MIDTERM 2 SOLUTIONS (18.05, SPRING 2003)

1) For a Poisson distribution with mean $\lambda > 0$ we have

$$P(X = n) = \frac{\lambda^n e^{-\lambda}}{n!}.$$

We have $P(X = 2) = \frac{\lambda^2 e^{-\lambda}}{2}$, $P(X = 4) = \frac{\lambda^4 e^{-\lambda}}{24}$. Now $P(X = 2) = 12\, P(X = 4)$ implies $\lambda^2 = 1$, and $\lambda = 1$. Therefore,

$$P(X = 17) = 1/(e \cdot 17!) \approx .1 \cdot 10^{-14}$$

and

$$P(X \geq 4 \mid X \geq 2) = \frac{1 - (P(0) + P(1) + P(2) + P(3))}{1 - (P(0) + P(1))} = \frac{1 - \frac{1}{e}(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!})}{1 - \frac{1}{e}(\frac{1}{0!} + \frac{1}{1!})}$$

$$= \frac{1 - \frac{1}{e} \cdot 2\frac{2}{3}}{1 - \frac{1}{e} \cdot 2} \approx \frac{0.019}{0.264} \approx 0.072$$

2) Recall that $\chi^2(7)$ has the same p.d.f. as Gamma distribution with $\alpha = 7/2$ and $\theta = 2$. For the Gamma distribution the p.d.f. is given by

$$\frac{1}{\Gamma(\alpha)\theta^\alpha}\, x^{\alpha - 1}\, e^{-x/\theta}$$

Now, $\Gamma(t) = \Gamma(t-1)\Gamma(t)$ for all t. Thus

$$\Gamma(\frac{7}{2}) = \frac{5}{2} \cdot \Gamma(\frac{5}{2}) = \frac{5}{2} \cdot \frac{3}{2} \cdot \Gamma(\frac{3}{2}) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma(\frac{1}{2}) = \frac{15}{8}\sqrt{\pi}$$

Together this gives a p.d.f. for $\chi^2(7)$:

$$\frac{1}{\frac{15}{8}\sqrt{\pi}\, 2^{7/2}}\, x^{\frac{7}{2} - 1}\, e^{-x/2} = \frac{1}{15\sqrt{2\pi}}\, x^{\frac{5}{2}}\, e^{-x/2}$$

Recall the mean $E(X) = \alpha\theta = 7$ and the variance $\mathrm{Var}(X) = \alpha\theta^2 = 14$. Therefore, $E(Y) = 16 \cdot 7 = 112$. By independence, $\mathrm{Var}(X) = 16 \cdot 14 = 224$.

3) Recall that $Y = \frac{X-8}{8}$ is $N(0,1)$. Therefore

$$P(X \leq 32) = P\left(Y \leq \frac{32 - 8}{8}\right) = P(Y \leq 3) \approx 0.9987 \ \text{(from the table)}$$

Similarly,

$$P(X \geq -4) = P\left(Y \geq \frac{-4-8}{8}\right) = P(Y \geq -1.5) = P(Y \leq 1.5) \approx 0.9332$$

$$P(0 \leq X \leq 12) = P\left(\frac{0-8}{8} \leq Y \leq \frac{12-8}{8}\right) = P(-1 \leq Y \leq .5)$$
$$= P(Y \leq .5) - P(Y \geq -1) = P(Y \leq .5) - (1 - P(Y \leq 1)$$
$$\approx 0.6915 - (1 - 0.8413) = 0.5328$$

On the other hand, we obviously have

$$P(X \geq 12 \mid X \geq 32) = 1$$

(the conditioning implies the event in the probability for any random variable X).

4) Suppose Z is $N(0,1)$. Recall that $X = Z^2$ is $\chi^2(1)$. Therefore

$$P(X \leq 1.44) = P(-\sqrt{1.44} \leq Z \leq \sqrt{1.44}) = P(-1.2 \leq Z \leq 1.2)$$
$$= P(Z \leq 1.2) - P(Z \leq -1.2) = P(Z \leq 1.2) - (1 - P(Z \leq 1.2))$$
$$= 2 \cdot P(Z \leq 1.2) - 1 \approx 2 \cdot 0.8849 - 1 \approx 0.7698$$

(from the table). Similarly,

$$P(X \leq 4.41 \mid X \geq 1.44) = P(|Z| \leq \sqrt{4.41} \mid |Z| \geq 1.2) = \frac{P(1.2 \leq |Z| \leq 2.1)}{P(|Z| \geq 1.2)}$$

$$= \frac{2 \cdot P(1.2 \leq Z \leq 2.1)}{2 \cdot P(Z \geq 1.2)} = \frac{\Phi(2.1) - \Phi(1.2)}{1 - \Phi(1.2)} \approx \frac{0.9821 - 0.8849}{1 - 0.8849} \approx 0.8444$$

c) For $\chi^2(1)$ we recall $\mu = 1$ and $\sigma^2 = 2$. Part a) gives $P(X \geq 1.44) \approx 1 - 0.7698 \approx 0.2302$. The Chebyshev bound gives

$$P(X \geq 1.44) \leq P(|X-1| \geq 0.44) \leq \frac{\sigma^2}{(0.44)^2} \approx 10.33$$

which obviously holds but unhelpful.

5) In other words, we have scores of 1000 seniors randomly selected from $N(500, 100^2)$ conditioned their scores are 600 and 750. Observe that score of ≤ 750 represents $\approx \Phi(\frac{750-500}{100}) = \Phi(2.5) \approx 0.9938$ portion of all students, while score of ≤ 600 represents $\approx \Phi(\frac{600-500}{100}) = \Phi(1) \approx 0.8413$ portion. Therefore, 20% students who receive financial aid are expected to represent the $(0.9938 - 0.8413)/5 = 0.1525/5 \approx 0.0305$ portion of *all* seniors. These did better than $0.9938 - 0.0305$ portion of all seniors. Now,

$$\Phi^{-1}(0.9938 - 0.0305) = \Phi^{-1}(0.9633) \approx 1.79$$

so $500 + 100 \cdot 1.79 = 679$ is the (expected) minimum SAT score for the financial aid in the incoming class.

Similarly, one would expect to have

$$\frac{\Phi(\frac{750-500}{100}) - \Phi(\frac{700-500}{100})}{\Phi(\frac{750-500}{100}) - \Phi(\frac{600-500}{100})} \approx \frac{0.9938 - 0.9772}{0.9938 - 0.8413} \approx 0.11$$

portion of the incoming class with the score ≥ 700. Note that these are 11% of the students - remaining 9% with financial aid have scores between 679 and 699.

Now, if the University spends $10K$ per each of these students, the total spent on these is $\approx 1,000 \cdot 0.11 \cdot 10K = 1,100K$.

6) Compute the probability p that two people are "surprised". If the first has a birthday on day x, the second has 5 possibilities: $x-2, x-1, x, x+1, x+2$. (Of course, days right before and right after New Years' also can differ by most 2.) Thus

$$p = \frac{1}{365^2} \sum_{x=1}^{365} 5 = \frac{5}{365}.$$

Now, the total number of pairs of two people is $\binom{60}{2}$. By the linearity of expectations, the expected number of "surprised" pairs is

$$\binom{60}{2} \cdot p = \frac{60 \cdot 59}{2} \cdot \frac{5}{365} \approx 24.25$$

Similarly, three people can have consecutive birthdays on days $x, x+1, x+2$ in $3! = 6$ ways, so the total number of possibilities is $365 \cdot 6$. Thus the probability of three people being "shocked" is

$$p' = \frac{365 \cdot 6}{365^3} = \frac{6}{365^2}.$$

There are $\binom{60}{3}$ ways to choose three people at a party. Therefore, by the linearity of expectations, we have

$$\binom{60}{3} \cdot p' = \frac{60 \cdot 59 \cdot 58}{6} \cdot \frac{6}{365^2} \approx 1.54$$

is the expected number of "shocked" triples of people.