HOMEWORK 4 (18.319, FALL 2006)

1) Let $Q \subset \mathbb{R}^2$ be any self-intersecting polygon. By a *reflection move* we mean choosing a portion of the polygon between points $x, y \in Q$ (not necessarily vertices) and reflecting it along the (x, y) line as in the Figure. Prove or disprove that every polygon can be made non-self-intersecting by a finite sequence of reflection moves.

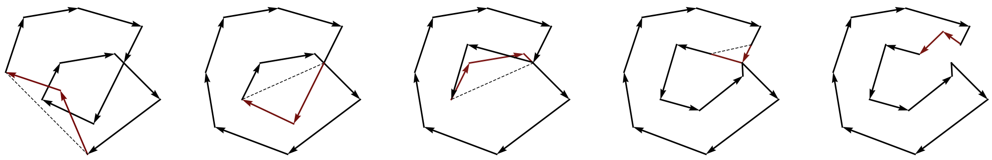

FIGURE 1. A sequence reflection of moves of a polygon.

2) Denote by $\alpha(\Gamma)$ the length of the longest cycle in graph Γ (cycles do not have repeated vertices). Construct a sequence of convex polytopes $\{P_n\}$ with graphs $\Gamma_n = \Gamma(P_n)$ on n vertices, such that $\alpha(\Gamma_n) = O(n^{1-\varepsilon})$ for some $\varepsilon > 0$ and $n \to \infty$.

3) Recall the mean curvature of a polyhedron $P \subset \mathbb{R}^3$:

$$M(P) = \sum_{e \in E(P)} \ell_e \theta_e \,,$$

where ℓ_e is the length of edge $e \in E$ and θ_e is the dihedral angle.

a) Prove the Schläfli formula for every deformation $\{T_t, t \in [0,1]\}$ of a tetrahedron $T = T_0$:

$$\sum_{e \in E} \ell_e(t) \cdot \theta'_e(t) = 0, \quad \text{for all } \ t \in [0,1].$$

b) Deduce from here the Schläfli formula for deformations of all (not necessarily convex) polyhedra.

4) Let $P \subset \mathbb{R}^3$ be a convex polyhedron and let $Q \subset S$ be a polygon on the surface $S = \partial P$. We say that the surface $A = S - Q$ is *convexly rigid* if for every convex polyhedron $P' \subset \mathbb{R}^3$ with surface S', if A is isometric to a region $A' = S' - Q'$ for some $Q' \subset S$, then A' is obtained from A by a rigid motion.

a) Prove that if Q is strictly inside a face of P, then every such $S - Q$ is convexly rigid.

b) Show that if Q lies in ε-neighborhood of a vertex of P, then $S - Q$ can be not convexly rigid, no matter how small is ε.

c) Prove or disprove part a) for all Q with no vertices of P inside.

5) Let $G = (V, E)$ be any connected graph, $V = \{v_1, \dots, v_n\}$. Fix an integer $k \geq 3$. A *drawing* of G is defined to be a maps $\varphi : V \to \mathbb{R}^2$ such that φ maps $v_1, \dots, v_k$ into vertices of a fixed convex k-gon. We say that a drawing of G is *barycentric* if every vertex v_i, $i \geq k+1$, is in the barycenter of its neighbors. Prove that such barycentric drawing is uniquely determined by G and the k-gon.

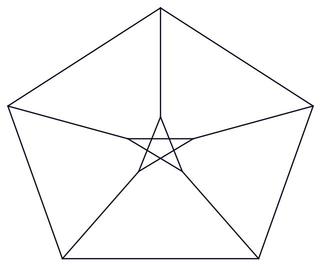

FIGURE 2. Barycentric drawing of the Petersen graph, $k = 5$.

6) (Generalized Cauchy Theorem) Consider an *unbounded* convex polyhedron P defined an intersection of a finitely many halfspaces. Suppose the origin $O \in P$. Denote by $C(P)$ the cone of all directions from O which do not intersect the boundary $S = \partial P$. Suppose two such polyhedra P, P' are combinatorially equivalent, have isometric corresponding faces and equal cones $C(P) = C(P')$. Prove that P can be obtained from P' by a rigid motion.

7) (Mean curvature again) Define

$$\mathrm{M}_d(P) = \int_{S^{d-1}} H(u)\, d\sigma(u),$$

where $P \subset \mathbb{R}^d$ is a convex polytope containing the origin O, $d\sigma$ is the invariant measure on S^{d-1}, and $H(u)$ is the support function defined by

$$H(u) = \max\{(x, u) \mid x \in P\}.$$

a) Prove that $\mathrm{M}_2(P) = \text{perimeter}(P)$.

b) Rewrite $\mathrm{M}_3(P)$ is a sum over vertices: we have:

$$\mathrm{M}_3(P) = -\sum_{v \in V} \Big(\int_{R_v} u\, d\sigma(u), \boldsymbol{r}_v \Big),$$

where $R_v = C_v^* \cap S^2$ and $\boldsymbol{r}_v = (v, O)$ as in the lectures.

c) For a simple cone $C \subset \mathbb{R}^3$ calculate the integrals in the summations on the right.

d) Use additivity to compute the integral above for general cones. Write the answer for each vertex:

$$\int_{R_v} u\, d\sigma(u) = -\sum_{e=(v,w) \in E} \theta_e\, \boldsymbol{u}_{v,e}\,.$$

e) Finally, relate $M_3(P)$ to the mean curvature.

8) Let $S = \partial P$ be the surface of a convex polytope $P \subset \mathbb{R}^3$. Prove that it is possible to subdivide S into triangles τ_i and then place these triangles on a plane in such a way that triangles that were adjacent by an edge are still adjacent by the same edge.

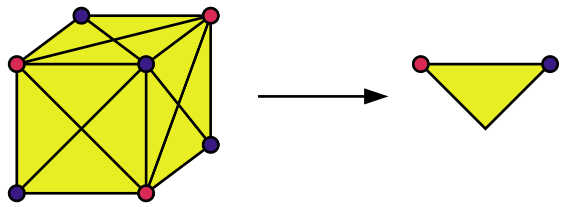

FIGURE 3. The surface of cube can be folded onto a plane.

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This is the last homework. It is due Wednesday December 6 at 11:05 am.

P.S. Do not forget: some of these problems are quite difficult. By no means you are expected to solve all or even most of them.