to apply an analogy of the method used in sec. 4.5. We shall give only a brief description.

If y is a positive number, then $\varphi(y)$ will denote the volume of that part of the cube $-1 \leq x_1 \leq 1, \ldots, -1 \leq x_n \leq 1$, the points of which satisfy $h(x_1, \ldots, x_n) \geq -\frac{1}{2}y^2$. Then we have

$$F(t) = \int_0^\infty e^{-\frac{1}{2}ty^2} d\varphi(y),$$

and so the problem has been reduced to a question about a single integral. Usually $\varphi(y)$ will be differentiable, and $\varphi'(y) \sim ny^{n-1}D^{-\frac{1}{2}}V_n$ $(y \to 0)$, where V_n is the volume of the unit sphere in n-dimensional euclidean space. For the main term we now obtain

$$F(t) \sim nD^{-\frac{1}{2}}V_n \int_0^\infty e^{-\frac{1}{2}ty^2}y^{n-1}\,dy = nD^{-\frac{1}{2}}V_n \cdot 2^{-1+\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)t^{-\frac{1}{2}n} \quad (t \to \infty).$$

As $V_n = \pi^{\frac{1}{2}n}/\Gamma(\frac{1}{2}n + 1)$, this gives the same result as (4.6.2).

4.7. An application

We shall discuss an instructive example of the multidimensional Laplace method. We consider the sum

$$S(s, n) = \sum_{k=0}^{2n} (-1)^{k+n} \binom{2n}{k}^s, \tag{4.7.1}$$

where s and n are positive integers. It is well-known that $S(1, n) = 0$, $S(2, n) = (2n)!/(n!)^2$, and a formula of Dixon [1]) gives $S(3, n) = = (3n)!/(n!)^3$. One of course expects similar formulas for larger values of s, but no such formula is known. A simple method to decide on the existence of such a formula is to determine the asymptotic behaviour of $S(s, n)$ as $n \to \infty$ (s fixed) and to investigate whether this corresponds to the behaviour of multiplicative combinations of factorials. It will turn out that the asymptotic formula for $S(s, n)$ involves $(\cos \pi/2s)^{2ns}$. The number $(\cos \pi/2s)^{2s}$ is rational if $s = 2$ or 3. If $s > 3$, however, this is no longer true, and it follows that $(\cos \pi/2s)^{2sn}$ does not occur in the Stirling formulas for $n!$, $2n!$, $3n!$, ... Therefore, we cannot expect simple extensions of the Dixon formula if $s > 3$.

Properly speaking, the discussion of $S(s, n)$ belongs to Ch. 3. We

[1]) See W. N. Bailey, Generalized Hypergeometric Series, Cambridge 1935, p. 13.

are, however, in the situation described in sec. 3.11: the sum is exponentially small compared to the largest term (i.e. the term with $k = n$). This fact is easily verified in the cases $s = 1, 2, 3$, and for general s it follows from our final result (4.7.4). (We notice that the term with $k = n$, which we denote by t_n, is asymptotically $(2^{2n}(\pi n)^{-\frac{1}{2}})^s$). This means, roughly, that the Euler-Maclaurin method (in the version of sec. 3.11, because of the alternating signs) gives a result of the type

$$S/t_n \approx 0 + 0 \cdot n^{-1} + 0 \cdot n^{-2} + \ldots,$$

and possibly (by the method of sec. 3.10) that S/t_n is exponentially small, but we will not be satisfied with a mere upper estimate. Moreover, in this case, the terms are, considered as functions of the summation variable k, quite awkward, and the Euler-Maclaurin analysis becomes involved. For these reasons it is worth while to try other explicit expressions for S. One possibility is used below, another one (not restricted to the case that s is an integer) will be used in sec. 6.4.

It is easily seen that the sum $S(s, n)$ is equal to the coefficient of $z_1^0 z_2^0 \ldots z_r^0$ in the product

$$(-1)^n (1 + z_1)^{2n} (1 + z_2)^{2n} \cdots (1 + z_r)^{2n} \{1 - (z_1 \cdots z_r)^{-1}\}^{2n},$$

where $r = s - 1$.

As $S(1, n) = 0$ is trivial, we henceforth assume $s \geq 2$, $r \geq 1$. By Cauchy's formula we have

$$S(r + 1, n) = (-1)^n (2\pi i)^{-r} \int \ldots \int (1 + z_1)^{2n} \cdots (1 + z_r)^{2n} \cdot \\ \cdot \{1 - (z_1 \cdots z_r)^{-1}\}^{2n} (z_1^{-1} dz_1 \ldots z_r^{-1} dz_r),$$

where the integrals are taken along the unit cicles in the complex z-planes.

On substituting $z_j = \exp(2i\varphi_j)$ we obtain

(4.7.2)
$$S(r + 1, n) = 2^{2rn+2n} \pi^{-r} \cdot \\ \cdot \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \ldots \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \{\cos \varphi_1 \cdots \cos \varphi_r \sin(\varphi_1 + \ldots + \varphi_r)\}^{2n} d\varphi_1 \ldots d\varphi_r,$$

and to this multiple integral we can apply the Laplace method. We put

$$G(\varphi_1, \ldots, \varphi_r) = \cos \varphi_1 \cdots \cos \varphi_r \sin(\varphi_1 + \ldots + \varphi_r),$$

and our first question concerns the extreme points of G. As $G = 0$ on the boundary of the cube

$$-\tfrac{1}{2}\pi \leq \varphi_1 \leq \tfrac{1}{2}\pi, \ \ldots, \ -\tfrac{1}{2}\pi \leq \varphi_r \leq \tfrac{1}{2}\pi,$$

whereas G takes both positive and negative values inside the cube, the boundary can be neglected. As to the inner points, we remark that G has continuous partial derivatives, and so we only need to consider points where $\partial G/\partial\varphi_1 = \ldots = \partial G/\partial\varphi_r = 0$. Excluding points where $G = 0$, we have, if $j = 1, \ldots, r$,

$$\text{(4.7.3)} \qquad \partial G/\partial\varphi_j = \{-\tan\varphi_j + \cot(\varphi_1 + \ldots + \varphi_r)\}\cdot G.$$

Hence our condition implies that $\tan\varphi_1 = \ldots = \tan\varphi_r$. The φ_j being restricted to the interval $(-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi)$ it follows that all φ_j are equal, $\varphi_1 = \ldots = \varphi_r = \alpha$, say. We obtain $\cot r\alpha = \tan\alpha$, and so $\alpha + r\alpha = \tfrac{1}{2}\pi + k\pi$, where k is an integer. In other words $\alpha = \nu\pi/2s$, where $s = r + 1$, and ν is an odd integer, $|\nu| < s$. The value of G in such a point is

$$G(\alpha, \ldots, \alpha) = (\cos\alpha)^r \sin(r\alpha) = \pm(\cos\alpha)^s.$$

So there are two absolute maxima of G^2, corresponding to $\nu = +1$ and $\nu = -1$. These are $\alpha = \beta$ and $\alpha = -\beta$, respectively, where $\beta = \pi/2s$. It is sufficient to consider only one of them, $\alpha = +\beta$, say. For, the integral in (4.7.2) can be split into two equal parts, according to $\varphi_1 + \ldots + \varphi_r > 0$ or < 0.

We shall write, in a neighbourhood Ω of $(\beta, \ldots, \beta)$,

$$G(\varphi_1, \ldots, \varphi_r) = G(\beta, \ldots, \beta) \exp h(\beta + x_1, \ldots, \beta + x_r),$$

and we have to deal with

$$2\int \ldots \int \exp(2nh(\beta + x_1, \ldots, \beta + x_r))dx_1 \ldots dx_r,$$

the integral being extended over a neighbourhood Ω' of $(0, \ldots, 0)$. As G has continuous partial derivatives of all orders, we have a multiple Taylor expansion for h (cf.(4.6.1)). As G is maximal at $x_1 = \ldots = x_r = 0$, and as $h = 0$ at that point, the constant term and all linear terms vanish:

$$h(\beta + x_1, \ldots, \beta + x_r) = -\tfrac{1}{2}\sum_{i=1}^{r}\sum_{j=1}^{r} a_{ij}x_ix_j + \ldots,$$

where $a_{ij} = -(\partial/\partial\varphi_i)(\partial/\partial\varphi_j)(\log G)$, evaluated at $x_1 = \ldots = x_r = 0$.

From (4.7.3) we infer

$$a_{ij} = (\partial/\partial\varphi_i)\{\tan\varphi_j - \cot(\varphi_1 + \ldots + \varphi_r)\} =$$
$$= \delta_{ij}\cos^{-2}\varphi_j + \sin^{-2}(\varphi_1 + \ldots + \varphi_r) = (\delta_{ij} + 1)\cos^{-2}(\pi/2s),$$

for at $\varphi_1 = \ldots = \varphi_r = \pi/2s$ we have $\sin(\varphi_1 + \ldots + \varphi_r) = \sin(r\pi/2s) = \cos(\pi/2s)$. Here δ_{ij} is the Kronecker symbol: $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \neq j$. The determinant of the matrix $(1 + \delta_{ij})$ $(i, j = 1, \ldots, r)$ has elements 2 in the main diagonal, and all other elements are 1. Its value equals s (the order of the matrix is r), which easily can be shown by induction. It can also be derived from eigenvalue theory: the numbers 1 and $r + 1$ are obviously eigenvalues, and as subtraction of the unit matrix from the given matrix leads to a matrix of rank 1, the multiplicity of the eigenvalue 1 equals $r - 1$. Therefore, there are no other eigenvalues. The determinant equals the product of the eigenvalues, whence the determinant equals $r + 1$.

The matrix $(1 + \delta_{ij})$ is positive definite, for it is the matrix of the quadratic form

$$x_1^2 + \ldots + x_r^2 + (x_1 + \ldots + x_r)^2.$$

We are now in a position to apply (4.6.2) and (4.6.3), and the result is that $S(s, n)$ is asymptotically equivalent to

$$2^{2rn+2n}\pi^{-r}2\cdot(2\pi)^{\frac{1}{2}r}D^{-\frac{1}{2}}\cdot(2n)^{-\frac{1}{2}r}\cdot\{G(\beta, \ldots, \beta)\}^{2n},$$

where $D = s\cos^{-2r}(\pi/2s)$, $G(\beta, \ldots, \beta) = \cos^s(\pi/2s)$. It results that

$$(4.7.4)\qquad S(s, n) \sim \{2\cos(\pi/2s)\}^{2ns+s-1}2^{2-s}(\pi n)^{\frac{1}{2}(1-s)}s^{-\frac{1}{2}}$$

if $n \to \infty$ and if s is fixed $(s = 2, 3, \ldots)$.

As a verification we take $s = 3$. Then we find

$$S(3, n) \sim 3^{3n+\frac{1}{2}}(2\pi n)^{-1} \qquad (n \to \infty),$$

and since

$$(3n)!/(n!)^3 \sim (3n)^{3n+\frac{1}{2}}(2\pi)^{\frac{1}{2}}e^{-3n}\{n^{n+\frac{1}{2}}(2\pi)^{\frac{1}{2}}e^{-n}\}^{-3},$$

this is in accordance with Dixon's formula $S(3, n) = (3n)!/(n!)^3$.

4.8. Exercises. 1. Prove the asymptotic equivalence

$$\int_0^\pi x^n \sin x\, dx \sim \pi^{n+2}n^{-2} \qquad (n \to \infty).$$