THE CAYLEY-MENGER DETERMINANT (18.319, FALL 2006)

Theorem (Cayley-Menger). *For every simplex* $\Delta = (v_0 v_1 \dots v_d) \subset \mathbb{R}^d$*, we have:*

$$\operatorname{vol}^2(\Delta) \,=\, \frac{(-1)^{d-1}}{2^d\, d!^2} \cdot \det \begin{pmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & \ell_{01}^2 & \ell_{02}^2 & \dots & \ell_{0d}^2 \\ 1 & \ell_{01}^2 & 0 & \ell_{12}^2 & \dots & \ell_{1d}^2 \\ 1 & \ell_{02}^2 & \ell_{12}^2 & 0 & \dots & \ell_{2d}^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \ell_{0d}^2 & \ell_{1d}^2 & \ell_{2d}^2 & \dots & 0 \end{pmatrix},$$

where $\ell_{ij} = |v_i v_j|$*, for all* $0 \le i < j \le d$*.*

Proof of the Theorem. Suppose $v_i = (x_{i1}, \dots, x_{id}) \in \mathbb{R}^d$, for all $0 \le i \le d$. Define matrices A and B as follows:

$$A \,=\, \begin{pmatrix} x_{01} & x_{02} & \dots & x_{0d} & 1 \\ x_{11} & x_{12} & \dots & x_{1d} & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{d1} & x_{d2} & \dots & x_{dd} & 1 \end{pmatrix}, \qquad B \,=\, \begin{pmatrix} x_{01} & x_{02} & \dots & x_{0d} & 0 \\ x_{11} & x_{12} & \dots & x_{1d} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{d1} & x_{d2} & \dots & x_{dd} & 0 \end{pmatrix}.$$

Clearly, $\det(B) = 0$, and

$$\det(A) \,=\, \det \begin{pmatrix} x_{01} & x_{02} & \dots & x_{0d} & 1 \\ x_{11} - x_{01} & x_{12} - x_{02} & \dots & x_{1d} - x_{0d} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{d1} - x_{01} & x_{d2} - x_{02} & \dots & x_{dd} - x_{0d} & 0 \end{pmatrix} \,=\, d! \cdot \operatorname{vol}(\Delta).$$

Note that

$$A \cdot A^T = \big((v_i, v_j) + 1\big)_{0 \le i,j \le d}, \qquad \text{and} \quad B \cdot B^T = \big((v_i, v_j)\big)_{0 \le i,j \le d}.$$

From here we have:

$$\det(A \cdot A^T) \,=\, \det \begin{pmatrix} (v_0, v_0) + 1 & (v_0, v_1) + 1 & \dots & (v_0, v_d) + 1 & 0 \\ (v_1, v_0) + 1 & (v_1, v_1) + 1 & \dots & (v_1, v_d) + 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (v_d, v_0) + 1 & (v_d, v_1) + 1 & \dots & (v_d, v_d) + 1 & 0 \\ 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

$$= -\det \begin{pmatrix} (v_0, v_0) & (v_0, v_1) & \dots & (v_0, v_d) & 1 \\ (v_1, v_0) & (v_1, v_1) & \dots & (v_1, v_d) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (v_d, v_0) & (v_d, v_1) & \dots & (v_d, v_d) & 1 \\ 1 & 1 & \dots & 1 & -1 \end{pmatrix}.$$

Since $\det\big((v_i, v_j)\big) = \det(B \cdot B^T) = \det^2(B) = 0$, in the determinant above we can replace the bottom right entry (-1) with 0. Denote the resulting matrix by M.

Now observe that $\ell_{ij}^2 = (v_i, v_i) - 2(v_i, v_j) + (v_j, v_j)$. Denote by C the matrix as in the theorem. Using row and column operations we obtain:

$$\det(C) \,=\, \det \begin{pmatrix} -2(v_0, v_0) & -2(v_0, v_1) & \dots & -2(v_0, v_d) & 1 \\ -2(v_1, v_0) & -2(v_1, v_1) & \dots & -2(v_1, v_d) & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2(v_d, v_0) & -2(v_d, v_1) & \dots & -2(v_d, v_d) & 1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix} \,=\, (-2)^d \det(M).$$

Since $\det(M) = -\det(A \cdot A^T) = -\big(d!\, \mathrm{vol}(\Delta)\big)^2$, we conclude that

$$\det(C) \,=\, (-1)^{d-1}\, 2^d\, d!^2\, \mathrm{vol}^2(\Delta),$$

as desired. □