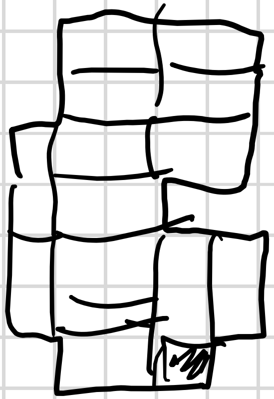
L6 (IUM, Spring) March 16, 2021

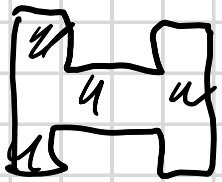
Convex Polytopes

Q: $\Gamma \subset \mathbb{Z}^2$ region, s-c

Can Γ be tiled w/ dominoes?

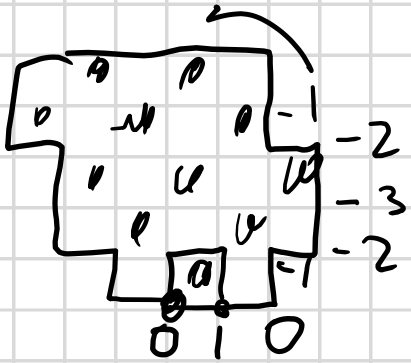


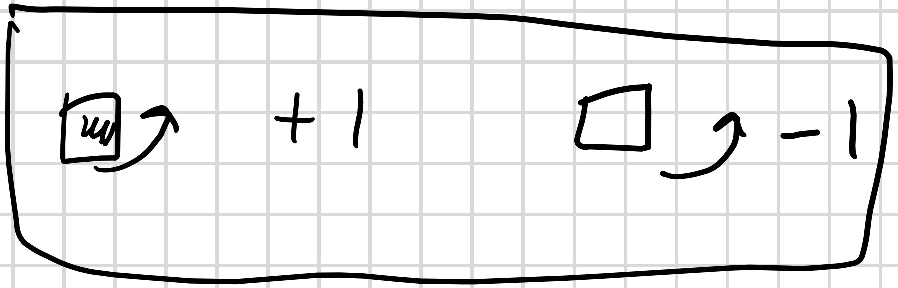
Today: Thurston's Algorithm

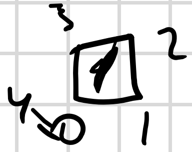


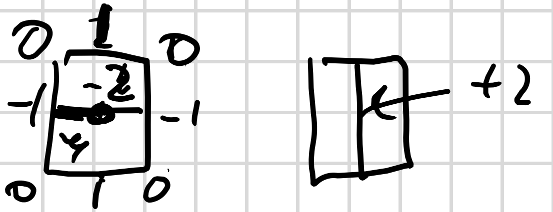
Def height function

$h: \partial\Gamma \to \mathbb{Z}$





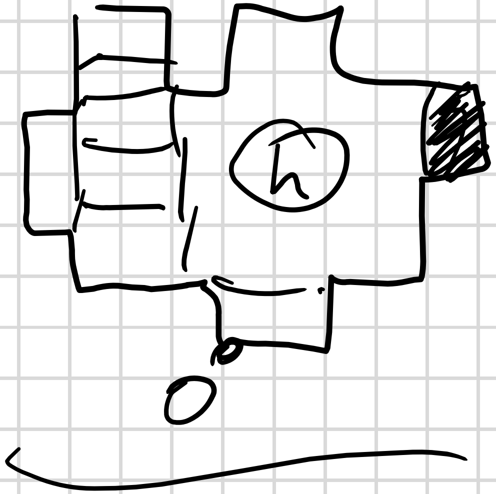




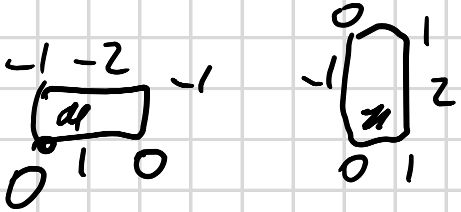
we get $h: \partial\Gamma \to \mathbb{Z}$

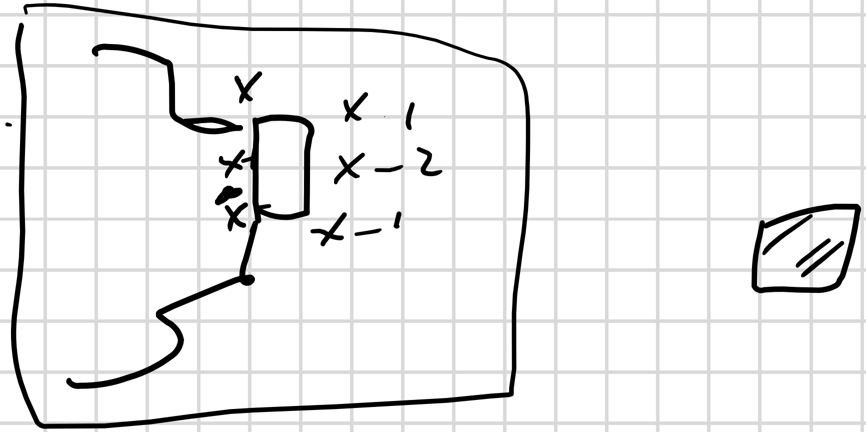
$\underline{L}$ Γ - tileable w/ dominoes $\Rightarrow$ h is well defined.

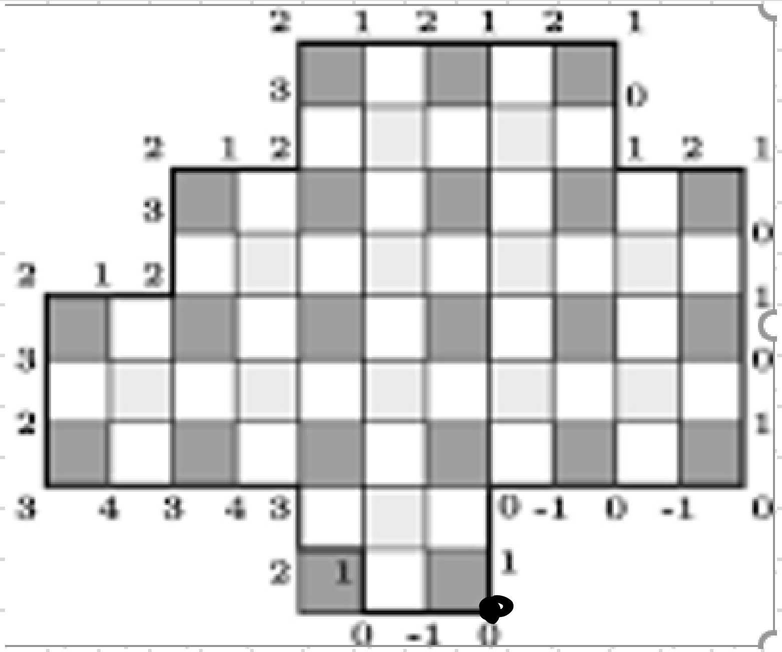
D



extend h into Γ







Def flip

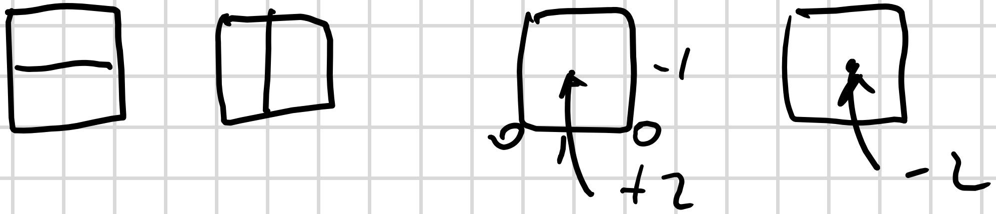
$\boxminus \leftrightarrow \boxed{\,|\,|\,}$

$$\boxed{\begin{array}{l}\text{Th} \text{ if } \Gamma \text{ - simply conn, } \Gamma \subset \mathbb{Z}^2\\ \text{Then all domino tilings}\\ \text{are flip-connected.}\end{array}}$$

Def T, T' – domino tilings

we say $\boxed{T \leqslant T'}$ if $h(x) \leq h'(x)$ $\forall x \in T$

h, h' – corresp height functions



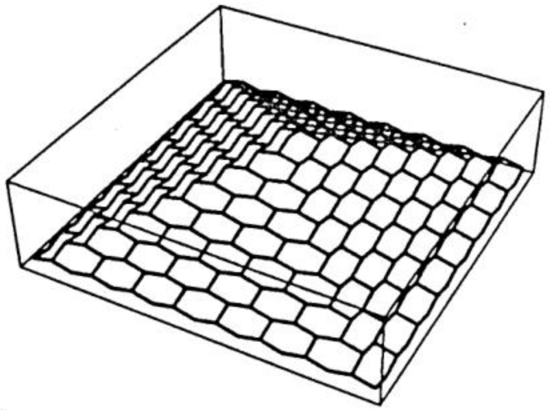
Th2 $\forall\ \Gamma \subset \mathbb{Z}^2$ – simply conn., tileable w/ dominos.

then $\exists$ domino tiling T

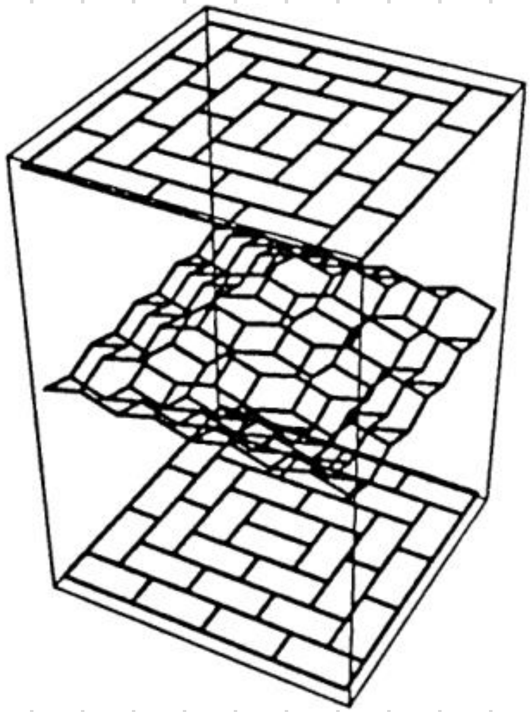
$T \leq$ all other T', T' – domino tiling of Γ

$\Updownarrow$

$$h_T = \min_{\text{all } T'} h_{T'} \quad \Leftarrow \exists \text{ global min}$$



← max



← max

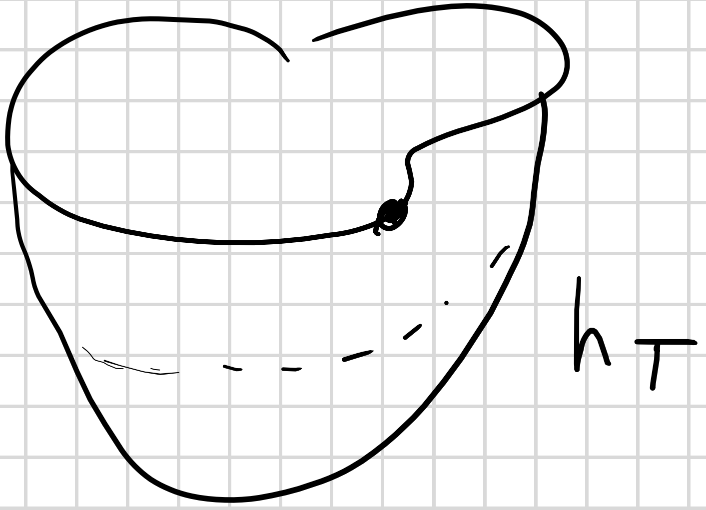
← min

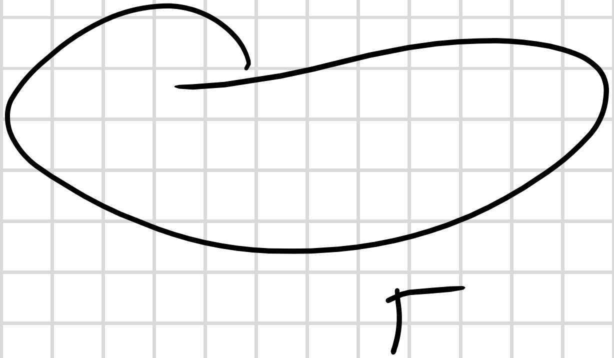
$\Gamma = \square$ k k

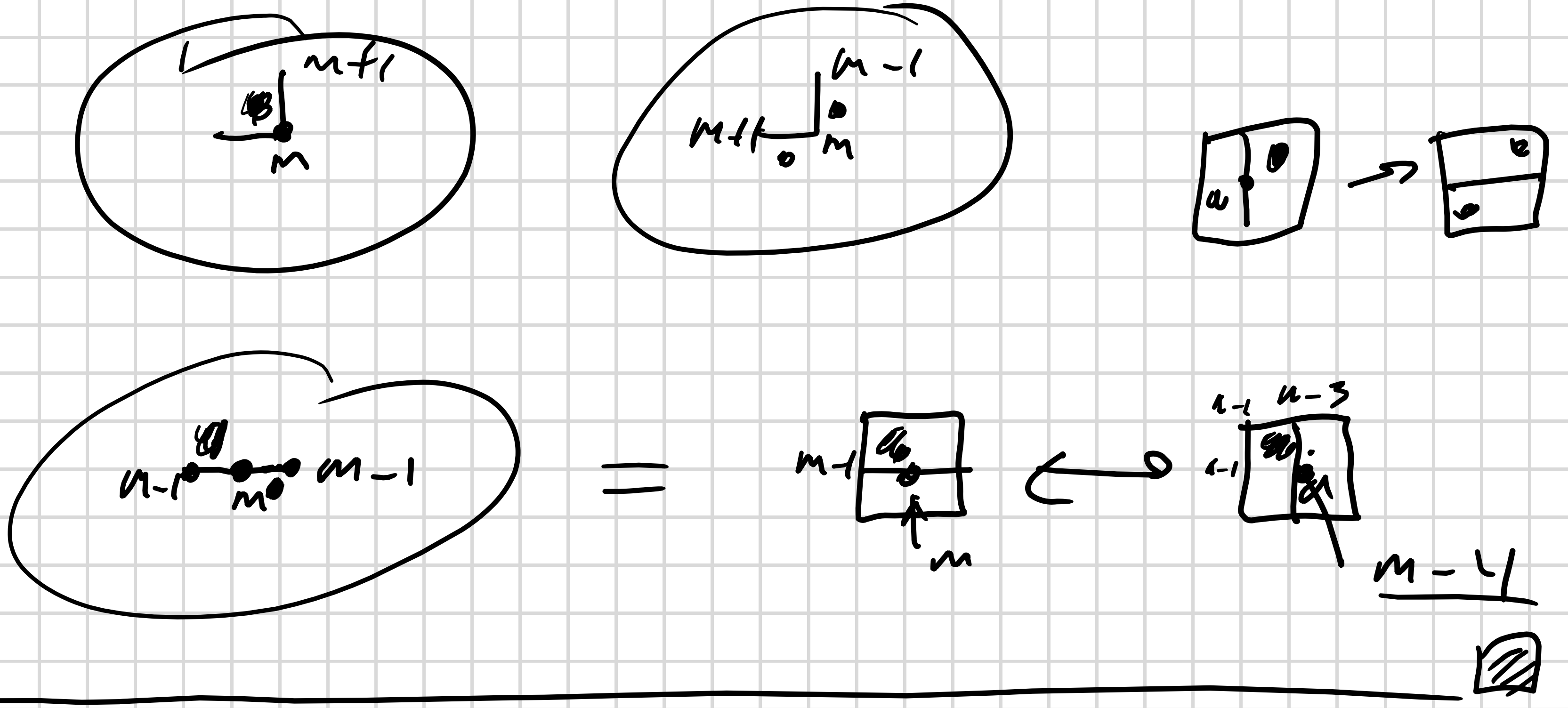
L Let T - local min of domino tilings

Then $\max h_T \in \partial \Gamma$

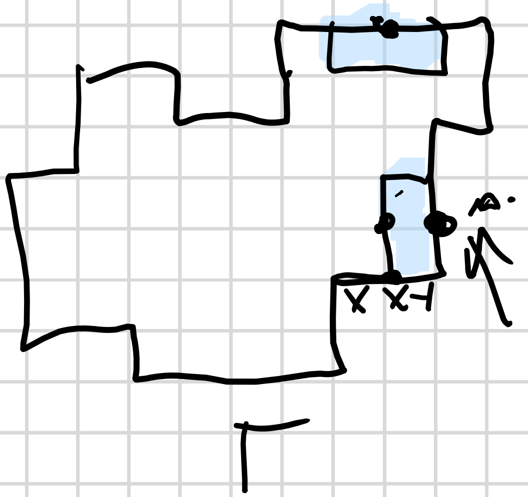
Def T - local min if $\nexists\, T'$ s.t. $T' < T$



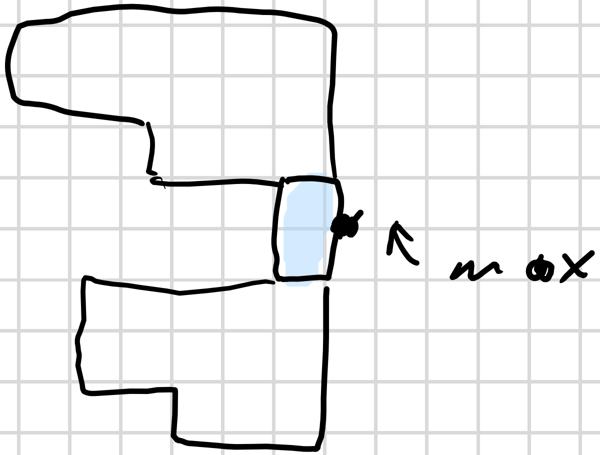


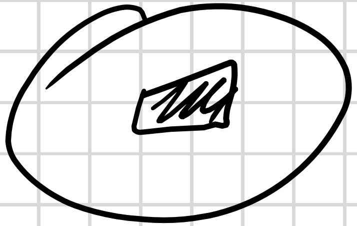


Cor (of the proof) every T is flop-connected to global min



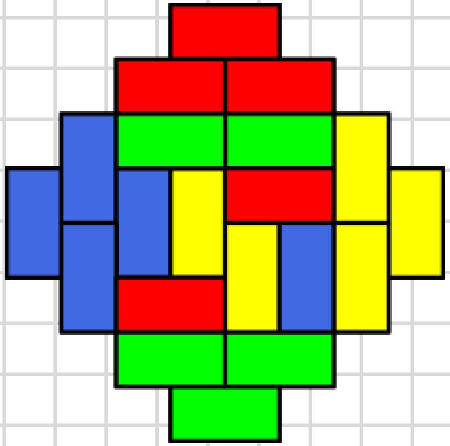
max h

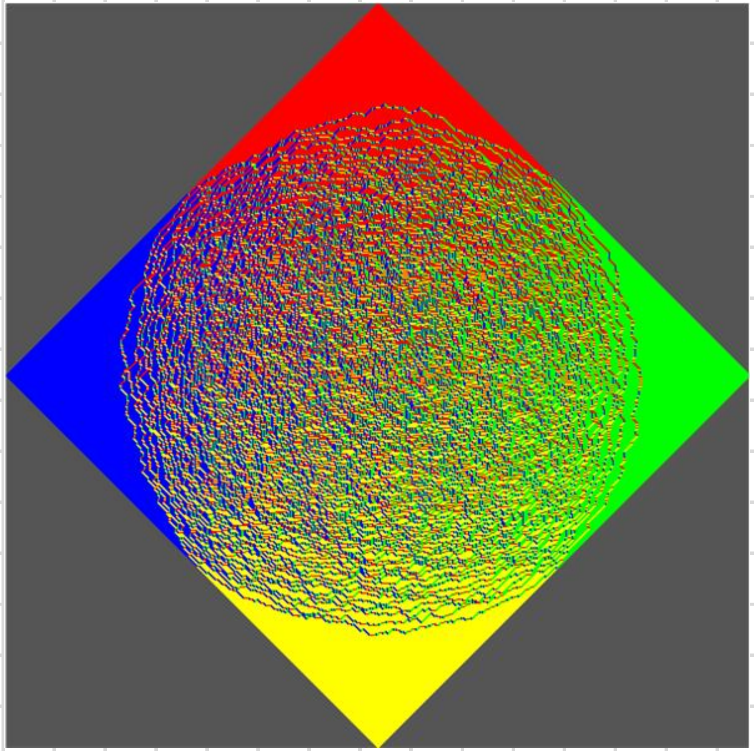




$\Rightarrow \exists!$ local min = ~~local~~ global min.

Proof of Thm for all T $\to$ global min $\boxed{\lightning}$





Q complexity of thurston's alg?

ThA $\sim O^*(\overline{\overline{\text{area}}}^{\,n})$

truth: $O(n \log n)$

Thurston's Alg

Input: Γ

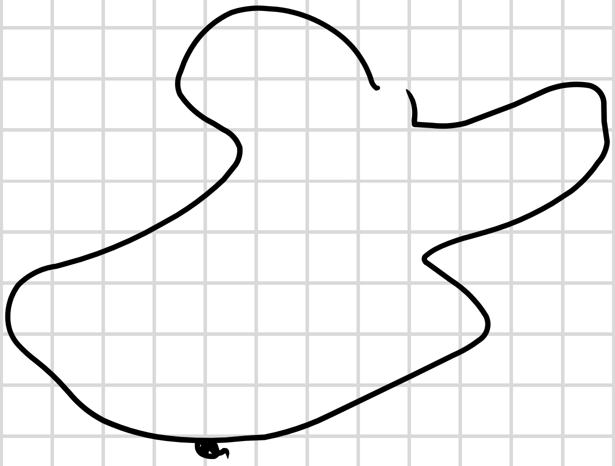
Output: yes: if $\exists$ domino tiling of Γ

NO: otherwise

Th (P-Sheffer-Tossy)

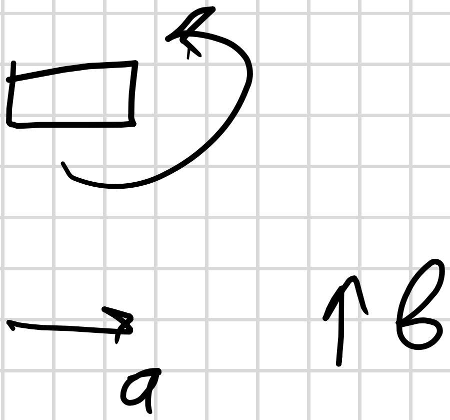
$\exists$ algorithm $O(p \log p)$

$p = |\partial \Gamma|$



$F = \langle a, b \rangle$

$F/(a^2 = b^2 = 1) \simeq D_\infty \leftrightarrow \mathbb{Z}$



Def

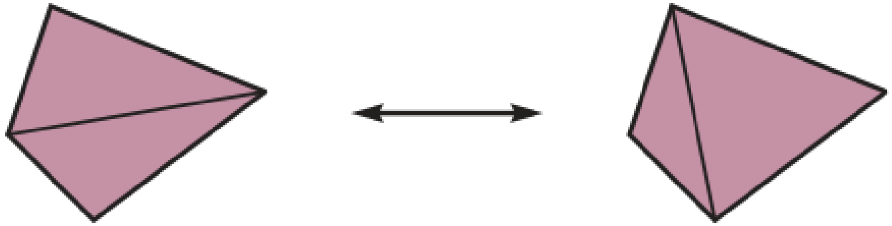


FIGURE 14.1. An example of a flip.

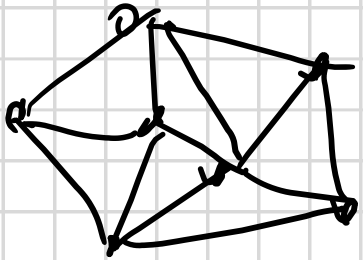
$$\boxed{Q_V = \mathrm{conv}\{v_1, \dots, v_n\}}$$

Def $V = \{v_1 \dots v_n\} \subset \mathbb{R}^2$

general position

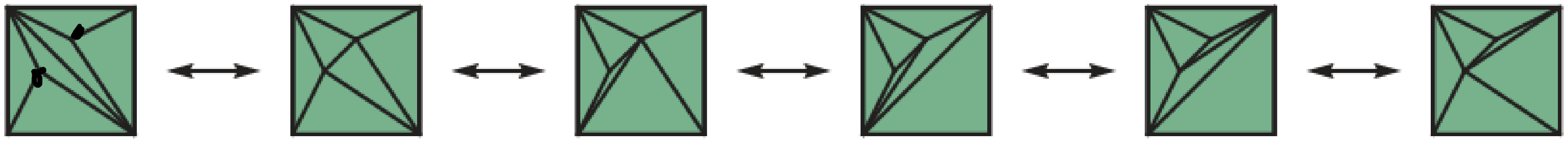
$\mathcal{T}_V$ = set of triang. of Q_V w/ vertices at V

T



$T \in \mathcal{T}_V$

Ex



Th $\forall V \subset \mathbb{R}^2$ general position all triangulations are connected by a seq of flips.

Moreover $\boxed{\#\text{ flips } = O(n^2)}$

Proof Idea $\forall T \in \mathcal{T}_V$ is conn to Delone triang.

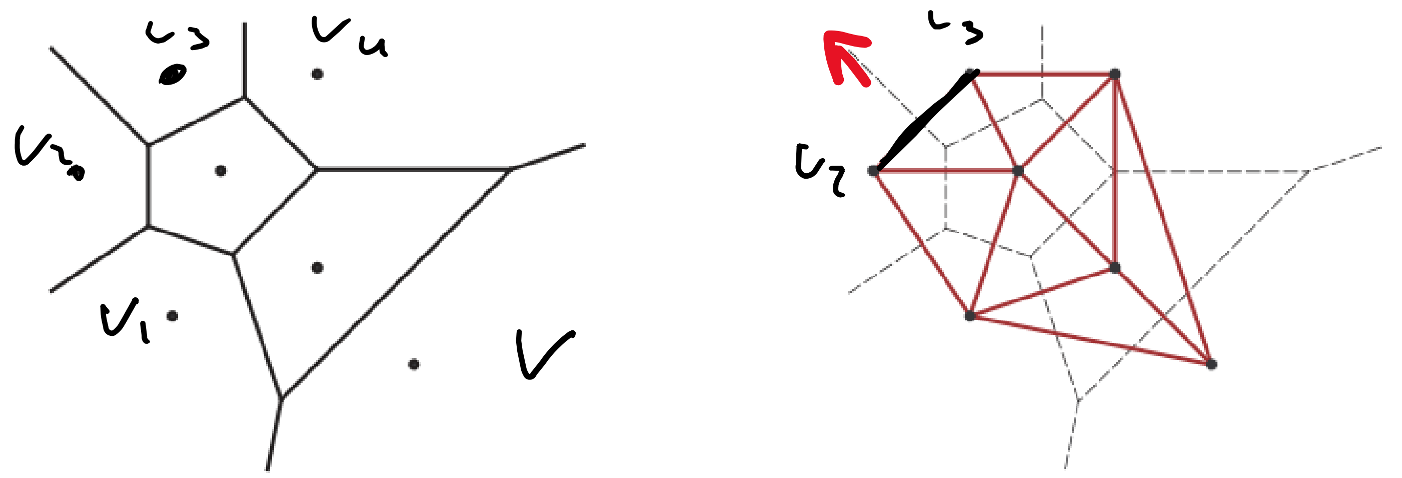


FIGURE 14.3. Voronoi diagram and the corresponding Delaunay triangulation.

Def Voronoi diagram

$$\mathbb{R}^2 = \bigsqcup_{i=1}^{n} B_i$$

$B_i \leftarrow$ set of pts in $\mathbb{R}^2$ closest to v_i

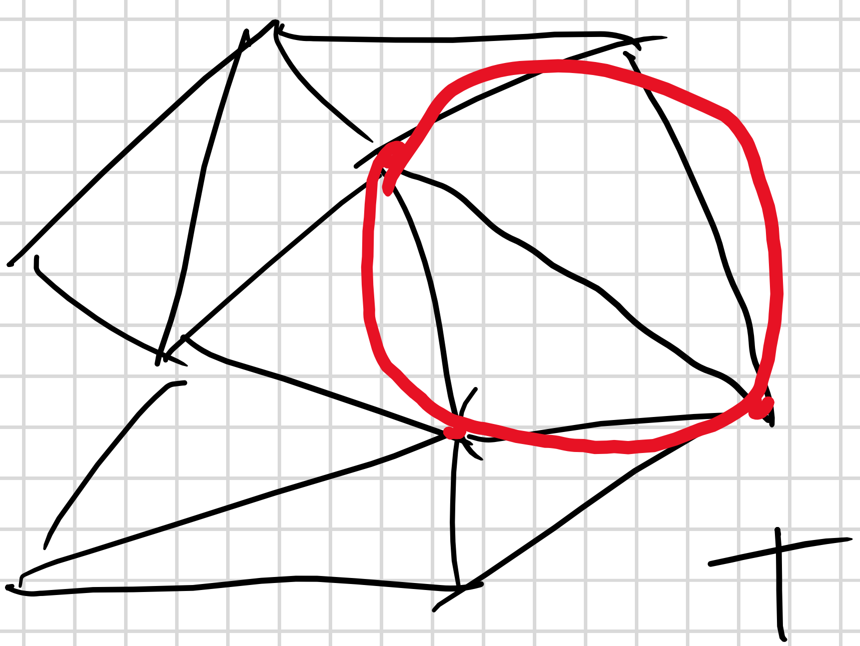
Def Del. triang $\leftarrow$ dual to Voronoi diagram

L $(v_i, v_j) \in \partial Q_V \Rightarrow B_i$ and B_j have common edge. $\square$

$Q_V = \mathrm{conv}\{v_i\}$

L (Delone criterion)

triang T is a Del. triang $\iff$ every circum circle is empty (around $v_i v_j v_k \in T$)



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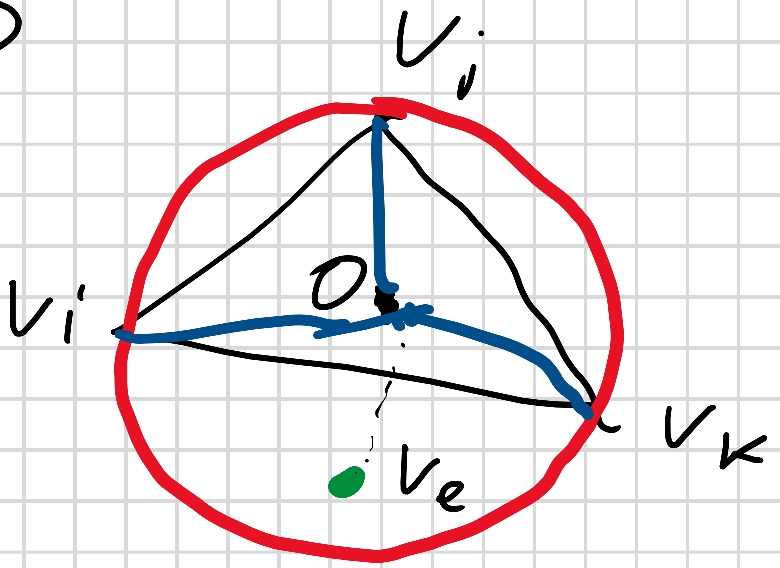
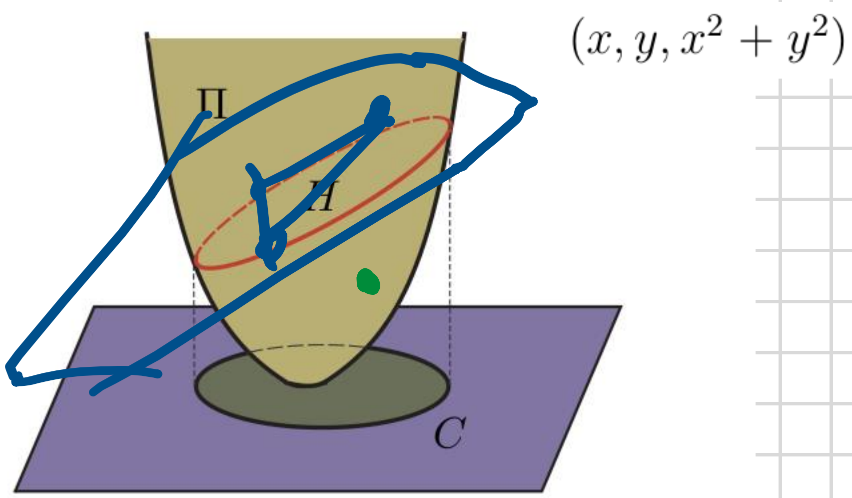
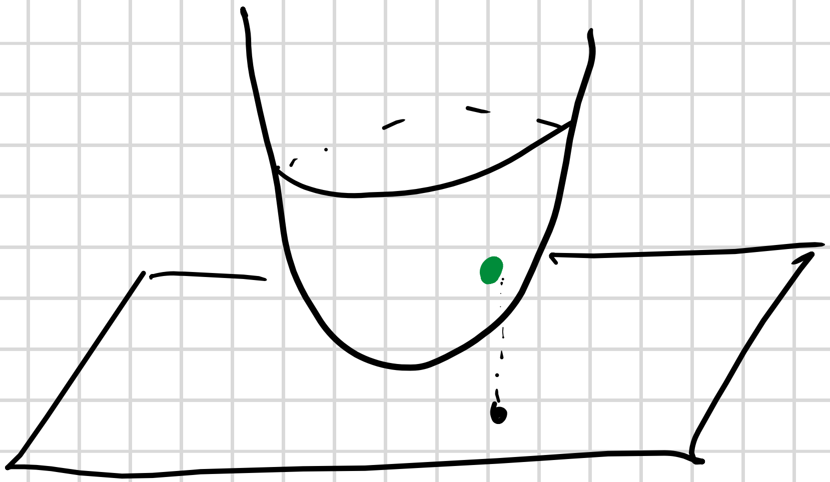






Construction $V \subset \mathbb{R}^2$

$h(x,y) := x^2 + y^2$

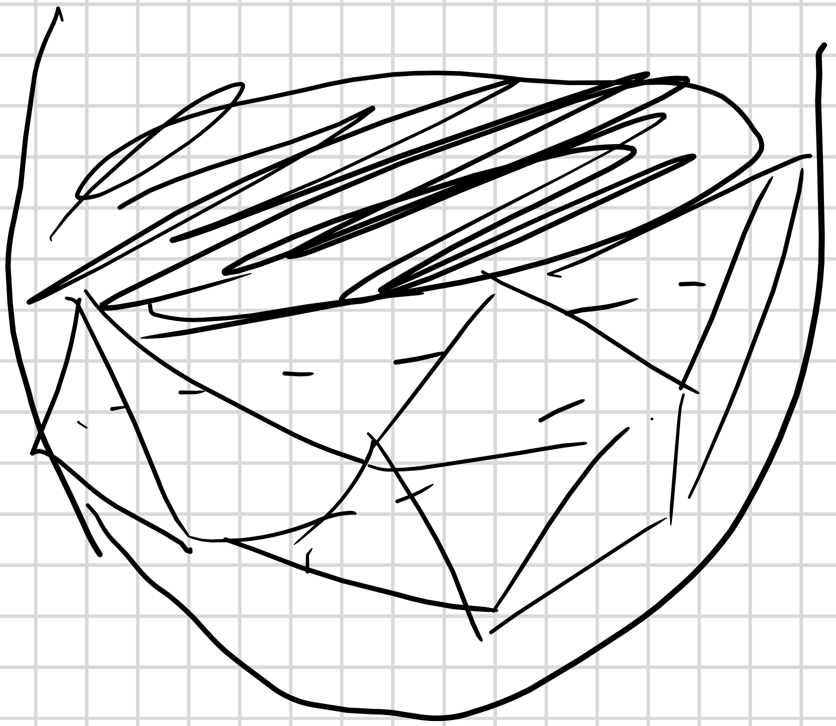
$v_i \to (v_i, h(v_i)) = \hat{v}_i$

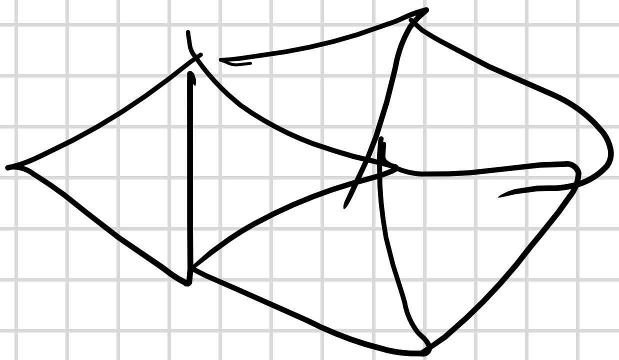
L Delaunay =

= lower convex hull of $\{\hat{v}_i\}$ P_V

Obs 1) P_V - simplicial

2)





Main Lemma

every T ⟷ Del. triang ∇

D check all circles

if NOT empty

make a flip

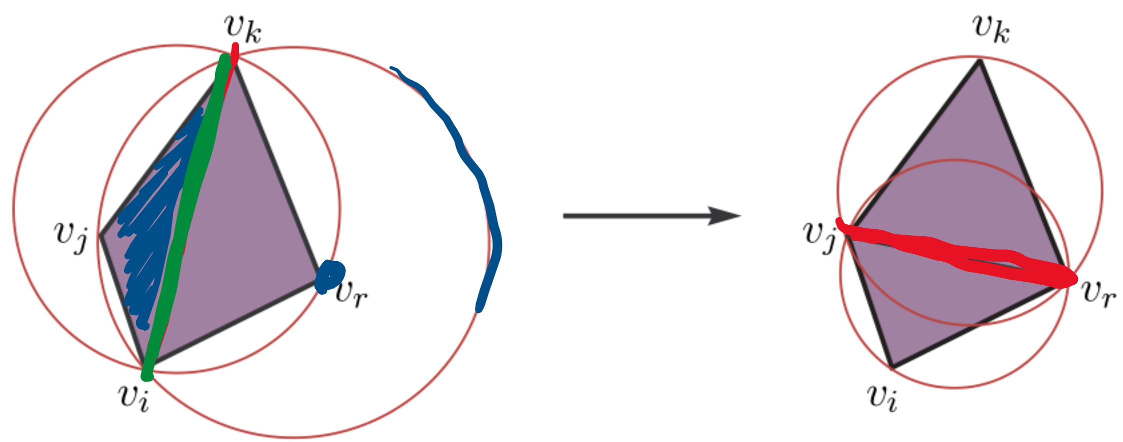
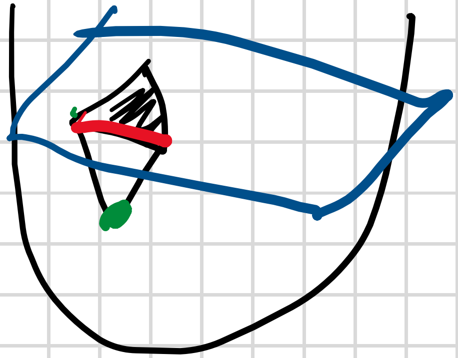


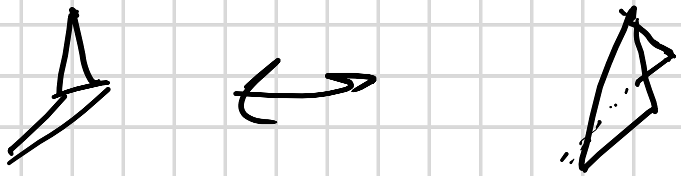
FIGURE 14.5. An increasing flip.

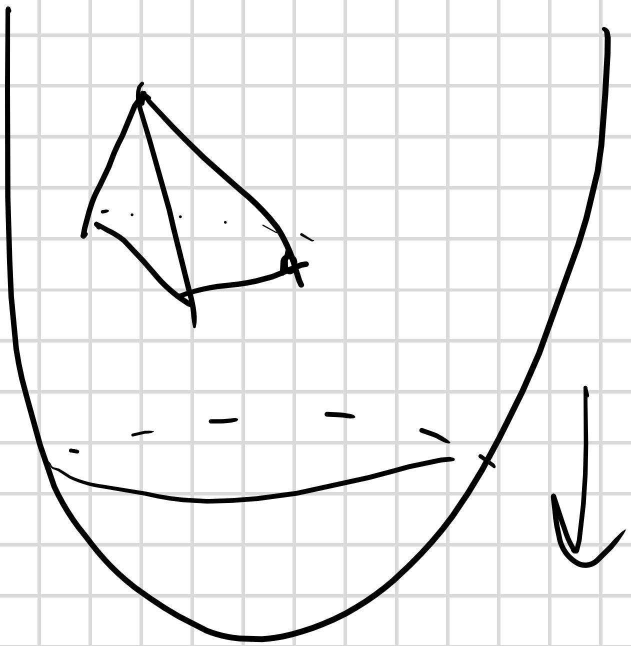
Claim increasing such flips eventually end at Del triang

Q: what happened in $\mathbb{R}^3$?



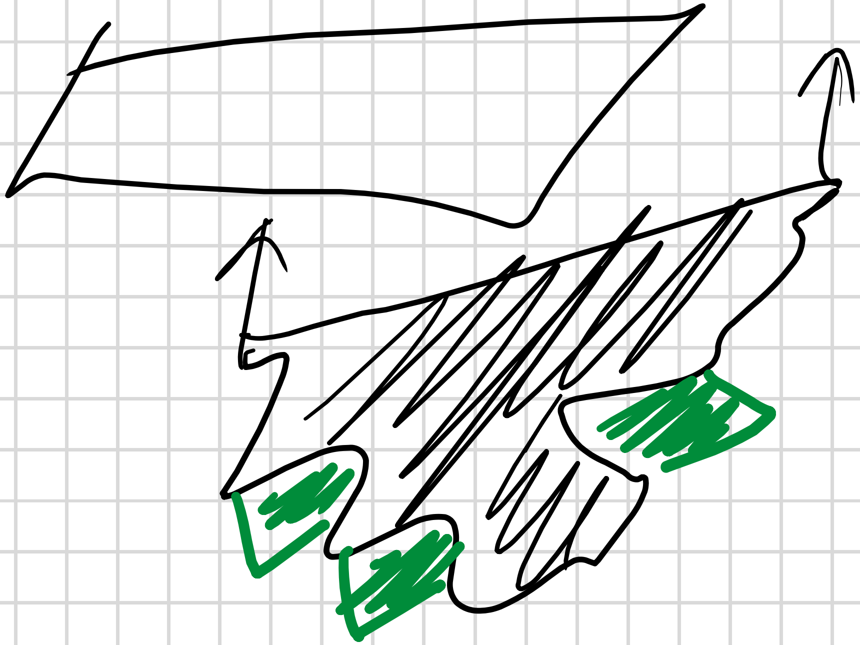
∂P_v





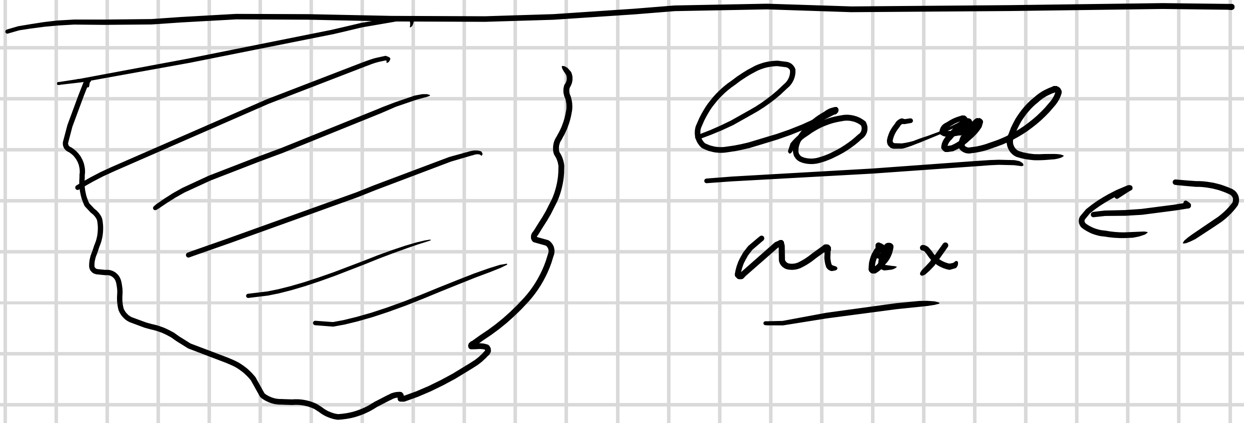
at every increasing flip

a tetrahedron is added to the ~~surface~~ upper hull



at the end → local max

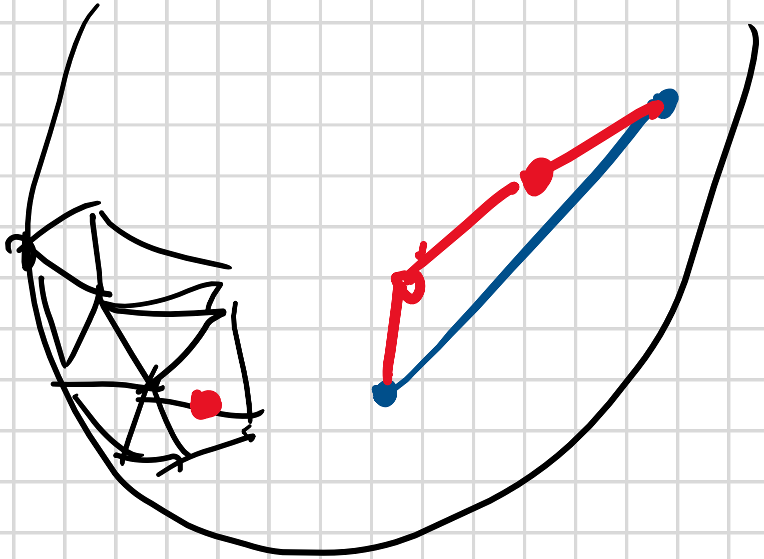
L



local max $\Leftrightarrow P_v$

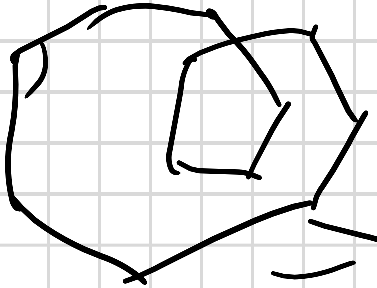
Proof   surfaces ← local max

$\Rightarrow$ S is convex at all edges



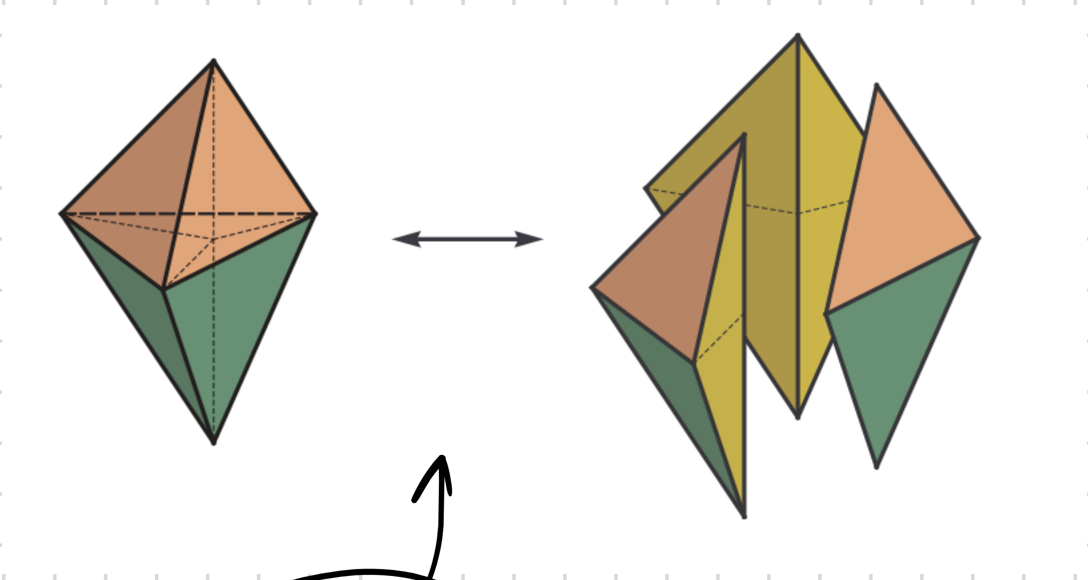
$\leq \binom{n}{2} \times 2 \leq n^2$

local convexity $\Rightarrow$ global convexity



$\Rightarrow$ Th!

$\boxed{///}$



flip

Conj $V \subset \mathbb{R}^3$ convex position

Then all triangs. are flip-conn.

$\boxed{\text{False in higher dim}}$