A Bijective Proof of The Quintuple Product Identity

Sun Kim

Department of Mathematics, University of Illinois,

1409 West Green Street, Urbana, IL 61801, USA

sunkim2@illinois.edu

Abstract

We give a bijective proof of the quintuple product identity using bijective proofs of Jacobi's triple product identity and Euler's recurrence relation.

Key Words: quintuple product identity, Jacobi triple product identity, Euler's pentagonal number theorem, theta functions

2000 Mathematical Reviews Classification Numbers: 05A17, 11P81, 11P82.

1 Introduction

The quintuple product identity is stated in the form

$$\sum_{n=-\infty}^{\infty} q^{n(3n+1)/2}(x^{3n} - x^{-3n-1}) = \prod_{n=1}^{\infty}(1-xq^n)(1-q^n)(1-x^{-1}q^{n-1})(1-x^2q^{2n-1})(1-x^{-2}q^{2n-1}). \quad (1)$$

It can be presented in many different forms and various proofs have been given. But, (1) seems to be the form which appears most frequently. Shaun Cooper [2] gave a comprehensive survey of the work on the quintuple product identity, and classified and discussed all known proofs. For historical notes and detailed proofs, the reader is directed to [2].

Although at least 29 proofs of the quintuple product identity have been given, no direct combinatorial proof has yet been shown. J. Lepowsky and S. Milne set $q = uv^2$, $x = v^{-1}$ in (1) to obtain

$$\sum_{n=-\infty}^{\infty} u^{n(3n+1)/2} v^{n(3n-2)} - \sum_{n=-\infty}^{\infty} u^{n(3n+1)/2} v^{(n+1)(3n+1)}$$
$$= \prod_{n=1}^{\infty} (1 - u^n v^{2n-1})(1 - u^{n-1} v^{2n-1})(1 - u^n v^{2n})(1 - u^{2n-1} v^{4n-4})(1 - u^{2n-1} v^{2n}),$$

and they gave the following combinatorial interpretation:

> The excess of the number of partitions of (m, n) into an even number of distinct parts of the type $(a, 2a)$, $(b, 2b-1)$, $(c-1, 2c-1)$, $(2d-1, 4d-4)$, $(2e-1, 4e)$ over those into an odd number of parts is 1 or -1 if (m, n) is of the type $(r(3r+1)/2, r(3r-2))$ or $(r(3r+1)/2, (r+1)(3r+1))$, respectively, and 0 otherwise.

They remarked that a direct combinatorial proof of it can be given. However, Cooper [2] states that "this proof was never published and the notes are most likely now lost."

M. V. Subbaro and M. Vidyasagar [5] deduced the following identities from the quintuple product identity:

$$1 + \sum_{n=1}^{\infty} q^{3n^2} x^{3n-1} (xq^{2n} - x^{-1} q^{-2n})$$
$$= 1 + \sum_{n=1}^{\infty} (-1)^n x^n q^n (1 + qx)(1 + q^3 x) \cdots (1 + q^{2n-1} x)$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n} q^{n(n+1)}}{(1 + qx)(1 + q^3 x) \cdots (1 + q^{2n+1} x)}, \tag{2}$$

and Subbarao [6] gave a combinatorial proof of (2). In [2], Cooper mentioned that it is not a completely combinatorial proof of the quintuple product identity because a lot of algebraic rearrangements are required to derive (2).

Thus, the goal of this paper is to give a bijective proof of the quintuple product identity, especially in the form (1). We remark that the right hand side of (1) can be viewed as a product of two different forms of Jacobi's triple product identity

$$\sum_{n=-\infty}^{\infty} q^{n^2} x^n = \prod_{n=1}^{\infty} (1 + xq^{2n-1})(1 + x^{-1}q^{2n-1})(1 - q^{2n}). \tag{3}$$

This naturally suggests that we can apply two bijections of (3) in different forms. In order to complete the proof, we also employ a bijective proof of Euler's pentagonal number theorem in the form

$$\prod_{n=1}^{\infty} (1 - q^n)^{-1} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = 1.$$

In the next section, we first derive a combinatorial interpretation from (1), and present the aforementioned three bijective proofs. Lastly, we give a bijective proof of the quintuple product identity using them.

2 A bijective proof the quintuple product identity

Let $\mathcal{D}$ be the set of partitions into distinct positive parts, $\mathcal{D}_0$ be the set of partitions into distinct nonnegative parts and $\mathcal{O}$ be the set of partitions into distinct odd parts. The weight $|\pi|$ and the length $\ell(\pi)$ of a partition π denote the sum of the parts and the number of parts of π, respectively.

We can easily see that (1) has the following combinatorial interpretation by comparing the coefficients of $x^m q^N$ on each side of (1) :

Theorem 1. *The excess of the number of partitions of N into an even number of parts in the form*

$$N = \pi_1 + \pi_2 + \pi_3 + \sigma_1 + \sigma_2,$$

where $\pi_1, \pi_2 \in \mathcal{D}$, $\pi_3 \in \mathcal{D}_0$, $\sigma_1, \sigma_2 \in \mathcal{O}$ *and* $\ell(\pi_1) - \ell(\pi_3) + 2\ell(\sigma_1) - 2\ell(\sigma_2) = m$, *over those into an odd number of parts is* 1 *or* -1 *if* $(m, N) = (3n, n(3n+1)/2)$ *or* $(m, N) = (-3n-1, n(3n+1)/2)$, *respectively, and* 0 *otherwise.*

Before proving Theorem 1, we first introduce two combinatorial proofs of Jacobi's triple product identity. J. Zolnowsky [7] made the substitutions $q^2 = uv$, $x = -(u/v)^{1/2}$ in (3) to obtain

$$\prod_{n=1}^{\infty}(1-u^n v^{n-1})(1-u^{n-1}v^n)(1-u^n v^n) = 1 + \sum_{n=1}^{\infty}(-1)^n \big(u^{\frac{n(n+1)}{2}} v^{\frac{n(n-1)}{2}} + u^{\frac{n(n-1)}{2}} v^{\frac{n(n+1)}{2}}\big),$$

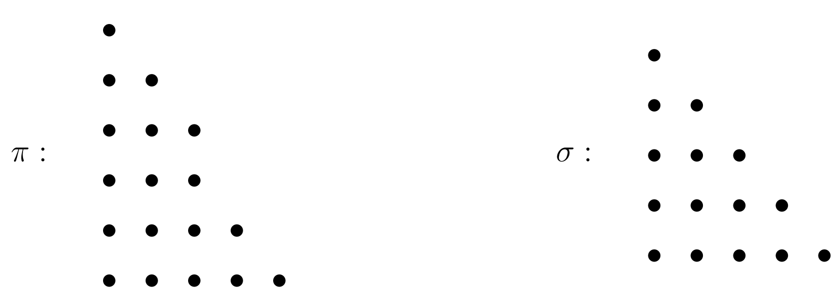
for which he gave a combinatorial proof. Using his bijection, we can also give a combinatorial proof of Jacobi's triple identity in the form

$$\sum_{n=-\infty}^{\infty}(-1)^n x^n q^{\frac{n(n+1)}{2}} = \prod_{n=1}^{\infty}(1-xq^n)(1-q^n)(1-x^{-1}q^{n-1}). \tag{4}$$

Comparing the coefficient of $x^m q^N$ on the both sides of (4), we obtain the following combinatorial interpretation.

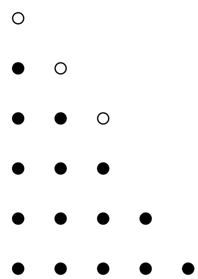
Theorem 2. *The excess of the number of partitions of N into an even number of parts in the form* $N = \tau_1 + \tau_2 + \tau_3$, *where* $\tau_1, \tau_2 \in \mathcal{D}$, $\tau_3 \in \mathcal{D}_0$ *and* $\ell(\tau_1) - \ell(\tau_3) = m$, *over those into an odd number of parts is* $(-1)^n$ *if* $(m, N) = (n, n(n+1)/2)$, *and* 0 *otherwise.*

For convenience, we follow Zolnowsky's notations and rules from [7]. We draw the Ferrers diagram of a partition placing parts left to right in decreasing order. For instance, the partitions $\pi = (6, 5, 4, 2, 1) \in \mathcal{D}$ and $\sigma = (5, 4, 3, 2, 1,) \in \mathcal{D}$ are represented as the following.

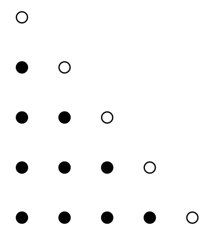


We define the slope of the diagrams to be the portion consisting of $\circ$ in the following graphs.

π :



σ :



Thus, the length of the slope is equal to the number of consecutive parts starting from the largest one. We say that the slope of a partition in $\mathcal{D}$ is nondetachable if the largest part is the same as the number of parts as in the graph of σ, and otherwise, we say the slope is detachable as in the graph of π. We define a slope of an empty partition to be nondetachable.

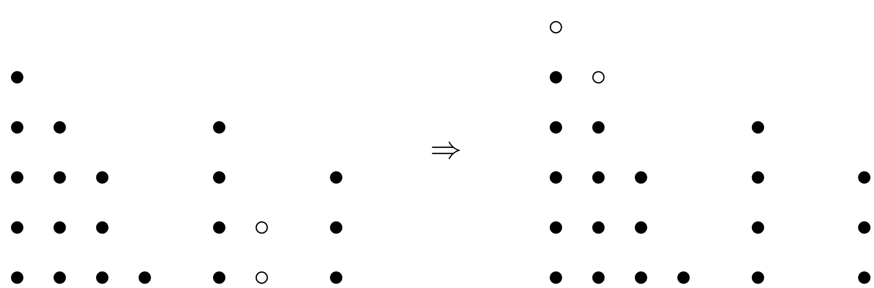
We can also define the slope of diagrams of partitions in $\mathcal{D}_0$ in a similar way. For example, the length of slope of $\pi = (5, 4, 3, 1, 0)$ is 3 and that of $\sigma = (4, 3, 2, 1, 0)$ is 5. Similarly, we say that the slope of π is detachable and the slope of σ is nondetachable. Note that if the slope of a partition $\in \mathcal{D}_0$ is nondetachable, then the largest part is the number of parts -1.

Proof of Theorem 2. First, we consider the case when $m \geq 0$, i.e. $\ell(\tau_1) \geq \ell(\tau_3)$.

Let LS denote the length of the slope of τ_1, HL designate the largest part of τ_1 (0 if τ_1 is empty) and HM and HR denote the smallest parts of τ_2 and τ_3, respectively (infinite if they are empty).

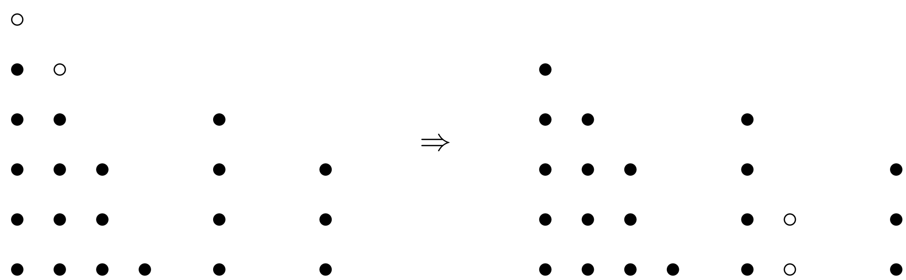
Case 1 : LS $\geq$ HM (Note that τ_2 is not empty.)

Move the least part of τ_2 onto the slope of τ_1 to create a new slope. For instance, $(5, 4, 3, 1) + (4, 2) + (3)$ corresponds to $(6, 5, 3, 1) + (4) + (3)$.



Case 2 : LS $<$ HM, and the slope is detachable.

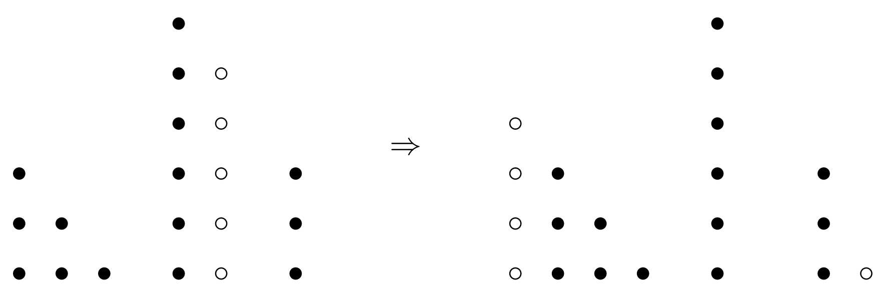
Remove the slope of τ_1 to create a new smallest part τ_2. For instance, $(6,5,3,1)+(4)+(3)$ corresponds to $(5,4,3,1)+(4,2)+(3)$.



Note that Case 1 and Case 2 correspond to each other.

Case 3 : LS $<$ HM, the slope is nondetachable, and HM $\leq$ HL+HR with nonempty τ_2.

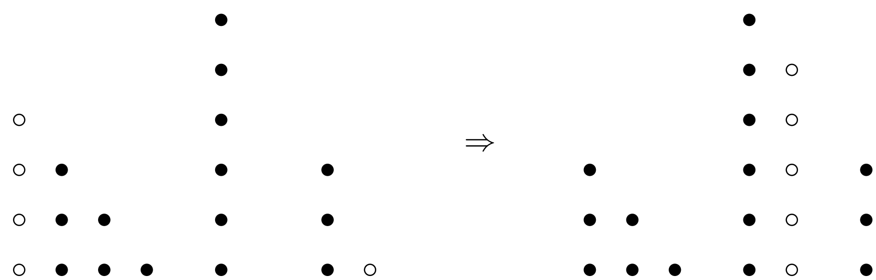
In this case, HM $>$ HL $=$ LS. Remove the smallest part of τ_2 to create a new largest part (=HL+1) and a new smallest part (since $0 \leq$ HM $-$ (HL $+1$) $<$ HR). For instance, $(3,2,1)+(6,5)+(3)$ corresponds to $(4,3,2,1)+(6)+(3,1)$.



Case 4 : LS $<$ HM, the slope is nondetachable, and HM $>$ HL+HR with nonempty τ_3. (Note that τ_1 is nonempty since $m \geq 0$.)

Add the largest part of τ_1 and the smallest part of τ_3 to form a new smallest part of τ_2.

For instance, $(4,3,2,1)+(6)+(3,1)$ corresponds to $(3,2,1)+(6,5)+(3)$.



Note that Case 3 and Case 4 correspond to each other. Also, note that all the four operations change the parity of partitions and none of the rules changes the condition $\ell(\tau_1) - \ell(\tau_3) = m$.

The bijection fails when the slope of τ_1 is nondetachable, and τ_2 and τ_3 are empty, i.e. for some $n \geq 0$,

$$N = \frac{n(n+1)}{2}, \qquad m = \ell(\tau_1) - \ell(\tau_3) = \ell(\tau_1) = n,$$

and the excess of the number of partitions of N into an even number of parts over those into an odd number parts is $(-1)^n$.

Now, consider the case when $m < 0$. In this case, we switch the roles of τ_1 and τ_3. In other words, LS is the length of the slope of τ_3, HL denotes the largest part of τ_3 and HM and HR designate the smallest parts of τ_2 and τ_1, respectively. Recall that if the slope of τ_3 is nondetachable, then LS= HL $+1$ (so, in Case 3, HM-(HL+1)≥ 1). Similarly, the bijection fails when τ_1 and τ_2 are empty and the slope of τ_3 is nondetachable, i.e. for some negative integer n,

$$m = \ell(\tau_1) - \ell(\tau_3) = -\ell(\tau_3) = n, \qquad N = 0 + 1 + \cdots + (-n-1) = \frac{n(n+1)}{2},$$

and the excess of the number of partitions of N into an even number of parts over those into an odd number parts is $(-1)^n$. Hence, we complete the proof. $\square$

Next, we introduce another combinatorial proof of Jacobi's triple product identity in the form

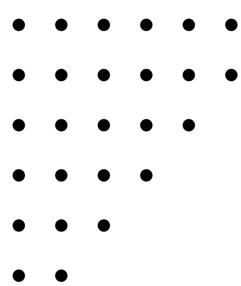
$$\prod_{n=1}^{\infty}(1+xq^{2n-1})(1+x^{-1}q^{2n-1}) = \Big(\sum_{n=-\infty}^{\infty} x^n q^{n^2}\Big)\Big(\prod_{n=1}^{\infty}(1-q^{2n})^{-1}\Big)$$
$$= \Big(\sum_{n=-\infty}^{\infty} x^n q^{n^2}\Big)\Big(\sum_{n=0}^{\infty} p_e(2n)q^{2n}\Big), \qquad (5)$$

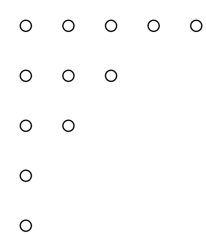
where $p_e(n)$ is the number of partitions of n into even parts. Comparing the coefficients of $x^k q^N$ on each side of (5), R. P. Lewis derived the following combinatorial interpretation and gave a bijective proof of it. We also present his proof here.

Theorem 3. *The number of partitions of N in the form $N = \pi + \sigma$, where $\pi, \sigma \in \mathcal{O}$ and $\ell(\pi) - \ell(\sigma) = k$ is equal to $p_e(N - k^2)$.*

Remark : Lewis proved Theorem 3 with $p((N-k^2)/2)$ instead of $p_e(N-k^2)$ in [4]. Theorem 3 implies that given a partition of $N = \pi + \sigma$ with $\pi, \sigma \in \mathcal{O}$ and $\ell(\pi) - \ell(\sigma) = k$, we can find a partition τ bijectively such that $N = k^2 + \tau$ and τ is a partition of $N - k^2$ into even parts.

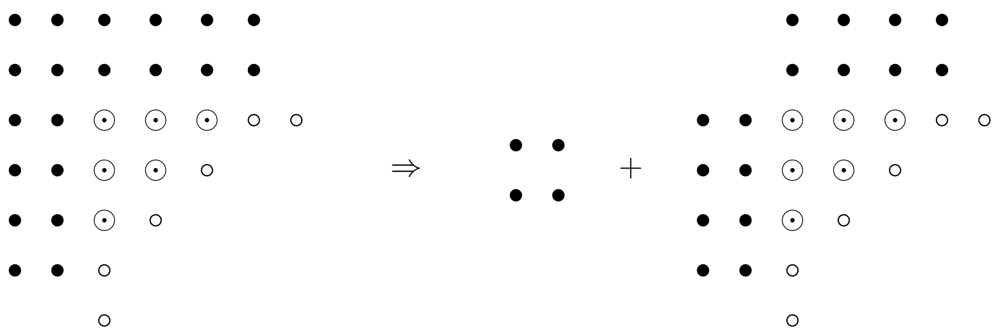
Proof. Let us consider the case when $k \geq 0$ only since we can exchange π and σ. Given $N = \pi + \sigma$ with $\pi, \sigma \in \mathcal{O}$ and $\ell(\pi) - \ell(\sigma) = k$, we draw the self-conjugate diagrams G_1 and G_2, respectively. For example, if $N = 38$, $\pi = (11, 9, 5, 1)$ and $\sigma = (9, 3)$, then

G_1 : 

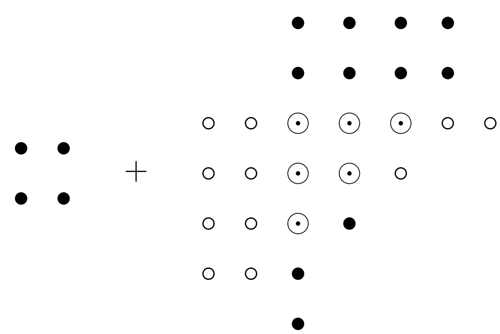
G_2 : 

Now, superimpose G_2 on G_1 with the top left corner of G_2 over the point $k+1$ places down the diagonal of G_1. And then, remove the top left square of size k^2. For our example, since

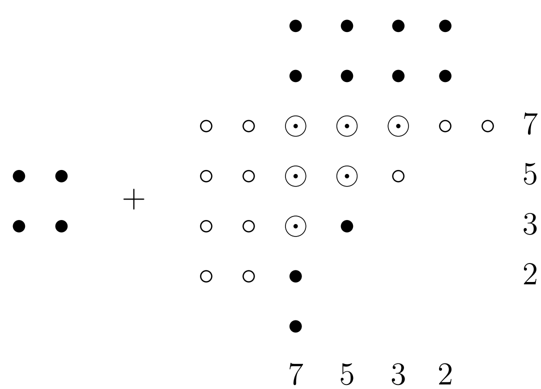
$k = 2$,



Lastly, switch • and ∘ below the diagonal of the diagram.



The new diagram is composed of the graph, drawn with •, of a partition of $(N - k^2)/2$ with the graph of its conjugate, drawn with ∘, superimposed.



Since we have the two same partitions of $(N - k^2)/2$, by multiplying each part by 2, we obtain a partition of $N - k^2$ into even parts. Thus, for our example, we obtain a partition

$14 + 10 + 6 + 4$ of $N - k^2 = 38 - 4 = 34$. This process is obviously reversible, so we complete the proof. □

Lastly, we introduce a bijective proof of Euler's recurrence relation by David M Bressoud and Doron Zeilberger [1]. From Euler's pentagonal number theorem in the form

$$\prod_{n=1}^{\infty}(1-q^n)^{-1}\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}} = \sum_{n=0}^{\infty} p(n)q^n \sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}} = 1, \tag{6}$$

we deduce the following theorem.

Theorem 4. *For* $n \geq 1$,

$$\sum_{i\ even} p(n - i(3i+1)/2) = \sum_{i\ odd} p(n - i(3i+1)/2),$$

where $i \in \mathbb{Z}$ *is allowed to be negative.*

For instance, if $n = 7$, then

$$\sum_{i\ even} p(n - i(3i+1)/2) = p(7) + p(2) + p(0) = 15 + 2 + 1 = 18,$$

$$\sum_{i\ odd} p(n - i(3i+1)/2) = p(6) + p(5) = 11 + 7 = 18.$$

Proof of Theorem 4. Let $a(i) = i(3i+1)/2$. Define the map γ by the following rule:
for a partition $\lambda : n - a(i) = \lambda_1 + \lambda_2 + \cdots + \lambda_t$,

$$\gamma(\lambda) = \begin{cases} \lambda' : n - a(i-1) = (t + 3i - 1) + (\lambda_1 - 1) + \cdots + (\lambda_t - 1) & \text{if } t + 3i \geq \lambda_1 \\ \\ \lambda' : n - a(i+1) = (\lambda_2 + 1) + \cdots + (\lambda_t + 1) + \underbrace{1 + \cdots + 1}_{\lambda_1 - t - 3i - 1} & \text{if } t + 3i < \lambda_1. \end{cases}$$

It is not hard to see that γ is an involution, so we complete the proof. □

Now, let us use three bijections that we showed above to prove Theorem 1.

Proof of Theorem 1. First, fix $\sigma_1,\ \sigma_2 \in \mathcal{O}$, and say $|\sigma_1| + |\sigma_2| = M$. Now, consider all the partitions $\pi_1 + \pi_2 + \pi_3$ of $N - M$ with $\pi_1, \pi_2 \in \mathcal{D}$, $\pi_3 \in \mathcal{D}_0$ and $\ell(\pi_1) - \ell(\pi_3) = m - 2(\ell(\sigma_1) - \ell(\sigma_2))$ so that $N = \pi_1 + \pi_2 + \pi_3 + \sigma_1 + \sigma_2$. By the bijective proof of Theorem 2, the excess of the number of partitions of N into an even number of parts in the form $N = \pi_1 + \pi_2 + \pi_3 + \sigma_1 + \sigma_2$, over those into an odd number of parts (with fixed σ_1 and σ_2) is nonzero only when $\pi_1 = 1 + \cdots + t$, $t \geq 0$ and $\pi_2 = \pi_3 = \emptyset$ or $\pi_3 = 0 + 1 + + \cdots + (-t-1)$, $t < 0$, and $\pi_1 = \pi_2 = \emptyset$.

Thus, we only have to consider the partitions of the form

$$N = 1 + \cdots + t + \sigma_1 + \sigma_2,\ t \geq 0, \qquad N = 0 + 1 + \cdots + (-t-1) + \sigma_1 + \sigma_2,\ t < 0,$$

where $\sigma_1, \sigma_2 \in \mathcal{O}$ and $2(\ell(\sigma_1) - \ell(\sigma_2)) = m - t$. By the bijection described in Theorem 3, each pair (σ_1, σ_2) corresponds to $(\ell(\sigma_1) - \ell(\sigma_2))^2 + \tau$, where τ is a partition of $N - t(t+1)/2 - (\ell(\sigma_1) - \ell(\sigma_2))^2$ into even parts. Thus, each partition of N of the form

$$\mu : N = t(t+1)/2 + \sigma_1 + \sigma_2, \quad t \in \mathbb{Z}, \tag{7}$$

is bijectively associated with

$$\mu' : N = t(t+1)/2 + (\ell(\sigma_1) - \ell(\sigma_2))^2 + \tau.$$

We consider three difference cases when $m \equiv 0, 1$ or $-1 \pmod 3$.

Case 1: $m = 3n$, $n \in \mathbb{Z}$.

Let $\ell(\sigma_1) - \ell(\sigma_2) = r$, then $t + 2r = 3n$ and $\ell(\mu) = t + \ell(\sigma_1) + \ell(\sigma_2) \equiv t + r \equiv n - r \pmod 2$. Also,

$$\mu' : N = \frac{t(t+1)}{2} + r^2 + \tau = \frac{n(3n+1)}{2} + 3(n-r)^2 + (n-r) + \tau. \tag{8}$$

So, if $N = n(3n+1)/2$, then we have $n = r = t$ and $|\tau| = 0$. Thus, the only possibilities for σ_1 and σ_2 for μ are $\sigma_1 = 1 + 3 + \cdots + 2n - 1$ and $\sigma_2 = \emptyset$ if $n \geq 0$, and $\sigma_1 = \emptyset$ and $\sigma_2 = 1 + 3 + \cdots + (-2n-1)$ if $n < 0$, since $n = r = \ell(\sigma_1) - \ell(\sigma_2)$. Considering $\ell(\mu) \equiv 2n \pmod 2$, we can see that the excess of the number of partitions of N into an even number of parts over those into odd number of parts in the form satisfying the condition of our theorem is 1.

Now, suppose $N \neq n(3n+1)/2$. Then, $L := N - n(3n+1)/2 \geq 1$ by (8). By the bijective relations between the solutions of μ and μ', the excess of the number of solutions of μ with $\ell(\mu)$ even over those with $\ell(\mu)$ odd is equal to the excess of the number of partitions of $L-(3(n-r)^2+(n-r))$ into even parts with $n-r$ even over the number of partitions of $L-(3(n-r)^2+(n-r))$ into even parts with $n-r$ odd since $\ell(\mu) \equiv n-r \pmod 2$. Using the fact that the number of partitions of a number a into even parts is equal to the number of partitions of $a/2$ and the bijection described in Theorem 4, we complete the proof of Case 1, because the previously described excess is equal to 0.

Case 2: $m = -3n-1$, $n \in \mathbb{Z}$.

If $\ell(\sigma_1) - \ell(\sigma_2) = r$, then $t + 2r = -3n-1$, $\ell(\mu) \equiv t + r \equiv n + r + 1 \pmod 2$ and

$$\mu' : N = \frac{t(t+1)}{2} + r^2 + \tau = \frac{n(3n+1)}{2} + 3(n+r)^2 + (n+r) + \tau.$$

Similarly, if $N = n(3n+1)/2$, then we have $n = -r = -t-1$ and $|\tau| = 0$. Since $|\sigma_1| + |\sigma_2| = N - t(t+1)/2 = (t+1)^2$ by (7) and $\ell(\sigma_1) - \ell(\sigma_2) = t+1$, we have $\sigma_1 = 1+3+\cdots+2t+1$ and $\sigma_2 = \emptyset$ if $t \geq -1$, and $\sigma_1 = \emptyset$ and $\sigma_2 = 1+3+\cdots+(-2t-3)$ if $t < -1$. Considering $\ell(\mu) \equiv 2t+1 \pmod 2$, we complete the proof when $N = n(3n+1)/2$.

By the same argument as in Case 2, we can also prove the theorem when $N \neq n(3n+1)/2$. (The only difference is that $\ell(\mu)$ has the opposite parity of $n+r$)

Case 3: $m = 3n+1$, $n \in \mathbb{Z}$.

Similarly, letting $\ell(\sigma_1) - \ell(\sigma_2) = r$, we have $t + 2r = 3n+1$, $\ell(\mu) \equiv t + r \equiv n - r + 1 \pmod 2$ and

$$\mu' : N = \frac{t(t+1)}{2} + r^2 + \tau = \frac{3n^2+3n+2}{2} + 3(n-r)(n-r+1) + \tau.$$

Let $L = N - (3n^2+3n+2)/2$. Then, the excess of the number of solutions of μ with $\ell(\mu)$ even over those with $\ell(\mu)$ odd is equal to the excess of the number of partitions of $L-(3(n-r)(n-r+1))$ into even parts with $n-r$ odd over those with $n-r$ even, which is

0, since $(n-r)(n-r+1) = \{-(n-r)-1\}\{(-(n-r)-1+1)\}$, and $n-r$ and $-(n-r)-1$ has opposite parity. Note that $L=0$ is not an exceptional case since $(n-r)(n-r+1)=0$ when $n-r=0$ or $n-r+1=0$. □

References

[1] D. M. Bressoud and D. Zeilberger, *Bijecting Euler's partitions-recurrence*, Amer. Math. Monthly **92** (1) (1985), 54–55.

[2] S. Cooper, *The Quintuple product identity*, Int. J. Number Theory **2** (2006), 115–161.

[3] J. Lepowsky and S. Milne, *Lie algebraic approaches to classical partition identities*, Adv. Math. **29** (1978), 15–59.

[4] R. P. Lewis, *A combinatorial proof of the Jacobi triple product identity*, Amer. Math. Monthly **91** (1984), 420–423.

[5] M. V. Subbarao and M. Vidyasagar, *On Watson's quintuple product identity*, Proc. Amer. Math. Soc. **26** (1970), 23–27.

[6] M. V. Subbarao, *Combinatorial proofs of some identities*, in Proceedings of the Washington State University Conference on Number Theory (Washington State Univ., Pullman, Washington, 1971), pp. 80–91.

[7] J. Zolnowsky, *A direct combinatorial proof of the Jacobi identity*, Discrete Math. **9** (1974), 293–298.