.

Then $m \geq n$.

Corollary 3.3 Let $\Gamma \subseteq P \times H$ be a bipartite graph satisfying

- (i) $(p, h) \notin \Gamma$ implies $\bar{p} \geq \bar{h} + \beta$
- (ii) $h \in H$ implies $\bar{h} < n$.

Then $m > n + \beta$.

Proof Let $\alpha = \min[h \in H, \bar{h} \neq 0] \{1 + \beta/\bar{h}\}$.

Now apply Theorem 3.1, to get $m \geq \alpha n > n + \beta$ since $\bar{h} < n$ for all $h \in H$.

Motzkin's proof of his result makes use of a version of Hall's theorem on partial transversals, and so it is reasonable to investigate the existence of matchings in the cases of equality in theorem 3.1. This leads us to

Theorem 3.3 Let $\Gamma \subseteq P \times H$ be a bipartite graph with $m = \alpha n$ satisfying

- (i) $(p, h) \notin \Gamma$ implies $\bar{p} = \alpha \bar{h}$

possible generalisations of Motzkin's lemma, together with a theorem giving the added information available in the case of equality.

We will be dealing with a bipartite graph $\Gamma \subseteq P \times H$ on finite disjoint sets P and H , and we shall always refer to m as $|P|$ and n as $|H|$. We also put

$$\begin{aligned}\bar{p} &= |\Gamma(p)| = |\{h \in H: (p, h) \in \Gamma\}| \\ \bar{h} &= |\Gamma(h)| = |\{p \in P: (p, h) \in \Gamma\}| \end{aligned}$$

We use Γ^c to describe the set of edges $P \times H \setminus \Gamma$.

Theorem 3.1 Let $\Gamma \subseteq P \times H$ be a bipartite graph satisfying $\bar{h} < n \forall h \in H$. Let $a_h = \text{ave}[p \notin \Gamma(h)](\bar{p})$ and suppose $a_h \geq \alpha \bar{h}$ for all $h \in H$, α a positive real. Then $m \geq \alpha n$, with equality only if $(p, h) \notin \Gamma$ implies $\bar{p} = \alpha \bar{h}$.

Proof Suppose, by way of contradiction, that $m < \alpha n$.

$$\text{Then } \frac{n}{m} > \frac{1}{\alpha} \geq \frac{\bar{h}}{a_h} \text{ for all } h \in H.$$

This means that

$$\frac{n - \bar{h}}{m - a_h} > \frac{n}{m} \text{ for all } h \in H.$$

Now let T be the set of $p \in P$ for which $\bar{p} = m$, and let $|T| = t$.

Then

$$\begin{aligned}n - t &= \sum_{p \in P - T} \frac{m - \bar{p}}{m - \bar{p}} \\ &= \sum_{p \in P - T} \frac{1}{h \notin \Gamma(p)} \frac{1}{m - \bar{p}} \\ &\geq \sum_{h \in H} \frac{n - \bar{h}}{m - a_h} > m \cdot \frac{n}{m} = n. \end{aligned}$$

- (ii) $\bar{h} < n \quad \forall h \in H$
 (iii) $\bar{p} < m \quad \forall p \in P$

Then Γ^C contains the union of m matchings of P into H such that every vertex of H is covered exactly n times.

Proof Form a new bipartite graph Δ with m copies of P and n copies of H . Each copy of an element p of P is joined in Δ to each of the n copies of each element of $\Gamma(p)$ in H . Thus Δ satisfies the hypotheses in place of Γ , with $\alpha = 1$. Once we know that the theorem is true for $\alpha = 1$, the general case follows immediately from our construction. We therefore assume $\alpha = 1$ and show that there is a matching in Γ^C between P and H .

Suppose not, and choose among all counterexamples, a bipartite graph Γ with minimal n . By Hall's theorem, there must exist a $t+1$ subset Z of P with $|\Gamma^C(Z)|$ a t subset of H . Clearly $t \neq 0$ since $\bar{p} < m$ for all p . Also $t \neq n-1$ since $\bar{h} < n$.

Now delete $\Gamma^C(Z)$ from H and Z from P , and let Δ be the restriction of Γ to $(P \setminus Z) \times H \setminus \Gamma^C(Z)$. In the new graph Δ , denote the new vertex degrees by p' and h' .

Then $p' \geq \bar{p} - t$ and $h' = \bar{h} - t + 1$.

It follows that $p' \geq h' - 1$. We can now apply corollary 3, since $h' < n - t + 1$, to get

$$m - t > (n - t + 1) - 1$$

which contradicts $m = n$. Thus Γ^C contains the required matching, and the theorem is proved.

It is too much to hope that such matchings should exist in general, as the following example shows.

Example 3.1 Choose $2 < 2s < t + 1$. Let $A = \{a_1, \dots, a_s\}$, $B = \{b_1, \dots, b_s\}$, $C = \{c_1, \dots, c_t\}$, $X = \{x_1, \dots, x_s\}$, $Y = \{y_1, \dots, y_{t+1}\}$ be pairwise disjoint sets. We put $P = Y \cup Y$ and $H = A \cup B \cup C$. Thus $m = 2s + t$ and $n = s + t + 1$. Γ is specified as

$$(Y \times A) \cup (Y \times B) \cup (X \times C) \\ \cup \{(x_i, a_i) : 1 \leq i \leq s\} \cup \{(x_i, b_i) : 1 \leq i \leq s\}$$

The degrees are given by

$$\bar{a}_i = \bar{b}_i = \bar{x}_i = t + 2, \quad \bar{y}_i = 2s, \quad \bar{c}_i = s$$

The hypotheses of theorem 3.3 are satisfied with $\alpha = 1$ but $m > n$ and neither Γ nor Γ^C contain a matching.

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SOME FAMILIES OF NON-HAMILTONIAN GRAPHS

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SUMMARY

We show how to construct $2n$ -connected regular m -valent non-Hamiltonian graphs for $n \geq 1$, $m \geq 2n + 1$, 3-connected graphs for $m \geq 4$, and 5-connected graphs for $m \geq 7$.

The connectedness is proved only for the $2n$ case. Similar methods can be used in the other cases.

Theorem

The edges of K_{m+1} can be partitioned to include $2(m - n) + 1$ sets each of n pairwise nonadjacent edges, for $n \geq 1$, $m \geq 2n - 1$.

Proof:

Consider first the case $m = 2n - 1$. K_{2n} is edge $(2n - 1)$ -colourable, (see [1]), and such a colouring clearly induces a partition of the edges with the required properties.

Now suppose $m = n + p - 1$, where $p > n$. Set $k = p - n$. Give K_{2p} an edge $(2p - 1)$ -colouring, and remove any k vertices. The resulting graph is K_{2p-k} with an edge colouring which induces a partition of the edges which has $2p - 1$ sets, each of which contains at least $p - k = n$ pairwise nonadjacent edges. Thus the theorem is true in this case.

$2n$ -CONNECTED GRAPHS

taking a K_{m+1} , where $m \geq 2n + 1$, we partition its edges as