Catwalks, Sandsteps & Pascal Pyramids

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Bill Sands [**9**, where the problem is stated without the result (1)] noticed that the number of different walks of n steps between lattice points, each in a direction N, S, E or W, starting from the origin and remaining in the upper half-plane, is

$$w_n = \binom{2n+1}{n} \tag{1}$$

and asked for a neat proof. What is wanted is a simple "choice" argument: any offers? This is sequence 1144 in [**10**]: my first attempt at any sort of proof was by induction from the formula

$$w_n = 4w_{n-1} - c_n \tag{2}$$

since a walk of length n is one more step in one of the four directions N, S, E or W, than a walk of length $n-1$, except that a southerly step is not allowed if the walk of length $n-1$ terminated on the x-axis, and it's well known that the number of such walks is the n-th Catalan number.

But it's *not* well known! It doesn't occur among the 31 manifestations listed by Kuchinski [**6**], nor can we immediately see any simple correspondence between the walks and any of the manifestations. However, first let's assume that it's true, and that (1) holds with $n-1$ in place of n. Then

$$\begin{aligned} 4w_{n-1} - c_n &= 4\binom{2n-1}{n-1} - \frac{1}{n+1}\binom{2n}{n} \\ &= \frac{4(2n-1)!}{n!(n-1)!} - \frac{(2n)!}{n!(n+1)!} \\ &= \frac{(2n-1)!}{n!(n+1)!}\{4n(n+1) - 2n\} \\ &= \frac{(2n)!(2n+1)}{n!(n+1)!} = \binom{2n+1}{n} \end{aligned}$$

What **is** well known is that the number of walks of $2n$ steps, each N or E, from (0,0) to (n,n), which don't cross the diagonal $y=x$, or the number of walks of $2n+2$ steps from (0,0) to $(n+1,n+1)$ which stay strictly above the diagonal, is c_n, the n-th Catalan number. This is clearly the same as the number of walks of $2n$ steps on the positive x-axis, starting and finishing at the origin.

Let us look at this one-dimensional analog of the Sands problem. We can exhibit the numbers of walks, $w(n,x)$, of n unit steps, starting at (0,0) and ending at $(x,0)$, $x \geq 0$, in a "Pascal semi-triangle" (Fig. 1).

Columns $x=0$ and $x=1$ contain the Catalan numbers, as already earnested; column $x=3$ also occurs in connexion with partitioning a polygon [**1**]. Columns $x = 2, 4, 6, 8, 10, 12$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	k	Total
0	1																	0	1
1		1																0	1
2	1		1															1	2
3		2		1														1	3
4	2		3		1													2	6
5		5		4		1												2	10
6	5		9		5		1											3	20
7		14		14		6		1										3	35
8	14		28		20		7		1									4	70
9		42		48		27		8		1								4	126
10	42		90		75		35		9		1							5	252
11		132		165		110		44		10		1						5	462
12	132		297		275		154		54		11		1					6	924
13		429		572		429		208		65		12		1				6	1716
14	429		1001		1001		637		273		77		13		1			7	3432
15		1430		2002		1638		910		350		90		14		1		7	6435
16	1430		3432		3640		2548		1260		440		104		15		1	8	12870

Figure 1: Numbers of walks, $w(n,x)$, on the positive x-axis.

are sequences 1130, 1602, 1866, 1981, 2048, 2104 in Sloane [**10**]: they are Laplace transform coefficients: more precisely, $w(2n,2k)$ is denoted in [**7**] by C_k, which is defined by:

$$(2\cos\theta)^{2n}\sin\theta = \sum_{k=0}^{n} C_k \sin(2k+1)\theta \tag{3}$$

Presumably there is an analogous formula for $w(2n+1,2k+1)$; compare equation (11) below.

The first table in Cayley's paper [**1**] is for the number of partitions of an r-gon into k parts by non-intersecting diagonals. His column $k=1$ is our main diagonal, and his column $k=2$ is our third diagonal (starting at $(n,x)=(4,0)$). More generally, his column k is our diagonal starting at $(2k,0)$, except that his entries contain an extra factor $(x+k-2)!/(x+1)!(k-2)!$, a generalized Catalan number: in fact, for $x=k-2$ it is c_{k-2}. Cayley attributes his results to [**4**] and [**11**]: the latter paper gives some history, mentioning Terquem, Lamé, Rodrigues, Binet & Catalan.

A near miss for column $x=9$ is Stirling numbers of the second kind, $\mathcal{S}_n^{(4)}$, but the entries (not shown here) for rows 17, 19 & 21 are deficient by 1, 18 & 190 respectively.

We omit zero values of $w(n,x)$ from our table: it's fairly obvious that $w(n,x)=0$ if n and x are of opposite parity, or if $x>n$. It's not too difficult to find formulas for the first few diagonals:

$$w(n,n)=1, w(n,n-2)=n-1, w(n,n-4)=\frac{1}{2}n(n-3), w(n,n-6)=\frac{1}{6}n(n-1)(n-5)$$

In fact there is a comparatively simple formula for all the entries in Figure 1:

$$w(n,x)=\binom{n}{r}-\binom{n}{r-1} \tag{4}$$

where $r=\frac{1}{2}(n-x)$. Indeed, the formula (4) also works in the apocryphal cases mentioned above, if we take the reasonable interpretation that $\binom{n}{r}=0$ if $r<0$, or if $n<r$, or if r is not

an integer. We shall do this: note that the usual formulas, such as

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \tag{5}$$

and (13) still hold in these cases. Formula (4) is easily proved by induction, since

$$\begin{aligned} w(n,x) &= w(n-1,x-1) + w(n-1,x+1) \\ &= \binom{n-1}{r} - \binom{n-1}{r-1} + \binom{n-1}{r-1} - \binom{n-1}{r-2} \\ &= \binom{n}{r} - \binom{n}{r-1} \end{aligned}$$

The well known result that we mentioned is the special case

$$w(2n,0) = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n} = c_n$$

The total number, $w(n)$, of walks of length n is

$$\begin{aligned} w(n,n) &+ w(n,n-2) + w(n,n-4) + \dots \\ &= \left[\binom{n}{0} - \binom{n}{-1}\right] + \left[\binom{n}{1} - \binom{n}{0}\right] + \dots + \left[\binom{n}{k} - \binom{n}{k-1}\right] = \binom{n}{k} \end{aligned}$$

where $n - 2k = 0$ or 1 according as n is even or odd: i.e.

$$\binom{2k}{k} \quad \text{or} \quad \binom{2k+1}{k}.$$

Here it is clear that the number of walks of even length is just twice the number of walks of (odd) length one less:

$$2\binom{2k+1}{k} = \frac{2(k+1)(2k+1)!}{(k+1)k!(k+1)!} = \binom{2k+2}{k+1}.$$

Is there a simple "choice" argument for walks of odd length? If you "know" the Catalan number result, then we can use a device similar to formula (2):

$$\begin{aligned} w(2k+1) &= 2w(2k) - c_k \\ &= 2\binom{2k}{k} - \frac{1}{k+1}\binom{2k}{k} \\ &= \frac{2k+1}{k+1}\binom{2k}{k} = \binom{2k+1}{k} \end{aligned} \tag{6}$$

but this has an air of circularity about it, or at best may be using a sledgehammer to crack a nut.

Return to the original problem: we can solve it if we go into more detail than most people would deem desirable. The numbers, $w_n(x,y)$, of walks of n steps from (0,0) to (x,y), which remain in the half-plane $y \geq 0$, may be exhibited in a "Pascal semi-pyramid" whose layers are shown in Fig. 2.

If we sum the rows in the layers of Fig. 2 we obtain the numbers, $w_n(y)$, of walks of n steps which start at (0,0) and end at distance y from the x-axis. These are shown in Fig. 3. We shall see that a special case is, as we have already earnested,

$$w_n(0) = c_{n+1}. \tag{7}$$

$n=0$:

1

$n=1$:

1
1 1

$n=2$:

1
2 2
1 3 1

$n=3$:

1
3 3
3 8 3
1 6 6 1

$n=4$:

1
4 4
6 15 6
4 20 20 4
1 10 20 10 1

$n=5$:

1
5 5
10 24 10
10 45 45 10
5 40 75 40 5
1 15 50 50 15 1

$n=6$:

1
6 6
15 35 15
20 84 84 20
15 105 189 105 15
6 70 210 210 70 6
1 21 105 175 105 21 1

$n=7$:

1
7 7
21 48 21
35 140 140 35
35 224 392 224 35
21 210 588 588 210 21
7 112 490 784 490 112 7
1 28 196 490 490 196 28 1

Figure 2: Layers of a Pascal semi-pyramid: values of $w_n(x, y)$.

n	w_n	$w_n(0)$	$w_n(1)$	$w_n(2)$	$w_n(3)$	$w_n(4)$	$w_n(5)$	$w_n(6)$	$w_n(7)$	$w_n(8)$	$w_n(9)$	$w_n(10)$
0	1	1										
1	3	2	1									
2	10	5	4	1								
3	35	14	14	6	1							
4	126	42	48	27	8	1						
5	462	132	165	110	44	10	1					
6	1716	429	572	429	208	65	12	1				
7	6435	1430	2002	1638	910	350	90	14	1			
8	24310	4862	7072	6188	3808	1700	544	119	16	1		
9	92378	16796	25194	23256	15504	7752	2907	798	152	18	1	
10	352712	58786	90440	87210	62016	33915	14364	4655	1120	189	20	1

Figure 3: Sums of rows of Fig. 2: values of $w_n(y)$.

In turn, the row sums of Fig. 3 are the total numbers, w_n, of Sands-type walks of length n. They are listed in column two of Fig. 3, and we will confirm another of our earlier statements:

$$w_n = \binom{2n+1}{n}. \tag{8}$$

At risk of losing some interesting heuristics, we again leap to the conclusion

$$w_n(x, y) = \binom{n}{r}\binom{n}{s} - \binom{n}{r-1}\binom{n}{s-1} \tag{9}$$

where $r = \frac{1}{2}(n + x - y)$, $s = \frac{1}{2}(n - x - y)$.

The obvious symmetry $w_n(x, y) = w_n(-x, y)$ is reflected in formula (9), since changing the sign of x is equivalent to interchanging r and s. It is also clear that

(a) if $n + x + y$ is odd, then r, s are not integers, and

(b) if $|x| + y > n$, then at least one of r, s is negative,

so that in either of these cases,

$$w_n(x, y) = 0.$$

We can prove (9) inductively from the recursion (10), which states that the last step was either N, S, E or W:

$$w_n(x,y) = w_{n-1}(x,y-1) + w_{n-1}(x,y+1) + w_{n-1}(x-1,y) + w_{n-1}(x+1,y) \tag{10}$$

Notice that the sums of the three arguments in the five terms are all of the same parity. If this is odd, then all the terms are zero. But if (r,s) are integers, then the corresponding values for the four terms on the right of (10) are

(r,s) $\quad(r-1,s-1)$ $\quad(r-1,s)$ $\quad(r,s-1)$

and if we assume that formula (9) holds with $n-1$ in place of n, then (10) yields

$$\begin{aligned} w_n(x,y) = & \binom{n-1}{r}\binom{n-1}{s} - \binom{n-1}{r-1}\binom{n-1}{s-1} + \binom{n-1}{r-1}\binom{n-1}{s-1} - \binom{n-1}{r-2}\binom{n-1}{s-2} \\ & + \binom{n-1}{r-1}\binom{n-1}{s} - \binom{n-1}{r-2}\binom{n-1}{s-1} + \binom{n-1}{r}\binom{n-1}{s-1} - \binom{n-1}{r-1}\binom{n-1}{s-2} \end{aligned}$$

which becomes formula (9) after some more or less tedious manipulation, depending on one's ingenuity or symbol manipulator.

To find $w_n(y)$, sum (9) over x:

$$\begin{aligned} w_n(y) &= \sum_{x=-n+y}^{x=n-y} w_n(x,y) = \sum_{r=0}^{2(n-y)} w_n(x,y) \\ &= \left[\binom{n}{0}\binom{n}{n-y} + \binom{n}{1}\binom{n}{n-y-1} + \ldots + \binom{n}{n-y}\binom{n}{0}\right] - \\ & \quad \left[\binom{n}{-1}\binom{n}{n-y-1} + \binom{n}{0}\binom{n}{n-y-2} + \ldots + \binom{n}{n-y-2}\binom{n}{0}\right] \end{aligned}$$

The two brackets are the coefficients of t^{n-y} and of t^{n-y-2} in the expansion of $(1+t)^n(1+t)^n$, so that

$$w_n(y) = \binom{2n}{n-y} - \binom{2n}{n-y-2}$$

which may be rewritten as

$$w_n(y) = \binom{2n+1}{n-y} - \binom{2n+1}{n-y-1}. \tag{11}$$

On comparing this with (4) we see that

$$w_n(y) = w(2n+1, 2y+1),$$

the number of odd length one-dimensional walks which finish, of course, at an odd distance from the origin.

In particular, (7) is the same as the number of walks from (0,0) to $(n, n+1)$ which begin with a northward step and do not cross the line joining start to finish, $w(2n+1,1) =$

$$\begin{aligned} w_n(0) &= \binom{2n+1}{n} - \binom{2n+1}{n-1} = \frac{(2n+1)!}{n!(n+2)!}(n+2-n) \\ &= \frac{2(n+1)(2n+1)!}{(n+1)!(n+2)!} = \frac{1}{n+2}\binom{2n+2}{n+1} = c_{n+1}, \end{aligned}$$

1

1	
	1

1		
	2	
2		1

1			
	3		
5		3	
	5		1

1				
	4			
9		6		
	16		4	
10		9		1

1					
	5				
14		10			
	35		10		
35		35		5	
	35		14		1

1						
	6					
20		15				
	64		20			
84		90		15		
	140		64		6	
70		84		20		1

1							
	7						
27		21					
	105		35				
168		189		35			
	378		189		21		
294		378		105		7	
	294		168		27		1

1								
	8							
35		28						
	160		56					
300		350		70				
	840		448		56			
840		1134		350		28		
	1344		840		160		8	
588		840		300		35		1

Figure 4: Layers of a Pascal quarter-pyramid: values of $w'_n(x,y)$.

the $(n+1)$th Catalan number.

Finally, summing (11) from $y=0$ to $y=n$, gives (8).

We could ask similar questions concerning walks which do not stray outside the positive quadrant. The numbers of such walks now form a "Pascal quarter-pyramid", which is exhibited in Fig. 4.

The entries in Fig. 4 are given, again without motivation, by

$$w'_n(x,y) = \binom{n}{r}\binom{n+2}{s} - \binom{n+2}{r+1}\binom{n}{s-1} \tag{12}$$

where $r=\frac{1}{2}(n+x-y)$, $s=\frac{1}{2}(n-x-y)$ as before. Notice that interchange of x and y keeps s fixed and replaces r by $n-r$. So the symmetry

$$w'_n(y,x) = w'_n(x,y)$$

follows from the symmetries

$$\binom{n}{r} = \binom{n}{n-r} \quad\text{and}\quad \binom{n+2}{r+1} = \binom{n+2}{n-r+1}. \tag{13}$$

We may prove (12) as we proved (9), since $w'_n(x,y)$ also satisfies the relation (10).

A remarkable coincidence is that

$$w'_{2k-1}(0,1) = \frac{1}{2}c_k c_{k+1}$$

is the number of inequivalent Hamiltonian rooted maps on $2k$ vertices (sequence 1647 in [**10**]) although Tutte [**12**] doesn't give the formula in terms of Catalan numbers. Is there yet another opportunity for a pure combinatorial proof?

Figure 5 is obtained by summing the rows of Fig. 4, and we may find $w'_n(y)$, the number of walks in the positive quadrant which finish at distance y from the x-axis, by summing (12)

n	$w'_n(0)$	$w'_n(1)$	$w'_n(2)$	$w'_n(3)$	$w'_n(4)$	$w'_n(5)$	$w'_n(6)$	$w'_n(7)$	$w'_n(8)$	$w'_n(9)$
0	1									
1	1	1								
2	3	2	1							
3	6	8	3	1						
4	20	20	15	4	1					
5	50	75	45	24	5	1				
6	175	210	189	84	35	6	1			
7	490	784	588	392	140	48	7	1		
8	1764	2352	2352	1344	720	216	63	8	1	
9	5292	8820	7560	5760	2700	1215	315	80	9	1

Figure 5: Sums of rows of Fig. 4: values of $w'_n(y)$.

from $x = 0$ to $x = n - y$.

$$\begin{aligned} w'_n(y) &= \left[\binom{n}{n-y}\binom{n+2}{0} - \binom{n+2}{n-y+1}\binom{n}{-1}\right] \\ &\quad + \left[\binom{n}{n-y-1}\binom{n+2}{1} - \binom{n+2}{n-y}\binom{n}{0}\right] \\ +\ldots &+ \left[\binom{n}{\frac{1}{2}(n-y)}\binom{n+2}{\frac{1}{2}(n-y)} - \binom{n+2}{\frac{1}{2}(n-y)+1}\binom{n}{\frac{1}{2}(n-y)-1}\right] \end{aligned}$$

if $n - y$ is even, but with the last term replaced by

$$\left[\binom{n}{\frac{1}{2}(n-y+1)}\binom{n+2}{\frac{1}{2}(n-y-1)} - \binom{n+2}{\frac{1}{2}(n-y+3)}\binom{n}{\frac{1}{2}(n-y-3)}\right]$$

if $n - y$ is odd.

Put $n - y = 2k$ or $2k + 1$ and

$$w'_n(y) = \begin{cases} \binom{n+1}{k}\binom{n}{k} - \binom{n+1}{k}\binom{n}{k-1} & \text{if } n-y=2k \\ \binom{n+1}{k}\binom{n}{k+1} - \binom{n+1}{k+1}\binom{n}{k-1} & \text{if } n-y=2k+1 \end{cases}$$

In particular, if $y = 0$,

$$\begin{aligned} w'_n(0) &= \frac{1}{k+1}\binom{2k}{k}\binom{2k+1}{k} \quad \text{or} \quad \frac{1}{k+2}\binom{2k+2}{k+1}\binom{2k+1}{k} \\ \text{i.e.} &\quad \binom{2k+1}{k}c_k \quad \text{or} \quad \binom{2k+1}{k}c_{k+1} \end{aligned}$$

according as $n = 2k$ or $n = 2k + 1$, where c_k is the k-th Catalan number.

For walks in the positive quadrant it's more natural and symmetrical to ask for the numbers of walks which terminate at various distances from the origin, using the "Manhattan metric", $x + y = n - 2s$. Figure 6 shows the sums of the diagonals of Fig. 4.

The entries in Fig. 6 are

$$\begin{aligned} w''_n(x+y) &= w''_n(n-2s) = \sum_{x+y=n-2s} w'_n(x,y) \\ &= \binom{n+2}{s}\left[\binom{n}{s} + \binom{n}{s+1} + \ldots + \binom{n}{n-s}\right] - \\ &\quad \binom{n}{s-1}\left[\binom{n+2}{s+1} + \binom{n+2}{s+2} + \ldots + \binom{n+2}{n-s+1}\right] \end{aligned}$$

w'_n	n	0	1	2	3	4	5	6	7	8	9	10	11
1	0	1											
2	1		2										
6	2	2		4									
18	3		10		8								
60	4	10		34		16							
200	5		70		98		32						
700	6	70		308		258		64					
2450	7		588		1092		642		128				
8820	8	588		3024		3414		1538		256			
31752	9		5544		12276		9834		3586		512		
116424	10	5544		31680		43230		26752		8194		1024	
426888	11		56628		141570		138424		69784		18434		2048

Figure 6: Sums of diagonals of Fig.4: values of $w''_n(x+y)$.

Except for small values of s, the truncated binomial expansions do not seem to have a simple closed form:

$$\begin{aligned}
w''_n(n) &= 2^n \\
w''_n(n-2) &= (n-2)2^n + 2 \\
w''_n(n-4) &= \frac{1}{2!}(n^2-5n+2)2^n + n^2 + 3n - 2 \\
w''_n(n-6) &= \frac{1}{3!}[n(n^2-9n+14)2^n + n(n^3+4n^2-n-28)] \\
w''_n(n-8) &= \frac{1}{4!}n(n-1)(n^2-13n+34)2^n + \frac{1}{72}n(n-1)(n^4+4n^3-n^2-64n-204)
\end{aligned}$$

An amusing curiosity is that $w''_n(n-2)$ is twice the genus of the $(n+2)$-dimensional cube [**8**, or see Theorem 14 in **3**].

The total number of walks, w'_n, the left hand column in Fig. 6, has, on the other hand, the comparatively simple formula

$$w'_n = \binom{n}{\lfloor n/2 \rfloor}\binom{n+1}{\lfloor (n+1)/2 \rfloor} \tag{14}$$

which again seems to beg for a simple proof.

If there is no restriction on the two-dimensional walks, i.e. if they may wander on either side of the x- and y-axes, then it is fairly easy to see that their number of length n, from (0,0) to (x, y), is

$$\binom{n}{r}\binom{n}{s} \tag{15}$$

where r and s are as before, but calculated using the absolute values of x and y.

Of course, the total number of walks of length n is 4^n.

Although we certainly haven't found the most aesthetic proofs, the comparative simplicity of the final results tempts us to ask what happens in three dimensions. Let $W_n(x, y, z)$ be the number of walks of n steps, each in a direction N, S, E, W, up, or down, from (0,0,0) to (x, y, z), which never go below the (x, y)-plane. We will not attempt to depict the four-dimensional "Pascal semi-pyramid", but the sums of its layers now give $W_n(z)$, the number of walks terminating at height z above the (x, y)-plane, and this satisfies the recurrence

$$W_n(z) = W_{n-1}(z-1) + 4W_{n-1}(z) + W_{n-1}(z+1) \tag{16}$$

which may be used to produce the array of Fig. 7.

Each entry in Fig. 7 is the sum of four times the entry immediately above it and the two neighbors of that entry, e.g.

$$W_5(2) = 4 \cdot 99 + 288 + 16 = 700.$$

W_n	n	0	1	2	3	4	5	6	7	8	9
1	0	1									
5	1	4	1								
26	2	17	8	1							
139	3	76	50	12	1						
758	4	354	288	99	16	1					
4194	5	1704	1605	700	164	20	1				
23460	6	8421	8824	4569	1376	245	24	1			
132339	7	42508	48286	28476	10318	2380	342	28	1		
751526	8	218318	264128	172508	72128	20180	3776	455	32	1	
4290838	9	1137400	1447338	1026288	481200	156624	35739	5628	584	36	1

Figure 7: Walks in three dimensions: values of $W_n(z)$.

We again suppress the details of discovery of the general formula, and of its inductive proof: these details seem to be more complicated than before, and we found no obvious manifestation of the Catalan numbers. The simplest expression for $W_n(z)$ that we have so far found is not in closed form:

$$W_n(n-v) = a_{n,0}\binom{n}{v} + a_{n,1}\binom{n}{v-1} + \ldots + a_{n,t}\binom{n}{v-t}$$

where $t = \lfloor (v+2)/2 \rfloor$ and the coefficients $a_{n,u}$ are of shape

$$a_{n,u} = \left[\binom{v-u}{u} - 4^2\binom{v-u}{u-2}\right] 4^{v-2u}$$

although there are, of course, closed form expressions for small values of v:

$$\begin{aligned}
W_n(n) &= 1 \\
W_n(n-1) &= 4n \\
W_n(n-2) &= (n-1)(8n+1) \\
W_n(n-3) &= \frac{4}{3}n(n-2)(8n-5) \\
W_n(n-4) &= \frac{1}{6}n(n-3)(64n^2-144n+83) \\
W_n(n-5) &= \frac{2}{15}n(n-1)(n-4)(64n^2-240n+239) \\
W_n(n-6) &= \frac{1}{90}n(n-1)(n-5)(512n^3-3648n^2+8872n-7233) \\
W_n(n-7) &= \frac{2}{315}n(n-1)(n-2)(n-6)(512n^3-4800n^2+15496n-17007)
\end{aligned}$$

We have not found a closed expression for W_n, the total number of walks of n steps which do not go below the (x, y)-plane, nor have we had an opportunity to examine the paper [**5**] which may contain such an expression and may overlap the present paper in other ways. The total number of n-step walks in d dimensions, without restriction, is, of course, $(2d)^n$.

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ADDENDUM

As a result of correspondence with Christian Krattenthaler and Bruce Sagan, and receipt of a referee's report, this paper will not seek formal publication, but will revert to its original intent — to form the basis of a solution to the problem of Bill Sands [**9.**]

Christian Krattenthaler writes that " ... the classes of lattice paths you have considered do not seem to have been treated before". On the other hand, both he and the referee point out that formula (15) appears in Theorem 2 in [**D1**]. The referee also notes that formula (4) is in [**Fe**] and that formulas (9), (11), (12), and that for $w'_n(y)$, follow easily from (15) and the reflexion principle. However, the exciting news is that Krattenthaler supplies combinatorial proofs of almost all of the formulas, of the kind appealed for, and that Bruce Sagan also gives very similar proofs. The main item that is missing is a combinatorial proof of the formula(s) for $w'_n(y)$.

I paraphrase the proofs below and leave their originators to publish them as they think fit. I will ask Bill Sands to list Krattenthaler and Sagan among the solvers of *Crux* Problem 1517 unless they write to one of us requesting that their names do not appear. The list already contains the names of Harvey Abbott and George Szekeres, incidentally, but they didn't give "pure" combinatorial proofs.

1. Proof of (15). Set up a correspondence between "NSEW" paths, p, and *pairs* (p_1, p_2) of "NE" paths: if the m-th step of p is N, S, E, W, then the m-th step of p_1 is respectively N, E, E, N, and that of p_2 is N, E, N, E. Then the "NSEW" paths of n steps from (0, 0) to (x, y) are in 1–1 correspondence with pairs (p_1, p_2) of "NE" paths, where p_1 runs from (0, 0) to (r, $n-r$) and p_2 from (0, 0) to (s, $n-s$), where $r=\frac{1}{2}(n+x-y)$ and $s=\frac{1}{2}(n-x-y)$ as before.

[Algebraic detail: If the numbers of N, S, E, W steps are respectively a, b, c, d, then $n=a+b+c+d$, $x=c-d$, $y=a-b$, $r=b+c$, $n-r=a+d$, $s=b+d$, $n-s=a+c$ and r, s are as stated.]

But the number of "NE" paths from (0, 0) to (k, l) is $\binom{k+l}{k}$, so the number of NSEW paths from (0, 0) to (x, y) is

$$\binom{n}{r}\binom{n}{s},$$

i.e., formula (15).

2. Proof of (9). To count NSEW paths of n steps from (0, 0) to (x, y) which don't go below the x-axis, use the reflexion principle. We must subtract off the number of paths which cross the x-axis. Each of these has a first point, say P, for which $y=-1$. Relect the initial portion OP in the line $y=-1$, giving a 1-1 correspondence between paths which cross the x-axis and paths from (0, -2) to (x, y). Their number is the same as the total number of paths already counted, except that y is replaced by $y+2$, i.e., r and s are each decreased by 1. This gives formula (9):

$$w_n(x,y)=\binom{n}{r}\binom{n}{s}-\binom{n}{r-1}\binom{n}{s-1}.$$

3. Proof of (12). Reflexion may also be used to count the number of NSEW paths which stay in the positive quadrant. A second relexion in $x = -1$ together with the inclusion-exclusion principle shows that this number is

#{paths from $(0, 0)$ to (x, y)} - #{paths from $(0, -2)$ to (x, y)} - #{paths from $(-2, 0)$ to (x, y)} + #{paths from $(-2, -2)$ to (x, y)}

$$= \binom{n}{r}\binom{n}{s} - \binom{n}{r-1}\binom{n}{s-1} - \binom{n}{r+1}\binom{n}{s-1} - \binom{n}{r}\binom{n}{s-2}$$

which easily manipulates into formula (12).

4. Proof of (11). To count the number of Sands-type paths, p, which finish at height y, use another correspondence with NE paths, $\bar{p}$, of twice the length. If the m-th step of p is N, S, E, W, then the $(2m-1)$-th and $2m$-th steps of $\bar{p}$ are respectively NN,EE,NE,EN. This sets up a bijection between NSEW paths with n steps from $(0, 0)$ to height y which do not cross the x-axis and NE paths from $(0, 0)$ to $(n-y, n+y)$ which do not cross the line $y = x - 1$. To enumerate these, use the reflexion principle again. A path which crosses the line $y = x - 1$ has a first point Q for which $y = x - 2$. Reflect the portion OQ of such a path in the line $y = x - 2$. This gives a correspondence with NE paths of $2n$ steps from $(2, -2)$ to $(n - y, n + y)$ whose number is $\binom{2n}{n-y-2}$. This confirms the formula displayed just before formula (11). To see (11) itself, adjoin a single N-step to the beginning of path $\bar{p}$ and again apply the reflexion principle.

Note that if we also adjoin a final E-step to the path $\bar{p}$ we see that the number of Sands-type paths with n steps from $(0, 0)$ which finish on the x-axis is the same as the number of NE walks from $(0, -1)$ to $(n+1, n)$ which do not cross the line $y = x - 1$, and this is well-known to be c_{n+1}.

5. Proof of (1). First use the correspondence of **4.** to map p (Sands-type, n steps from $(0, 0)$ to height y, not crossing the x-axis) onto $\bar{\bar{p}}$ (NE path, $2n+1$ steps from $(0, -1)$ to $(n-y, n+y)$, not crossing $y = x - 1$). Consider the last meeting point of $\bar{\bar{p}}$ with $y = x - 1$. The next step is an N-step, which we change to an E-step, obtaining a new path from $(0, -1)$ to $(n - y + 1, n + y - 1)$. Repeat this procedure, this time considering the last meeting point with the line $y = x - 2$. Next consider the line $y = x - 3$, &c. After y changes we arrive at a path from $(0, -1)$ to (n, n). Since this sets up a bijection between NE paths of $2n + 1$ steps starting from $(0, -1)$ and not crossing the line $y = x - 1$, and NE paths from $(0, -1)$ to (n, n) (last meeting points become first crossing points!), we obtain the total number of Sands-type paths of n steps,

$$\binom{2n+1}{n}.$$

AUGMENTED AND ANNOTATED BIBLIOGRAPHY

[Items referred to by pure numbers pertain to the original article. When a final selection of appropriate references is made, they are easily renumbered by "query replace", starting at the end (e.g., replace `bf Ze` by `bf 37`, say).]

A1. D. André, Solution directe du problème résolu par *M.* Bertrand, *C.R. Acad. Sci. Paris* **105**(1887) 436–437; Jbuch **19**, 200.

[cf. **B1** and **Ze**.]

A2. George E. Andrews, Catalan numbers, q-Catalan numbers and hypergeometric series, *J. Combin. Theory Ser. A* **44**(1987) 267–273; MR **88f**:05015.

[The Catalan numbers may be defined as solutions to (1) $C_1 = 1$, $C_n = \sum_{k=1}^{n-1} C_k C_{n-k}$, $n > 0$. The author introduces a new q-analog of the Cats via $C_1(q) = (1+q)/2$, $((1+q^n)/2)C_n(q) = \sum_{k=1}^{n} C_k(q)C_{n-k}(q)q^k \ldots$ and the more general numbers $\mathcal{C}_n(a,q) := q^{2n}(-aq^{-1};q^2)_n/(q^2,q^2)_n$ and gives two proofs of $\sum_{j=0}^{n} \mathcal{C}_j(a^{-1},q)\mathcal{C}_{n-j}(a,q)(-aq)^j = 0$, $n > 0$. He also establishes the explicit formula $C_n(q) = -2^{2n-1}\mathcal{C}_n(-1,q)$. He notes that his q-Cats are the only known q-analog of the Cats which have both a simple representation as a finite product and satisfy an exact q-analog of (1). He also interprets $\mathcal{C}_n(a,q)$ and $\mathcal{C}_n(a^{-1},q)$ as generating functions of numbers of certain partitions. *Mourad E.H. Ismail*]

B1. T. Bertrand, Solution d'un problème, *C.R. Acad. Sci. Paris* **105**(1887) 369.

[cf. **A1** and **Ze**.]

B2. M.T.L. Bizley, Derivation of a new formula for the number of minimal lattice paths from (0,0) to (km, kn) having just t contacts with the line $my = nx$ and having no points above this line; and a proof of Grossman's formula for the number of paths which may touch but do not rise above this line, *J. Inst. Actuar.*, **80**(1954) 55–62; MR **15**, 846d.

[Write $\phi(k,t)$ for the lattice paths with t contacts as described in the title and, following **G5**, write

$$F_j = [j(m+n)]^{-1}\binom{jm+jn}{jm};$$

then the author's new formula may be stated as

$$\sum \phi(k,t)x^k = [1 - \exp(-F - 1x - F_2x^2 - \ldots)]^t.$$

The corresponding formula for $\phi_k = \sum \phi(k,t)$, namely

$$\sum \phi(k)x^k = \exp(F_1x + F_2x^2 + \ldots) - 1$$

is equivalent to Grossman's formula ... *J. Riordan*] [presumably $\phi_k = \phi(k)$ and the summation is over t.]

B3. David Blackwell & J.L. Hodges, Elementary path counts, *Amer. Math. Monthly*, **74**(1967) 801–804; Zbl. **155**, 29c.

[Consider a sequence $x_0, \ldots, x_n$ with entries $x_i = \pm 1$, and let $s_i = x_0 + \ldots + x_i$ be the partial sums. The authors furnish an elementary combinatorial proof of two known theorems concerning the enumeration of such sequences relative to two parameters: the sum s_n, and the lead, or number of indices i for which both s_{i-1} and s_i are non-negative. *H.H. Crapo*]

Ca. L. Carlitz & J. Riordan, Two element lattice permutation numbers and their q-generalization, *Duke Math. J.*, **31**(1964) 371–388; MR **29** #5752.

[A two-element lattice permutation can be described in a two-dimensional lattice as a path leading from (0,0) to a point (m,n) (where $0 \le m \le n$) with the conditions that the path has minimum length, viz., $m+n$, and that it does not contain points (a,b) with $a < b$. It can also be described as an election for two candidates A, B with final vote (n,m), which is such that none of the partial results gives a majority for B (in the paper it is less accurately said

that all partial results correctly predict the winner). The number $a_{n,m}$ of such two-element lattice permutations was determined in 1887 by J. Bertrand [**B1**, see **Ma**] as

$$a_{n,m} = (n+1-m)(n+1)^{-1}\binom{n+m}{m}.$$

The authors consider $a_n(x) = \sum_{m=0}^{n} a_{n,m}x^m$ and show that this nth degree polynomial is characterized by the property that $(1-x)^{n+1}a_n(x)$ has the form $1 - xP_n(x(1-x))$, where P_n is again an nth degree polynomial; in fact, $P_n(u) = \sum_{m=0}^{n} a_{m,m}u^m$. A number of relations are derived for the generating function $a(x,y) = \sum_{n=0}^{\infty} x^n a_n(y)$.

The authors consider these formulas as the special case ($q = 1$) of a q-generalization. They define the nth degree polynomial $a_n(x,q)$ by the condition that there exist $c_0, \ldots, c_n$ such that

$$(x)_{n+1}a_n(x,q) = 1 - x\sum_{m=0}^{n} c_m(qx)^m(x)_m,$$

where $(x)_k$ denotes $(1-x)(1-qx)\ldots(1-q^{k-1}x)$. The coefficients $a_{n,m}(q)$ of this polynomial are studied in several ways.

{The reviewer remarks that $a_{n,m}(q)$ has the following combinatorial interpretation: It is the sum of all $q^{k_1+\cdots+k_m}$, where $k_1, \ldots, k_m$ are integers subject to the conditions $1 \le k_1 \le \ldots \le k_m \le n$, $k_1 \ge 1, \ldots, k_m \ge m$.} *N.G. de Bruijn*]

1. Arthur Cayley, On the partitions of a polygon, *Proc. London Math. Soc.* **22**(1891) 237–262 = Coll. Math. Papers **13**(1897) 93–113.

Ci. J. Cigler, Some remarks on Catalan families, *European J. Combin.*, **8**(1987) 261–267; MR **89a**:05010.

[The author first gives a simple proof that the r-Catalan number $(1/((r-1)n+1))\binom{rn}{n}$ is the number of ways of parenthesizing an r-ary product of $(r-1)n+1$ factors, or equivalently, the number of r-ary trees with n points. Next he shows, using a variant of the Dvoretzky-Motzkin cycle lemma, that the number of r-ary trees with n points and t_i edges from a point to its ith subtree, where $\sum t_i = n-1$, is $(1/n)\binom{n}{t_1}\binom{n}{t_2}\ldots\binom{n}{t_r}$. From this formula he deduces that the number of nonnegative paths from (0,0) to $(rn, 0)$ with $k+1$ peaks, using the steps (1,1) and $(1, 1-r)$, is the generalized Runyon number $(1/n)\binom{(r-1)n}{k}\binom{n}{n-1-k}$. Finally he discusses the connexion between these results, noncrossing partitions and Sperner's theorem. *Ira Gessel*]

C0. E. Csáki, On the number of intersections in the one-dimensional random walk, *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, **6**(1961), 281–286; MR **26** #5642.

[Consider a discrete one-dimensional random walk in which, with the usual notation, $p_{i,i+1} = p_{i,i-1} = \frac{1}{2}$, and let $\lambda_j^{(k)}$ be the number of passages through the point k in a walk of j steps. Write $\lambda_j^{(0)} = \lambda_j$. The following results are proved. (i) $P(\lambda_{2n-1} = l) = P(\lambda_{2n} = l) = 2^{-2n+2}\binom{2n-1}{n+l}$ for $l \neq 0$, and is $2^{-2n+1}\binom{2n}{n}$ if $l = 0$. (ii) For fixed positive even k, $P(\lambda_{2n-1}^{(k)} = l) = P(\lambda_{2n}^{(k)} = l) = 2^{-2n+1}\binom{2n}{n+l+k/2}$. (iii) For fixed positive odd k, $P(\lambda_{2n}^{(k)} = l) = P(\lambda_{2n+1}^{(k)} = l) = 2^{-2n}\binom{2n+1}{n+l+\frac{1}{2}(k+1)}$. The proofs are entirely combinatorial. *J. Gillis*]

C1. Endre Csáki & Sri Gopal Mohanty, Some joint distributions for conditional random walks, *Canad. J. Statist.* **14**(1986) 19–28; MR **87k**:60168.

[Joint distributions of maxima, minima and their indices are determined for certain conditional random walks called Bernoulli excursion and Bernoulli meander. The distribution of the local time of these processes is treated by a generating function technique. Limiting distributions are also given, providing some partial results for Brownian excursion and meander. For instance the authors conjecture the joint limit distributions of the local time and the maximum for these two processes. Similar investigations are carried out for the unconditional random walk and for the Bernoulli bridge. *G. Louchard* (Brussels)]

C2. Endre Csáki, Sri Gopal Mohanty & Jagdish Saran, On random walks in a plane, *Ars Combin.* **29**(1990) 309–318. [Not yet seen]

C3. E. Csáki & I. Vincze, On some problems connected with the Galton test, *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, **6**(1961), 97–109; MR **26** #3138.

[The authors give the probability of the number of waves relative to the horizontal line in the sequence of the sum S_i of the first i members of a random sequence of n $+1$'s and n -1's, where $i = 1, 2, \ldots, 2n$, with limiting distribution in $n \to \infty$ and the joint probability distribution for the number of waves mentioned above and the Galton statistic in the sequence with the limiting distribution. Then they give the joint probability distribution for the number of waves relative to the height $k > 0$ from the horizontal line and the length of time spent above this height expressed by the number of positive members in the well-defined sequence with the limiting distribution. They suggest the statistical tests based on these theorems in a two-sample problem. *C. Hayashi* (Tokyo)]

C4. E. Csáki & I. Vincze, On some distributions connected with the arc-sine law (Russian summary), *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, **8**(1963), 281–291; MR **29** #4078.

[Several results extending those of the reviewer and Feller, and generalized by E.S. Andersen, are given. The combinatorial formulae are obtained by one-to-one correspondence, from which asymptotic ones are computed. For instance, let s_i, $i = 0, 1, 2, \ldots$, be the successive sums in the coin-tossing game with stakes ± 1, $s_0 = 0$, and let $2\gamma_{2n}^{(2k)}$ denote the number of indices i ($i = 1, \ldots, 2n$) for which either $s_i > 2k$ or $s_i = 2k$ but $s_{i-1} = 2k+1$, where k is a non-negative integer; then

$$\mathbf{P}(\gamma_{2n}^{(2k)} = g) = \frac{1}{2^{2n}}\binom{2g}{g}\binom{2n-2g}{n-g+k}, \quad g = 1, 2, \ldots, n-k,$$

$$\mathbf{P}(\gamma_{2n}^{(2k)} = 0) = \frac{1}{2^{2n}}\sum_{j=-k}^{k}\binom{2n}{n+j}.$$

K.L. Chung

De. Nachum Dershowitz & Schmuel Zaks, The cycle lemma and some applications, *European J. Combin.*, **11**(1990) 35–40. [not yet seen]

D1. Duane W. DeTemple & Jack M. Robertson, Equally likely fixed length paths in graphs, *Ars Combin.* **17**(1984) 243–254; MR **86h**:60103.

[give equation (15) in the original paper.] [The authors investigate the unique stochastic process whose realizations are the set of paths of given length joining two given vertices of a given graph and which has the property that all such paths are equally likely to occur. There is an application to the design of experiments. *G.R. Grimmett* (Bristol)]

D2. Duane W. DeTemple, C.H. Jones & Jack M. Robertson, A correction for a lattice path counting formula, *Ars Combin.* **25**(1988) 167–170; MR **89i**:05017.

[gives equation (15) in the original paper.] [In **D1**, DeT & R gave the formula

$$\frac{a-kb}{d}\binom{d}{\frac{d-a-b}{2}}\binom{d}{\frac{d-a+b}{2}}$$

for the number of lattice paths in the plane length d, with unit steps in the positive and negative coordinate directions, starting at the origin and ending at (a, b) which touch the line $x = ky$ only at the initial point. In the present paper the authors note that this formula is correct if $d = a + b$, $a - kb = 1$, or $k = 1$; in other cases it is an upper bound.

The authors use the method of cyclic permutation or "penetrating analysis" due to **Dv**. A method which yields an exact, but complicated, formula for this kind of problem was described in **G1**. *Ira Gessel*]

2. C. Domb, On the theory of cooperative phenomena in crystals, *Advances in Physics* **9**(1960) 149–361.

[didn't find this in MR or in Zbl. I think I must have got it from Sloane [**10**], and I can no longer find a reference to it in the text, so let's forget it?]

Dv. A. Dvoretzky & Th. Motzkin, A problem of arrangements, *Duke Math. J.*, **14**(1947) 305–313; MR **9**, 75e.

[... In an election, candidates P and Q receive p and q votes, respectively; required the probability that the ratio of the ballots for P to those for Q will, throughout the counting, be larger than (larger than or equal to) α.]

E. O. Engleberg, On some problems concerning a restricted random walk, *J. Appl. Probability*, **2**(1965) 396–404; MR **32** #475.

[The restricted random walk in question is on a line (vertical for convenience) with unit steps up and down and prescribed numbers of each. In the terminology of Feller [**Fe**] such a walk is a polygonal path from (0,0) to $(n+m, n-m)$; it is also in one-to-one correspondence with a two-candidate election return with final vote (n,m). In election return terms, the author's main results are as follows: If $c(x;n,m)$ is the enumerator of election returns with final vote (n,m), $n \geq m$, by number of changes of lead, if $t(x;n,m)$ is the corresponding enumerator by number of ties, then $c(x;n,m) = \sum_{k=0}^{m} a_{n+k,m-k}x^k$, $n > m$, $c(x;n,n) = 2c(x;n,n-1)$, $t(x;n,m) = \sum_{k=0}^{m} a_{n-1,m-k}(2x)^k$, where

$$a_{n,m} = \binom{n+m}{m} - \binom{n+m}{m-1} = \frac{n+1-m}{n+1}\binom{n+m}{m}.$$

The numbers $a_{n,m}$ are the oldest ballot numbers. (For $n < m$, the enumerators in the two cases, by symmetry, are those above with n and m interchanged, a result not noticed by the author.) These results are used with obvious summations to verify the results of Feller [**Fd**] for the enumeration of unrestricted random walks by number of axis crossings and by number of zeros (of their polygonal paths). Also the limiting distribution functions of tie returns $(n = m)$ are determined. Finally, the application of the results to the comparison of two empirical distributions is sketched.

{In equation (11) there are two typographical slips whose corrections will probably be evident to most readers.} *J. Riordan*]

Fd. W. Feller, The number of zeros and of changes of sign in a symmetric random walk, *Enseignement Math.*(2), **3**(1957) 229–235; MR **20** #4329.

[Let $S_n = X_1 + X_2 + \ldots + X_n$ where the X_j are independent and assume the values ± 1 with probability $\frac{1}{2}$. The author derives for this symmetric random walk explicit formulas for the probability distribution of the number of returns to the origin, the number of changes of sign and other related quantities. The derivations are of a very elementary nature and the paper is self-contained. A more exhaustive treatment appears in Chapter III of the 2nd ed. of **Fe** (1957). *J.L. Snell*]

Fe. W. Feller, An Introduction to Probability Theory and its Applications, Vol. 1, Wiley, 1968, p. 73

[The reference gives equation (4) in the original paper. On p. 82 it is noted that $\binom{2k}{k}\binom{2n-2k}{n-k}$ is the number of paths of length $2n$ with exactly $2k$ steps lying above $y = x$. On p. 96 is given a bijection between paths from (0,0) to (k,k) and paths of length $2k$ which do not pass below $y = x$.]

Fl. Philippe Flajolet, Combinatorial aspects of continued fractions, *Discrete Math.*, **32**(1980) 125–161; *Ann. Discrete Math.*, **9**(1980) 217—222; MR **82f**:05002ab.

[referred to in review of **G3**.]

Fu. J. Fürlinger & J. Hofbauer, q-Catalan numbers, *J. Combin. Theory Ser. A*, **40**(1985) 248–264; MR **87e**:05017.

[The Catalan numbers C_n are defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$ and satisfy $z = \sum_{n=1}^{\infty} C_n z^n/(1+z)^{2n}$ and $z = \sum_{n=1}^{\infty} C_n z^n(1-z)^n$. This paper surveys several q-analogs of the Catalan numbers. The authors first consider the q-Catalan numbers of **Ca**. These satisfy

$$z = \sum_{n=1}^{\infty} q^{\binom{n-1}{2}} C_n(q) \frac{z^n}{(1+z)(1+qz)\dots(1+q^{2n-1}z)}$$

and

$$z = \sum_{n=1}^{\infty} C_{n-1}(q) z^n (1-z)(1-qz)\dots(1-q^{n-1}z).$$

There is no simple explicit formula for $C_n(q)$, but there is a combinatorial interpretation in terms of inversions of certain 0-1 sequences.

Next the authors consider the q-Catalan numbers given by

$$c_n(\lambda; q) = \sum_{k=0}^{n} \frac{1}{[n]} \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} n \\ k+1 \end{bmatrix} q^{k^2+\lambda k},$$

in which the terms are q-Runyon numbers. Here one defines $[n] = (q^n - 1)/(q-1)$ and $\begin{bmatrix} n \\ k \end{bmatrix}$ is the q-binomial coefficient. For $\lambda = 1$ these reduce to $c_n(1; q) = C_n$. These q-Catalan numbers satisfy

$$z = \sum_{n=1}^{\infty} \frac{c_n(\lambda; q) z^n}{q^{\binom{n}{2}}(-q^{-n}z)_n(-q^{\lambda}z)^n},$$

where $(a)_n = (1-a)(1-qa)\dots(1-q^{n-1}a)$, and they have a combinatorial interpretation in terms of major indices and descents of 0-1 sequences.

A generalization which includes both types of q-Catalan numbers is considered next. These satisfy

$$z = \sum_{n=1}^{\infty} \frac{a^{-\binom{n}{2}} C_n(x; a, b) z^n}{(1+a^{-1}z)\dots(1+a^{-n}z)(1+xbz)\dots(1+xb^n z)}$$

and are also given a combinatorial interpretation. They also include as a special case the q-Catalan numbers studied in **Po**. *Ira Gessel*]

Fv. Harry Furstenberg, Algebraic functions over finite fields, *J. Algebra*, **7**(1967) 271–277; MR **35** #6655.

[referred to in review of **G1**.]

G1. Ira M. Gessel, A factorization for formal Laurent series and lattice path enumeration, *J. Combin. Theory Ser. A* **28**(1980) 321–327; MR **81j**:05012.

[A powerful and striking factorization for certain formal Laurent series is proved, namely that the series is a product of a constant, a series in only negative powers and a series in only positive powers. Lagrange's formula for series reversion is treated as an application. Other applicationa are to the problems of enumerating restricted lattice paths (a novel interpretation of Laurent series in combinatorial theory) and to H. Furstenberg's theorem [**Fv**] that the diagonal of a rational power series in two variables is algebraic (giving a new formal method of showing that certain series are algebraic. *D.G. Rogers*]

G2. Ira M. Gessel, A probabilistic method for lattice path enumeration, *J. Statist. Plann. Inference* **14**(1986) 49–58; MR **87h**:05017.

[Some lattice path counting problems may be converted into problems of deriving distributions on random walks which give rise to functional equations. Solutions of these equations provide a probabilistic approach to the lattice path enumeration problems. The approach is illustrated by a few examples. *Sri Gopal Mohanty*]

G3. Ira M. Gessel, A combinatorial proof of the multivariable Lagrange inversion formula, *J. Combin. Theory Ser. A* **45**(1987) 178–195; MR **88h**:05011.

[Using the exponential generating function, the author gives a combinatorial proof of one form of the multivariable Lagrange inversion formula (MLIF). An outline of the proof is: (1) interpret the defining functional relations as generating functions for colored trees, (2) interpret the desired coefficient as the generating function for functions from a set to a larger set, (3) decompose the functional digraph from (2) into two types of connected components, whose generating functions give the MLIF. Labelle had given such a proof in one variable [**L**].

The author also gives a useful survey of forms of the MLIF given by Jacobi, Stieltjes, Good, Joni, and Abhyankar. He shows that Jacobi's form implies Good's form and gives a simple form generalizing that of Stieltjes, Joni, and Abhyankar.

The paper includes some historical information on the Jacobi formula for matrices, det exp A = exp trace A. *Dennis Stanton*]

Go. Henry W. Gould, Final analysis of Vandermonde's convolution, *Amer. Math. Monthly*, **64**(1957) 409–415; MR **19**, 379c.

[referred to in review of **Ra**.]

GJ. I.P. Goulden & D.M. Jackson, Path generating functions and continued fractions, *J. Combin. Theory Ser. A*, **41**(1986) 1–10; MR **87i**:05020.

[The authors consider paths along the nonnegative integers in which each step consists of an increase of altitude of 1 (a rise), 0 (a level), or -1 (a fall). The paths are weighted to record the number of rises and levels at each altitude. The main result of the paper answers the following question: What is the sum of the weights of all paths with given initial and terminal altitudes, and with given bounds on maximum and minimum altitudes? The answer is expressed in terms of continued fractions and extends P. Flajolet's combinatorial theory of continued fractions [**Fl**]. Some classical identities for continued fractions are obtained as corollaries. *Ira Gessel*]

G4. Dominique Gouyou-Beauchamps & G. Viennot, Equivalence of the two-dimensional directed animal problem to a one-dimensional path problem, *Adv. in Appl. Math.*, **9**(1988) 334–357; MR **90c**:05009.

[A set P of lattice points in the plane is called a directed animal if there is a subset of P, whose elements are called root points, such that the root points lie on a line perpendicular to the line $y = x$, and every point of P can be reached from a root point by a path in P using only north and east steps. This paper is concerned with the enumeration of compact-rooted animals, which are animals in which the root points are consecutive. Animals which differ only by translation are considered to be equivalent.

The main result is a bijection between compact-rooted animals of size $n + 1$ and paths of length n on the integers, with steps $+1$, 0 and -1. This bijection implies that there are 3^n compact-rooted animals of size $n + 1$. Moreover the bijection allows the compact-rooted animals to be counted according to the number of root points. Further consequences are that the generating function for directed animals with one root point is $\frac{1}{2}((1+t)/\sqrt{1-2t-3t^2}-1)$ and that the number of directed animals of size n rooted at the origin and contained in the first octant $0 \le x \le y$ is the Motzkin number m_{n-1}.

The paper also contains many references to work by physicists on problems of counting animals, which arise in studying thermodynamic models for critical phenomena and phase transitions. *Ira Gessel*]

G5. Howard D. Grossman, Fun with lattice points, *Scripta Math.*, **15**(1945) 79–81; MR

12, 665d.

[Suppose an election results in km votes for A and kn for B. In how many orders may votes be cast so that A's vote is always at least m/n times B's? The author gives without proof the formula

$$p_k = \sum F_1^{k_1} F_2^{k_2} \dots [k_1! k_2! \dots]^{-1}$$

with $k_1 + 2k_2 + \ldots = k$, $F_j = j^{-1}(m+n)^{-1}\binom{jm+jn}{jn}$ and the sum over all partitions of k. He also gives a short introduction to enumerations in three-dimensional and derives a solution to a corresponding election problem, noting its agreement with MacMahon's. *J. Riordan*]

H1. B.R. Handa & Sri Gopal Mohanty, Enumeration of higher-dimensional paths under restrictions, *Discrete Math.* **26**(1979) 119–128; MR **81b**:05012.

[The authors consider the problem of counting lattice paths in k-dimensional space under restrictions. They obtain k-dimensional analogs of the familiar results in two dimensions. *D.P. Roselle*]

H2. B.R. Handa & Sri Gopal Mohanty, On a property of lattice paths, *J. Statist. Plann. Inference* **14**(1986) 59–62; MR **87i**:05023.

[The authors give an algebraic discussion of the implications on lattice paths of the following fact: increasing sequences of integers such that the jth is at least $b(j)$ greater than the $(j-1)$st, and is at most $a(j)$, are in one-to-one correspondence to increasing sequences such that the jth is at most $a(j)$ minus the sum of the first j $b(k)$'s. *D.J. Kleitman*]

3. Frank Harary, Topological concepts in graph theory, in Harary & Beineke, A Seminar on Graph Theory, Holt, Reinhart & Winston, New York & London, 1967, pp.13–17.

[a nonce-reference on p. 10 of the original paper.]

Hi. Terrell L. Hill, Steady-state kinetics of a linear array of interlocking reactions, Statistical Mechanics & Statistical Methods in Theory and Applications (Rochester NY), Plenum, New York, 1977, pp. 521–577; MR **57** #15112.

[referred to in review of **Sh**; the MR reference gives no further information.]

J. André Joyal, Une théorie combinatoire des séries formelles, *Adv. in Math.*, **42**(1981) 1–82; MR **84d**:05025.

["We present a combinatorial theory of formal power series. The combinatorial interpretation of formal power series is based on the concept of species of structures. A categorical approach is used to fomulate it. A new proof of Cayley's formula for the number of labelled trees is given as well as a new combinatorial proof (due to G. Labelle) of Lagrange's inversion formula. Pólya's enumeration theory of isomorphism classes of structures is entirely renewed. Recursive methods for computing cycle index polynomials are described. A combinatorial version of the implicit function theorem is stated and proved. The paper ends with general considerations on the use of coalgebras in combinatorics."]

K. S. Karlin & G. McGregor, Coincidence probabilities, *Pacific J. Math.*, **9**(1959) 1141–1164; MR **22** #5072.

[among Gessel's references, but may be marginal; cf. immediately preceding paper and review.]

4. T.P. Kirkman, On the k-partitions of the r-gon and r-ace, *Phil. Trans.* **147**(1857) 225.

[v. p. 3, l. 1 of original paper.]

Kl. Daniel Kleitman, A note on some subset identities, *Studies in Appl. Math.*, **54**(1975) 289–292; MR **56** #8386.

[gives combin. proof of

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n,$$

19 of the "Erdős-Pósa" problems – see **Mi.**]

5. Christian Krattenthaler, Counting lattice paths with a linear boundary. I. *Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II* **198**(1989) 87–107.

K1. Christian Krattenthaler, Enumeration of lattice paths and generating functions for skew plane partitions, *Manuscripta Math.*, **63**(1989) 129–155..

K2. G. Kreweras & H. Niederhausen, Solution of an enumerative problem connected with lattice paths, *European J. Combin.*, **2**(1981) 55–60; MR **82d**:05014.

[The authors consider lattice paths of p horizontal and q vertical steps from $(0,q)$ to $(p,0)$. Let $W(C)$ denote the number of paths below C in the dominance partial ordering. Theauthors prove

$$\sum (W(C))^2 = \frac{(p+q+1)!(2p+2q+1)!}{(p+1)!(2p+1)!(q+1)!(2q+1)!}$$

S.G. Williamson]

6. Mike Kuchinski, Catalan Structures and Correspondences, M.Sc. thesis, West Virginia University, Morgantown WV 26506, 1977.

L. Gilbert Labelle, Une nouvelle démonstration combinatoire des formules d'inversion de Lagrange, *Adv. in Math.*, **42**(1981) 217–247; MR **83e**:05016.

[The purpose of this paper is to examine connexions between two classical results — the Lagrange inversion formula for power series and Cayley's formula for the number of labelled trees on n vertices — in the light of the combinatorial theory of formal series recently presented by **J** and founded on the notion of "species of structures". Some combinatorial operations are defined over species and correspond to analytic operations over their generating functions: hence, properties of the operations over species yield identities for formal series. The authors introduce two canonical constructions which associate a new species — "arborescence R-enrichie" and "endofonction R-enrichie", respectively — to any given species R and proves some deep isomorphism results. These yield, as simple corollaries, some generalized versions of Cayley's formula and the Lagrange inversion formula. *Andrea Brini*]

L1. Jacques Labelle & Yeong-Nan Yeh, Dyck paths of knight moves, *Discrete Appl. Math.*, **24**(1989) 213–221; MR **90g**:05017.

[This paper enumerates lattice paths from the origin to a point along the x-axis which do not go below the x-axis, where the allowable moves are knight moves from left to right. The resulting generating function satisfies a fourth-degree polynomial equation. This compares with so-called Dyck paths, where the allowable moves are one-step diagonal moves to the right. In this classical case the resulting generating function satisfies a quadratic polynomial equation, whose solution yields the Catalan number generating function.

The authors apply their methods to paths with (r,s) knight moves, and obtain polynomial equations of higher degree. *Dennis White*]

L2. Jacques Labelle & Yeong-Nan Yeh, Generalized Dyck paths, *Discrete Math.*, **82**(1990) 1–6. [Not yet seen]

L3. Jack Levine, Note on the number of pairs of non-intersecting routes, *Scripta Math.* **24**(1959) 335–338; Zbl. **93** 13a.

[Soit 2 permutations $X_n = x_1x_2\dots x_n$ et $Y_n = y_1y_2\dots y_n$ de p éléments a et q éléments b, $p+q=n$. Si le nombre des a est différent du nombre des b dans chaque paire de séquences partielles correspondantes $x_1x_2\dots x_k$ et $y_1y_2\dots y_k$ pour $k = 1, 2, \dots, n-1$, les permutations sont dites une paire de permutations non intersectantes, pour une raison qui provient d'une interprétation graphique. Les 2 paires (X_n, Y_n) et (Y_n, X_n) sont à considérer comme équivalentes. L'A. obtient le nombre suivant des paires distinctes de telles permutations de p éléments a et q éléments b, $p+q=n$, $N_n(p,q) = \frac{n-1}{pq}\binom{n-2}{p-1}\binom{n-2}{q-1}$, $0<p<n$, $0<q<n$. *S. Bays*] [The Strens Collection has a run of *Scripta Math.* if you'd like me to send a copy of the paper.]

Li. N. Linial, A new derivation of the counting formula for Young tableaux, *J. Combin. Theory Ser. A*, **33**(1982) 340–342; MR **83m**:05016.

[The well-known hooklength formula for the number of standard Young tableaux of a given shape can be written in determinantal form (Frobenius) [see, e.g., D.E. Knuth, The Art of Computer Programming, Vol. 3, pp. 60–63, Addison-Wesley, 1973]. A short proof of this result is given by observing that the expansion of the determinant yields an alternating sum of mutinomial coefficients which obviously satisfies the difference equation for the numbers in question, together with the initial conditions. *Volker Strehl*]

Ly. R.C. Lyness, Al Capone and the death ray, *Math. Gaz.*, **25**(1941) 283–287.

[neither this nor **Pe** appears to have made it into MR or Zbl. I enclose photocopies of each of these, as they predate many other papers on the subject. I don't have immediate access to *Math. Gaz.*, **18**(1934) 124 or to *Chess Amateur*, mentioned by **Pe**, but they may also be of some historical interest.]

Ma. Major P.A. MacMahon, Combinatory Analysis, Vol. I, Section III, Chapter V, Cambridge Univ. Press, Cambridge, 1915.

[referred to in review of **Ze** and perhaps **G5**.]

Mi. E.C. Milner, Louis Pósa — a mathematical prodigy, *Nabla, Bull. Malayan Math. Soc.*, **7**(1960) 61–64; Solutions to the Erdős-Pósa problems I, II, *ibid.*, 107–112, 154–159.

[Problem 19 was to prove the identity mentioned under **Kl**.]

M1. Sri Gopal Mohanty, Lattice Path Counting and Applications. Probability and Mathematical Statistics. Academic Press, 1979, xi+185 pp.

[p. 2 gives the reflexion principle, whereby equations (9), (11), (12) and the formula at the foot of p. 8 of the original paper all follow from equation (15). I haven't had access to Mohanty's book — there may be several other relevant references therein.]

M2. Sri Gopal Mohanty, On some generalization of a restricted random walk, *Studia Sci. Math. Hungar.*, **3**(1968) 225–241; MR **39** #1022.

[The author considers the paths of a restricted random walk starting from the origin, which at each step moves either one unit to the right or μ (positive integer) units to the left, and reaches the point $m - \mu n$ in $m + n$ steps. Random walks are considered schematically by representing each movement of the particle to the right or to the left by a horizontal or a vertical unit so that the restricted random walk corresponds to the minimal lattice paths the particle describes from the origin to (m, n). Expressions are obtained for total numbers of distinct paths under certain further conditions as follows. For a given path, say C, of such a random walk the total number of paths is found which, after each step, do not lie to the left of the corresponding point of C, and which touch C in a prescribed way in exactly r of the last s left steps. Expressions are obtained also for the number of paths crossing r times (but not necessarily reaching) a point $\alpha \geq 0$, for the number of paths reaching α, r times, and concerning the joint distribution of the numbers of times and steps in the region to the right of α. The last is shown to lead to a result connected with a ballot theorem of L. Takács [**T**]. The author mentions also related results due to E. Csáki [**C0**], E. Csáki & I. Vincze [**C3, C4**], K. Sen [**Se**] and O. Engleberg [**E**]. *C.J. Ridler-Rowe*]

7. Athanasios Papoulis, A new method of inversion of the Laplace transform, *Q. App. Math.* **14**(1957) 405–414; MR **18**, 602e.

[finds Legendre coefficients; done before by Widder, *Duke Math. J.*, **1**(1935) 126–136; and by Shohat *ibid.*, **6**(1940) 615–626; MR **2**, 98.]

P. J. Peacock, On "Al Capone and the Death Ray", Note **1633** *Math. Gaz.*, **26**(1942) 218–219.

[see note under **Ly**.]

Po. G. Pólya, On the number of certain lattice polygons, *J. Combin. Theory* **6**(1969) 102–105; MR **38** #4329.

[A closed polygon without double points that consists of segments of length one joining neighboring lattice points is called a lattice polygon. Two lattice polygons are considered as not different if and only if there exists a parallel translation superposing one to the other. The number of "different lattice polygons" is indicated by dlp. A closed plane curve without double points is termed convex with respect to the direction d if for the intersection of the closed domain surrounded by the curve with any straight line of direction d, only three cases are possible: The intersection is either the empty set, or consists of just one point or of just one segment. A curve is convex in the usual sense if and only if it is convex with respect to all directions [a square is not convex under this strict definition — RKG].

Let a_m denote the number of dlp convex with respect to the vertical direction with area m, b_n the number of dlp convex with respect to the $-45°$ direction with perimeter $2n$, and c_{mn} the number of dlp convex with respect to the $-45°$ direction with area m and perimeter $2n$; evidently $\sum_m c_{mn} = b_n$. In this paper explicit expressions are given for the numbers a_m, b_n and c_{mn}. For example, $\sum_1^\infty a_m x^m = x + 2x^2 + 6x^3 + 19x^4 + 61x^5 + \ldots = x(1-x)^3/(1-5x+7x^2-4x^3)$, $b_n = (1/(4n-2))\binom{2n}{n}$. The proofs of the results stated will be presented in a subsequent paper. *A.L. Whiteman*]

Ra. George N. Raney, Functional composition patterns and power series reversion. *Trans. Amer. Math. Soc.*, **94**(1960) 441–451; MR **22** #5584.

[Let a_1, a_2, ... be an infinite sequence of natural numbers 0, 1, 2, ... , such that $\sum a_i$ is finite. The author defines the numbers $L = L(n; a_1, a_2, \ldots)$ combinatorially and shows that

$$L(n; a_1, a_2, \ldots) = \frac{(\sum_{i=0}^\infty a_i)!n}{\prod_{i=0}^\infty (a_i!)m}$$

where $m = n + \sum_{i=1}^\infty i a_i$, $a_0 = n + \sum_{i=1}^\infty (i-1)a_i$, and $L = 1$ if $m = n = 0$. He then derives some identities involving the numbers L, and uses them to prove a Lagrange inversion formula on formal power series and a convolution formula given by **Go**. *Rimhak Ree*]

8. Gerhard Ringel, Färbungsprobleme auf Flächen und Graphen, VEB Deutscher Verlag der Wissenschaften, Berlin, 1959.

R1. Don Rawlings, The Euler-Catalan identity, *European J. Combin.*, **9**(1988) 53–60; MR **89g**:05017.

[This paper is an attempt to unify the various generalizations of Catalan and Eulerian numbers by using q-theory. The author denotes the generating function for permutations by descents, major index, inversions and patterns by

$$(*) \qquad A(n; t, q, p, u, v) = \sum_{\sigma \in s} t^{d(\sigma)} q^{m(\sigma)} p^{i(\sigma)} u^{a(\sigma)} v^{b(\sigma)}.$$

Denoting $A_n(t) = A(n; t, q, p, u, v)$, the author shows that the recurrence

$$(**) \quad A_{n+1}(t) = A_n(tq) + t \sum_{k=1}^{n} q^k p^k u^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix} A_k(t) A_{n-k}(tq^{k+1})$$

holds. Defining $C(n; t, q, p) = A(n; t, q, p, 0, 1)$ and $K(n; t, q, p) = A(n; t, q, p, 1, 0)$, the author deduces recurrences for C and K from which two classic q-Catalan numbers defined by **Ca** follow by taking $t = q = 1$. Taking $u = 1$, $v = 1$ in $(**)$ leads to a new recurrence involving $E(n; t, q, p) = A(n; t, q, p, 1, 1)$ which defines generalized Eulerian numbers. A conjecture involving the q-Catalan numbers is posed at the end of Section 5. *R.N. Kalia*]

R2. John Riordan, Combinatorial Identities,

9. Bill Sands, Problem 1517*, *Crux Mathematicorum* **16**#2(Feb. 1990) 44.

Se. Kanwar Sen, On some combinatorial relations concerning the symmetric random walk, *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, **9**(1964), 335–357; MR **33** #6715.

[The author considers sequences $(\vartheta_1, \vartheta_2, \ldots \vartheta_{2n})$ of $n+k$ 1's and $n-k$ -1's, in other words the polygonal paths from (0,0) to $(2n, 2k)$ through the points (i, s_i) with $s_i = \vartheta_1 + \vartheta_2 + \ldots + \vartheta_i$ in a rectangular coordinate-system, where each possible array (path) has the same probability. In connexion with this restricted random walk, there are joint distribution laws and joint limiting distribution laws determined ofr the number of intersections (number of changes of sign of the s_i) and the number of positive steps, as well as for the number of intersections at the height r (the number of changes of sign of $s_i - r$) and the number of steps above the height r. Applying the results obtained, the author proves some known relations for the unrestricted case also. The proofs of the theorems are of a combinatorial character, some of them involving one-to-one correspondences of paths. The paper has some points in common with **E.** *I. Vincze*]

Sh. Louis W. Shapiro, A lattice path lemma and an application in enzyme kinetics, *J. Statist. Plann. Inference* **14**(1986) 115–122; MR **87j**:05021.

[Consider the rectangular lattice from (0,0) to (a, b). Choose z integers $1 \le a_1 < a_2 < \ldots < a_z \le a$ and z integers $1 \le b_1 < b_2 < \ldots < b_z \le b$, and place stones on the squares $(a_1, b_1), \ldots, (a_z, b_z)$. The author first shows that if the integers are chosen randomly, and a random path from (0,0) to (a, b) with unit steps east and north is chosen, then the probability that the path passes beneath all the stones is $1/(z+1)$.

The author next considers a model for enzyme kinetics closely related to that studied by **Hi** and by **SZ**. In these models the states are all 0-1 sequences of length M. Each 0 or 1 represents an enzyme, which is either reduced (1) or oxidized (0). The transition rules essentially allow a 01 subsequence to become 10. Using the result of the first part of the paper, the author computes the steady-state distribution under transition probabilities which are different from those used in the earlier papers. *Ira Gessel*]

SZ. Louis W. Shapiro & Doron Zeilberger, *J. Math. Biol.*, **15**(1982) 351–357; MR **84f**:92011.

[The authors investigate a continuous time Markov chain modelling the diffusion of a ligand across a membrane. The states of the chain are strings of 0's and 1's of length M and the transitions are $0\alpha \to 1\alpha$, $\alpha 10\beta \to \alpha 01\beta$, $\beta 1 \to \beta 0$, where all the transition rates are equal. The authors give a formula for the steady state probabilities that the rth component of the string is zero. *Petr Kürka*]

10. Neil J.A. Sloane, A Handbook of Integer Sequences, Academic Press, New York & London, 1973.

Su. Robert A. Sulanke, A recurrence restricted by a diagonal condition: generalized Catalan numbers, *Fibonacci Quart.*, **27**(1989) 33–46; MR **90c**:05012.

[This paper contains a proof via lattice paths of the Lagrange inversion theorem for ordinary generating functions. It also includes many examples of the Catalan-Motzkin-Schröder sequence variety. A translation from lattice paths to planar trees is given along with several planar tree examples.

Basically this paper considers paths where each step is of the form $(x, y) \to (x+j, y+1)$, $j \in \{0, 1, 2, \ldots\}$. Such a path from (0,0) to (k, l) is good if after leaving (0,0) all points on the path must lie above the line $y = \mu x$, $\mu \in \mathbb{Z}$.

The number of good paths from (0,0) to $(k\mu + d)$ [sic!] is shown to be $d/(1 + k\mu)$ of all paths between the same points. These paths are then weighted and generating functions are introduced, leading, eventually, to a combinatorial proof of the Lagrange inversion theorem.

Other combinatorial proofs include those of **Ra** (ordinary generating functions), **L** (exponential generating functions) and **G**. *Louis Shapiro*]

Sv. Marta Sved, *Math. Intelligencer*

[cf. **Kl, Mi** and see Gessel correspondence.]

T1. Lajos Takács, Ballot problems, *Z. Wahrscheinlichkeitstheorie und verw. Gebiete*, **1**(1962) 154–158; MR **26** #3131.

[The author proves a discrete variant of Spitzer's lemma, to the effect that $P\{\Delta_n = j|\nu_1 + \ldots + \nu_n = 1\} = 1/n$, where Δ_n is the number of positive partial sums $\nu_1 + \ldots + \nu_m$, $(m \le n)$ and the ν_i are integer-valued with permutation-symmetric distribution. He uses this to prove that, if all possible voting sequences are equally likely and the two candidates get a and b votes with $(a, b) = 1$, and if $\Delta^*_{a,b}$ denotes the number of the $a + b$ vote-count times when the ratio of votes is $\ge a/b$, then $P\{\Delta^*_{a,b} = j\} = 1/(a + b)$. The probability that the amount by which the first candidate is ahead stays strictly between $c - d$ and c, where c and d are positive integers, $c < d$, and $c - d < b - a < c$, is shown to be

$$\binom{a+b}{a}^{-1} \sum_k \left[\binom{a+b}{a-kd} - \binom{a+b}{a+c+kd}\right].$$

The results extend those of many authors, some of the more recent being Chung, Feller, Hodges, Gnedenko, Rvačeva. *J. Kiefer*]

T2. Lajos Takács, The distribution of majority times in a ballot, *Z. Wahrscheinlichkeitstheorie und verw. Gebiete*, **2**(1963) 118–121; MR **28** #3490.

[In this sequel to a previous paper [**T1**] further probabilities are calculated concerning the number of times one candidate leads over another during the successive stages of a ballot. The combinatorial proofs are based on lemmas which are relevant also to fluctuation theory, order statistics and the theory of queues. The author also gives a generalization to processes with independent increments. *F.L. Spitzer*]

Ta. Lajos Takács, Some asymptotic formulas for lattice paths, *J. Statist. Plann. Inference* **14**(1986) 123–142; MR **87k**:60082.

[The author proves asymptotic formulas for the area under lattice paths starting at (0,0) with unit steps in the positive x- and y-directions. Two of the simpler formulas are as follows.

Let $\theta(n)$ be the area under a random path of length n. Then

$$\lim_{n\to\infty} \left[\frac{n^{3/2}}{\sqrt{48}} P\{\theta(n) = j\} - \phi\left(\frac{\sqrt{48}(j - \frac{1}{8}n^2)}{n^{3/2}}\right)\right] = 0$$

uniformly in j for $j = 0, 1, \ldots, \lfloor \frac{1}{2}n^2 \rfloor$, where $\phi(x) = e^{-x^2/2}/\sqrt{2\pi}$.

Let $\phi(a, b)$ be the area under a random path from (0,0) to (a, b). Then

$$\lim_{\substack{a\to\infty \\ b\to\infty}} \left[\sigma_{a,b} P\{\theta(a, b) = j\} - \phi\left(\frac{j - \frac{1}{2}ab}{\sigma_{ab}}\right)\right] = 0$$

uniformly in j for $|j - \frac{1}{2}ab| \ge \sigma_{a,b}\epsilon$, where ϵ is a positive real number and $\sigma_{a,b} = \sqrt{ab(a + b + 1)/12}$.

Let $w(n, j)$ be the number of lattice paths from (0,0) to (n, n) with area j which stay below the line $y = x$ for $0 < x < n$. Let C_n be the Catalan number $(1/(n + 1))\binom{2n}{n}$, and let η_n be the discrete random variable whose probability distribution is given by $P\{\eta_n = j\} = w(n, \binom{n}{2} - j)/C_{n-1}$. The author gives empirical evidence that suggests

$$P\left\{\frac{a\eta_n}{\left(\frac{4^n}{8C_{n-1}} - \frac{n}{2}\right)} \le x\right\} \approx \frac{1}{\Gamma(a)} \int_0^x e^{-u} u^{a-1}\, du,$$

where $a = 3\pi/(10 - 3\pi)$. *Ira Gessel*]

[I'm sure there are misprints in MR — check against original paper — RKG.]

11. H.M. Taylor & R.C. Rowe, Note on a geometrical theorem, *Proc. London Math. Soc.* **13**(1882) 102–106.

12. W.T. Tutte, A census of Hamiltonian polygons, *Canad. J. Math.*, **14**(1962) 402–417; MR **25** #1108.

[number of inequivalent rooted maps of $2n$ vertives is

$$\frac{2^{n+1}(3n)!}{n!(2n+2)!}$$

and number of inequivalent Hamiltonian rooted maps of $2n$ vertices is

$$\frac{(2n)!(2n+2)!}{2n!((n+1)!)^2(n+2)!}.]$$

W1. Toshihiro Watanabe & Sri Gopal Mohanty, On an inclusion-exclusion formula based on the reflection principle, *Discrete Math.* **64**(1987) 281–288; MR **88d**:05012.

[**A1** first used the refexion principle to count paths in the plane, with unit steps in the positive horizontal and vertical directions, that never touch the line $x = y$. Suppose that lattice points p and q lie on the same side of this line. The number of "good paths" from p to q is the total number of paths from p to q minus the number of "bad paths" from p to q (those which touch the line). By reflecting in the line $x = y$ the segment of a bad path from its starting point to its first meeting with this line, we find that the number of bad paths from p to q is equal to the total number of paths from p' to q, where p' is the reflexion of p in $x = y$.

The authors show here how André's reflexion principle can be used to solve the multi-dimensional generalization of the ballot problem, which is equivalent to counting paths not touching the hyperplanes $x_i = x_j$, i, $j = 1, \ldots, n$. Here all reflexions in the hyperplanes $x_i = x_j$ are used and there are $n!$ terms in the resulting formula, with alternating signs, corresponding to the $n!$ permutations of the coordinates generated by the reflexions in the hyperplanes. A similar proof was given by **Ze**; the authors' proof describes in more detail the successive reflexions applied to a path. *Ira Gessel*]

W2. Toshihiro Watanabe, On a determinant sequence in the lattice path counting, *J. Math. Anal. Appl.* **123**(1987) 401–414; MR **88g**:05015.

[The number of lattice paths connecting two given lattice points and staying between upper and lower boundaries can be expressed by a determinant involving binomial coefficients, due to G. Kreweras. The recursive nature of this problem leads to a system of difference equations, and the same type of solution (determinants involving polynomials of binomial type) applies to a much larger class of operator equations. The author makes a new approach by associating such determinants with random tableaux. He obtains determinant sequences which satisfy a convolution identity similar to sequences of binomial type. The theory is then applied to Hill's enzyme model, reproducing a result of Shapiro and Zeilberger. *Heinrich Niederhausen*] [cf. **K2, Hi, SZ**]

W3. Toshihiro Watanabe, On a generalization of polynomials in the ballot problem, *J. Statist. Plann. Inference* **14**(1986) 143–152; MR **87j**:05024.

[The ballot-polynomials $p_m(x) = ((x-\mu m)/(x+m))\binom{x+m}{m}$ are Sheffer polynomials for the backwards difference operator ∇. They are the solutions of the system of operator equations $\nabla p_m(x) = p_{m-1}(x)$, uniquely determined by the initial conditions $p_0(x) = 1$ for all x, and $p_m(\mu m) = 0$ for all $m \geq 1$. Using his earlier results on multivariate umbral calculus, the author shows how to solve the n-dimensional system $P_i(\delta)s_{\mathbf{m}}(\mathbf{x}) = s_{\mathbf{m}-\mathbf{e}_i}(\mathbf{x})$ for all $i = 1, \ldots, n$, where $\{P_1(\delta), \ldots, P_n(\delta)\}$ is a delta set and $\mathbf{e}_i$ stands for the ith unit vector. The general solution for such a system is then specialized to give an n-dimensional version of the ballot polynomials.

A generalized version of the classical ballot problem, attributed to Takács, is solved by the polynomials

$$\frac{x}{x+\sum_{k=1}^{r}(\mu_k+1)n_k}\binom{x+\sum_{k=1}^{r}(\mu_k+1)n_k}{n_1,\ldots,n_r}.$$

The n-dimensional analog is obtained from a very general setting, leading to a so-called "multinomial basic polynomial sequence". *Heinrich Niederhausen*]

W4. Toshihiro Watanabe, On the Littlewood-Richardson rule in terms of lattice path combinatorics, Proc. First Japan Conf. Graph Theory & Appl., *Discrete Math.* **72**(1988) 385–390; MR **90b**:05010.

[This paper gives a proof of the Littlewood-Richardson rule for multiplying Schur functions by using the characterization of Schur functions as collections of nonintersecting lattice paths. The proof is based on Robinson's recomposition rule for transforming non-lattice paths into lattice paths. *Dennis White*]

Ze. Doron Zeilberger, André's reflection proof generalized to the many-candidate ballot problem, *Discrete Math.* **44**(1983) 325–326; MR **84g**:05016.

[The n-candidate ballot problem is the problem of counting those lattice walks from the origin to the point $(m_1,\ldots,m_n)$, $m_1 \geq m_2 \geq \ldots \geq m_n \geq 0$, which never touch any of the hyperplanes $x_i - x_{i+1} = -1$. The problem is equivalent to that of counting Young tableaux of a given shape. D. André [**A1**] showed that for $n=2$ the answer is $\binom{m_1+m_2}{m_1} - \binom{m_1+m_2}{m_1+1}$ as follows: $\binom{m_1+m_2}{m_1}$ counts all paths from (0,0) to (m_1,m_2). If a path touches $x_1 - x_2 = -1$, reflect the initial segment up to the first such touch in this line to get a path from $(-1,1)$ to (m_1,m_2). This gives a bijection between "bad" paths from (0,0) to (m_1,m_2) and all paths from $(-1,1)$ to (m_1,m_2) which proves the formula.

The author generalizes this argument to obtain the determinant formula (due to Frobenius and MacMahon) $(m_1+\ldots+m_n)!\det(1/(m_i-i+j)!)$. Here a term corresponding to a permutation π counts paths from $(1-\pi(1),\ldots,n-\pi(n))$ to $(m_1,\ldots,m_n)$. The "bad" paths are cancelled in pairs by reflexion in the hyperplanes $x_i - x_{i+1} = -1$. A related approach, using recurrences instead of reflexion, was recently given by N. Linial [**Li**]. *Ira Gessel*]