MIDTERM 1 SOLUTIONS (18.05, SPRING 2003)

For grading, see comments at the end.

1)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{2}{3} - \frac{1}{2} \cdot \frac{2}{3} = \frac{5}{6}$$

$$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{5}}{\frac{3}{5}} = \frac{1}{3}$$

$$P((\overline{A} \cap \overline{B}) \cup C) = P(C) + P(\overline{A} \cap \overline{B} \cap \overline{C}) = \frac{3}{5} + \left(1 - \frac{1}{2}\right)\left(1 - \frac{2}{3}\right)\left(1 - \frac{3}{5}\right)$$

$$= \frac{3}{5} + \frac{1}{15} = \frac{9+1}{15} = \frac{2}{3}$$

2) The total number of students is $2+6+6+10+6 = 30$. Let p_i be the fraction of students with i books. Then $p_0 = \frac{1}{15}$, $p_1 = p_2 = p_4 = \frac{1}{5}$, $p_3 = \frac{1}{3}$. We have:

$$\mu = E(X) = 1 \cdot p_1 + 2 \cdot p_2 + 3 \cdot p_3 + 4 \cdot p_4 = \frac{1}{5} + \frac{2}{5} + \frac{3}{3} + \frac{4}{5} = 2\frac{2}{5} = 2.4$$

$$Var(X) = E(X^2) - \mu^2 = \left(1^2 \cdot \frac{1}{5} + 2^2\frac{1}{5} + 3^3\frac{1}{3} + 4^2\frac{1}{4}\right) - \left(2\frac{2}{5}\right)^2 = \frac{36}{25} = 1.44$$

3) We have:

$$1 = \int_1^{10} f(x)dx = c\int_1^{10} \frac{dx}{x} = c \ln x \,\Big|_1^{10} = c \ln 10.$$

Therefore $c = 1/\ln 10 \approx 0.43$.

$$E = \int_1^{10} x \cdot f(x)\, dx = \int_1^{10} c\, dx = 9 \cdot c = \frac{9}{\ln 10} \approx 3.91$$

$$Var = \int_1^{10} x^2 \cdot f(x)\, dx - E^2 = \int_1^{10} c \cdot x\, dx \; - \; E^2 = \frac{cx^2}{2}\,\Big|_1^{10} - \left(\frac{9}{\ln 10}\right)^2 =$$

$$= c\frac{10^2 - 1^2}{2} - \left(\frac{9}{\ln 10}\right)^2 = \frac{99}{2\ln 10} - \frac{81}{(\ln 10)^2} \approx 6.22$$

4) a) Each player gets 6 balls. The total number of ways of arranging 3 green among 12 is $\binom{12}{3}$. Of these only $\binom{6}{3}$ ways have the first player collect all three green balls. We conclude:

$$P = \frac{\binom{6}{3}}{\binom{12}{3}} = \frac{20}{220} = \frac{1}{11}$$

b) Probability that both players end up with the same number or red balls is 0 since the total number of white balls is 9 which is not divisible by 2.

5) The probability that both teams win on the same week is $p = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$. Thus the expected wait time until both teams win on the same week is $E = \frac{1}{p} = \frac{1}{\frac{1}{12}} = 12$.

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In 3) either approximate or exact answer suffice.