SOLUTION 3 (18.05, SPRING 2003)

3.1-8 a) $f(i) = 1/12$, for all $i = 0, 1, 2, \ldots, 11$. (i.e. f - uniform distribution on $\{0, 1, 2, \ldots, 11\}$)

3.1-10

$$p = P(\text{exactly one}) = \frac{\binom{3}{1} \cdot \binom{47}{9}}{\binom{50}{10}} = \frac{39}{98} \approx 0.398.$$

$$P(\text{at most one}) = p + P(\text{exactly zero}) = p + \frac{\binom{3}{0} \cdot \binom{47}{10}}{\binom{50}{10}} \approx 0.398 + 0.504 = 0.902$$

3.2-2

$$E\big(X(11 - X)\big) = \frac{1}{10}(1 \cdot 10 + 2 \cdot 9 + \cdots + 10 \cdot 1) = \frac{2(10 + 18 + 24 + 28 + 30)}{10} = 22.$$

3.2-6 The number of three digit integers is 900. Thus

$$E = \frac{1}{900}(499) + \frac{899}{900}(-1) = -\frac{499 - 899}{900} = -\frac{4}{9}.$$

3.2-14 a)

$$\text{average class size} = \frac{16 \cdot 25 + 3 \cdot 100 + 1 \cdot 300}{16 + 3 + 1} = \frac{1000}{20} = 50.$$

b) Let X be the class size of a random student. Then

$$P(X = 25) = \frac{16 \cdot 25}{1000} = \frac{4}{10},$$

$$P(X = 100) = \frac{3 \cdot 100}{1000} = \frac{3}{10},$$

$$P(X = 300) = \frac{1 \cdot 300}{1000} = \frac{3}{10}.$$

c) The mean (whether surprising or not) is

$$\mu = 25 \cdot \frac{4}{10} + 100 \cdot \frac{3}{10} + 300 \cdot \frac{3}{10} = 10 + 30 + 90 = 130.$$

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3.2-12 The number of ways to get sum equal 7 is 6. By the symmetry, Low can be obtained $(6^2 - 6)/2 = 15$ ways. The same for the High. The expected payoff in either case is

$$(+1) \cdot \frac{15}{36} + (-1) \cdot \frac{21}{36} = -\frac{1}{6}$$

As for the betting on 7, the expected payoff is

$$(+4) \cdot \frac{6}{36} + (-1) \cdot \frac{30}{36} = -\frac{1}{6}$$

Bad deal either way...

H3-1 Note that $X = Y + Z$ where Y is first time Tails occur, Z is the first time Heads occur *after* the first Tails occur (at X). Thus

$$E(X) = E(Y) + E(Z) = \frac{1}{1-p} + \frac{1}{p} = \frac{p + (1-p)}{(1-p)p} = \frac{1}{p(1-p)}.$$

H3-2 a) Clearly,

$$P_1 = P(\text{first player wins after 1 flip of both coins}) = p_1(1 - p_2)$$

$$P_2 = P(\text{second player wins after 1 flip of both coins}) = p_2(1 - p_1)$$

Regardless of the repetitions, the probability that the first player wins is

$$\frac{P_1}{P_1 + P_2} = \frac{p_1(1-p_2)}{p_1(1-p_2) + p_2(1-p_1)}.$$

b) The probability that either player wins after the first flip of both coins is $Q = P_1 + P_2$. Thus the expected number of repetitions is

$$\frac{1}{Q} = \frac{1}{P_1 + P_2} = \frac{1}{p_1(1-p_2) + p_2(1-p_1)}.$$